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FINITE DIMENSIONAL ATTRACTORS FOR THE CAGINALP SYSTEM WITH SINGULAR POTENTIALS AND DYNAMIC BOUNDARY CONDITIONS

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Abstract

Our aim in this paper is to prove the existence of finite dimensional attractors for the Caginalp system with dynamic boundary conditions and singular potentials.

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1 Introduction

We consider, in a smooth and bounded domain $\Omega \subset \mathbb{R}^3$ with boundary $\partial\Omega = \Gamma$, the phase-field system

$$\begin{cases} \eta \dfrac{\partial w}{\partial t} - \Delta w = -\dfrac{\partial u}{\partial t}, \text{ in } \Omega,\ t > 0, \\ \dfrac{\partial u}{\partial t} - \Delta u + f(u) = w, \text{ in } \Omega,\ t > 0, \\ \dfrac{\partial w}{\partial \nu}\Big|_{\Gamma} = 0,\ u|_{\Gamma} = \psi,\ t > 0, \\ \dfrac{\partial \psi}{\partial t} - \Delta_{\Gamma}\psi + \lambda\psi + g(\psi) + \dfrac{\partial u}{\partial \nu} = 0, \text{ on } \Gamma,\ t > 0, \\ u|_{t=0} = u_0,\ w|_{t=0} = w_0, \text{ in } \Omega,\ \psi|_{t=0} = \psi_0, \text{ on } \Gamma, \end{cases} \tag{1}$$

where ν is the unit outer normal to the boundary, $\eta \in (0,1)$, $\lambda > 0$ and Δ_{Γ} is the Laplace-Beltrami operator. Moreover, w represents the (relative) temperature, while u is the order parameter with trace ψ on Γ.

This system, proposed in [1] in order to model melting-solidification phenomena in certain classes of materials, has been extensively studied, for various types of boundary conditions and for regular potentials f, see, e.g., [1], [5], [6], [12] and the references therein.

Now, singular potentials f are also important from a physical point of view; in particular, we have in mind the thermodynamically relevant logarithmic potentials

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$$f(s) = -2\kappa_0 s + \kappa_1 \ln \frac{1+s}{1-s}, \ s \in (-1,1), \ 0 < \kappa_0 < \kappa_1. \tag{2}$$

The Caginalp system, with singular potentials and various types of boundary conditions, has also been extensively studied (see [3], [10] and [11]). We can note that, contrary to regular potentials, singular potentials allow to prove that the order parameter remains strictly between, say, -1 and 1, as it is expected from the physical point of view.

In this paper, we supplement the equations with the so-called dynamic boundary conditions for the order parameter (in the sense that the kinetics, i.e., the time derivative of the order parameter, appears explicitly in the boundary conditions). Such boundary conditions have been proposed by physicists, in the context of the Cahn-Hilliard equation, in order to account for the interactions with the walls in confined systems. In particular, the Caginalp system, endowed with dynamic boundary conditions and with regular potentials, was considered in [5], [7], [8] and [9].

We proved in [2] the existence of strong solutions to (1), but were not able to obtain dissipative L^∞-estimates on the order parameter. We were able to derive such estimates under sign assumptions on g in [4], namely, g is nonnegative (resp., nonpositive) close to 1 (resp., -1). These dissipative estimates are essential in order to study the asymptotic behavior of the system and, more precisely, to prove the existence of attractors.

In this paper, we are able to derive dissipative L^∞-estimates under more optimal (sign) assumptions on the (dissipation) parameter λ and the function g and then to prove the existence of finite dimensional attractors. We also give, for the sake of completeness, the proof of existence and uniqueness of strong solutions to (1) for general potentials g; indeed, there is a mistake in the proof given in [2]. Unfortunately, this requires rather restrictive assumptions on f which are not satisfied by the usual logarithmic potentials. However, these assumptions seem to be natural in order to have a strict separation property and the existence of strong solutions (see Remark 7 and [15]). Furthermore, they are no longer necessary under the aforementioned conditions on λ and g.

Assumptions and notation

We make the following assumptions on f and g:

$$f \in C^3(-1,1), \ \lim_{s\to\pm 1} f(s) = \pm\infty, \ \lim_{s\to\pm 1} f'(s) = +\infty, \tag{3}$$

$$g \in C^2(\mathbb{R}), \ \liminf_{|s|\to+\infty} g'(s) \geq 0 \text{ and} \tag{4}$$

$$\text{either } \exists \mu > 0, \ \mu' \geq 0 \text{ such that } g(s)s \geq \mu s^2 - \mu', \ \forall s \in \mathbb{R}, \ \text{ or } g \text{ is constant.}$$

In particular, there exist $K_1 > 0$ and $K_2 > 0$ such that

$$f'(s) \geq -K_1, \ g'(s) \geq -K_2. \tag{5}$$

We agree to denote the Lebesgue spaces of square summable functions in Ω and Γ by $(L^2(\Omega), \langle\cdot,\cdot\rangle, \|\cdot\|)$ and $(L^2(\Gamma), \langle\cdot,\cdot\rangle_\Gamma, \|\cdot\|_\Gamma)$, respectively. We also introduce the average $\langle w\rangle = \frac{1}{|\Omega|}\int_\Omega w dx, \ \forall w \in L^1(\Omega)$.

Finally, embodying the boundary condition for w in $H^2_N(\Omega)=\left\{w\in H^2(\Omega):\ \frac{\partial w}{\partial\nu}\Big|_\Gamma=0\right\}$, we introduce the spaces $\Phi=\{z=(u,\psi,w)\in H^2(\Omega)\times H^2(\Gamma)\times H^2_N(\Omega):\ 0<D[u]<+\infty,\ \|\psi\|_{L^\infty(\Gamma)}<1,\ \psi=u|_\Gamma\}$ and $\Phi_M=\{z\in\Phi:\ |I_0|\le M\}$ which we endow with the $H^2(\Omega)\times H^2(\Gamma)\times H^2_N(\Omega)$-norm. In the above, $D[u]=(1-\|u\|_{L^\infty(\Omega)})^{-1},\ u\in L^\infty(\Omega),\ \|u\|_{L^\infty(\Omega)}\neq 1$, $I_0=\langle\eta w+u\rangle$ (note that this quantity is conserved) and $M>0$.

In general, throughout the paper, c stands for a positive constant which is allowed to also vary in the same line.

2 Existence and uniqueness of solutions

Theorem 2.1. *We assume that* (3)-(4) *hold and that*

$$f(1-s)\sim\frac{c_+}{s^{p_+}}\ \text{and}\ f(-1+s)\sim-\frac{c_-}{s^{p_-}}\ \text{as}\ s\to 0^+,\ p_\pm>1,\ c_\pm>0. \tag{6}$$

Then, for any initial datum $z_0=(u_0,\psi_0,w_0)\in\Phi$, problem (1) *possesses a unique solution $z(t)=(u(t),\psi(t),w(t))\in\Phi$, for every $t\ge 0$.*

Remark 1. As mentioned in the introduction, assumption (6) is not satisfied by the thermodynamically relevant logarithmic potentials (2).

In order to prove this theorem, we first obtain several a priori estimates. To do so, we a priori assume that the first component $u(t)$ of the solution is separated from the singularities of f, i.e., that $\|u(t)\|_{L^\infty(\Omega)}<1,\ \forall t\ge 0$. In particular, these estimates allow to prove that $u(t)$ is actually strictly separated from the singularities of f, i.e., that $\|u(t)\|_{L^\infty(\Omega)}\le c<1,\ \forall t\ge 0$.

First, repeating word by word the proof of [4, Theorem 3.1], we have the

Theorem 2.2. *Given any initial datum $z_0=(u_0,\psi_0,w_0)\in\Phi$, every solution $z(t)=(u(t),\psi(t),w(t))\in\Phi$ to* (1) *satisfies*

$$\begin{aligned}&\|u(t)\|^2_{H^1(\Omega)}+\|\psi(t)\|^2_{H^1(\Gamma)}+\|w(t)\|^2_{H^2(\Omega)}+\|\frac{\partial u}{\partial t}(t)\|^2+\|\frac{\partial\psi}{\partial t}(t)\|^2_\Gamma\\&\le Q_\eta(D[u_0],\|z_0\|_\Phi)e^{-kt}+C_\eta,\ \forall t\ge 0,\ k>0,\end{aligned}$$

where the increasing function Q_η and the positive constant C_η depend on η (C_η also depends on I_0).

Remark 2. Assumption (6) is not necessary for the proof of Theorem 2.2.

The next task consists in obtaining estimates on u and ψ in $H^2(\Omega)$ and $H^2(\Gamma)$, respectively. These cannot be achieved directly, due to the singular values of the potential f, and we first need to derive L^∞-estimates on u and ψ.

Theorem 2.3. *Given any $z_0=(u_0,\psi_0,w_0)\in\Phi$, the first two components of any solution $z(t)=(u(t),\psi(t),w(t))\in\Phi$ to* (1) *are strictly separated from the singularities of f, namely, there exists $\delta\in(0,1)$ such that*

$$\|u(t)\|_{L^\infty(\Omega)}\le\delta\ \text{and}\ \|\psi(t)\|_{L^\infty(\Gamma)}\le\delta,\ \forall t\ge 0. \tag{7}$$

Proof. Thanks to Theorem 2.2, there exists a constant $\beta > 0$ such that $\|w(t)\|_{L^\infty(\Omega)} \leq c\|w(t)\|_{H^2(\Omega)} \leq \beta,\ \forall t \geq 0$. We then have the

Theorem 2.4. *Given $\varepsilon > 0$ small enough, there exists a function $u_\varepsilon \in H^2(\Omega)$ such that*

$$- \Delta u_\varepsilon + f(u_\varepsilon) \geq \beta, \tag{8}$$

$$u_\varepsilon|_\Gamma = 1 - \delta(\varepsilon),\ \ \delta(\varepsilon) \in (0,1), \tag{9}$$

$$\frac{\partial u_\varepsilon}{\partial \nu}\Big|_\Gamma = \gamma(\varepsilon). \tag{10}$$

Furthermore, $u_\varepsilon \in [1 - 2\delta(\varepsilon), 1 - \delta(\varepsilon)]$ and the constants $\delta(\varepsilon)$ and $\gamma(\varepsilon)$ satisfy $\delta(\varepsilon) \to 0$ and $\gamma(\varepsilon) \to +\infty$ as $\varepsilon \to 0^+$.

The proof of this theorem will be given in Section 4.

Now, the function $U = u - u_\varepsilon$ satisfies

$$\frac{\partial U}{\partial t} - \Delta U + f(u) - f(u_\varepsilon) \leq w - \beta \leq 0,\ \text{in } \Omega,\ t > 0. \tag{11}$$

Furthermore, $\Psi = \psi - u_\varepsilon|_\Gamma$ satisfies $\Psi = U|_\Gamma$ and solves

$$\frac{\partial \Psi}{\partial t} - \Delta_\Gamma \Psi + \lambda\Psi + g(\psi) - g(u_\varepsilon|_\Gamma) + \frac{\partial U}{\partial \nu} = G,\ \text{on } \Gamma,\ t > 0, \tag{12}$$

where (see (9)-(10) and note that $\Delta_\Gamma u_\varepsilon|_\Gamma = 0$) $G = -\lambda(1 - \delta(\varepsilon)) - g(1 - \delta(\varepsilon)) - \gamma(\varepsilon)$. Due to Theorem 2.4, $G \leq 0$, provided that we fix ε small enough. Multiplying then (11) by $U^+ = \max\{U, 0\}$ and (12) by $\Psi^+ = \max\{\Psi, 0\}$, we have, owing to (5), $\frac{1}{2}\frac{d}{dt}(\|U^+\|^2 + \|\Psi^+\|_\Gamma^2) \leq K_1\|U^+\|^2 + K_2\|\Psi^+\|_\Gamma^2$. Noting that $U^+(0) = 0$ in Ω and $\Psi^+(0) = 0$ on Γ, Gronwall's lemma yields $u \leq u_\varepsilon \leq 1 - \delta(\varepsilon)$ in $\Omega \times [0, +\infty)$ and $\psi \leq u_\varepsilon|_\Gamma = 1 - \delta(\varepsilon)$ on $\Gamma \times [0, +\infty)$ (note that, for $u(0)$ given, $\|u(0)\|_{L^\infty(\Omega)} < 1$, we have, for ε small enough, $u(0) \leq 1 - 2\delta(\varepsilon)$, hence $u(0) \leq u_\varepsilon$; the same holds for $\psi(0)$). The lower estimates are proved analogously.

□

Theorem 2.5. *The assertions of Theorems 2.1 and 2.3 also hold if* (6) *is replaced by*

$$\lambda > \max\{-g(1), g(-1)\}. \tag{13}$$

The proof of this theorem is similar to the one performed in Section 3 below and we thus omit it.

Remark 3. In particular, Theorem 2.5 holds for the logarithmic potentials (2).

Lemma 2.6. *There exists $M_\delta > 0$ depending on the constant δ introduced in Theorem 2.3 (and, through δ, on $D[u_0]$ and $\|z_0\|_\Phi$) such that the first two components of any solution $z(t) = (u(t), \psi(t), w(t)) \in \Phi$ to* (1) *satisfy*

$$\|u(t)\|_{H^2(\Omega)} + \|\psi(t)\|_{H^2(\Gamma)} \leq M_\delta,\ \forall t \geq 0.$$

To prove this lemma, it suffices to write the equations for u and ψ as a suitable elliptic system and then apply the elliptic regularity result [13, Lemma A.1].

As far as the uniqueness and the continuous dependence of the solutions on the initial data are concerned, we have, arguing as in [3, Lemma 3.1] and [4, Lemma 3.3], the

Theorem 2.7. *Under the assumptions of Theorems 2.1 or 2.5, if $z_i(t) = (u_i(t), \psi_i(t), w_i(t)) \in \Phi$ is a solution to* (1) *departing from the initial data $z_{0i} = (u_{0i}, \psi_{0i}, w_{0i}) \in \Phi,\ i = 1, 2$, there holds, $\forall t \geq 0$,*

$$\begin{aligned} &\|u_1(t) - u_2(t)\|^2 + \|\psi_1(t) - \psi_2(t)\|_\Gamma^2 + \|w_1(t) - w_2(t)\|^2 \\ &\leq C_1\big(\|u_{01} - u_{02}\|^2 + \|\psi_{01} - \psi_{02}\|_\Gamma^2 + \|w_{01} - w_{02}\|^2\big)e^{C_2 t}, \end{aligned} \tag{14}$$

where the constants $C_1, C_2 > 0$ depend on η, but are independent of the initial data.

It is now not difficult to prove the existence of a solution (see [2], [3] and [4] for details).

Remark 4. Note that Lemma 2.6 does not prevent M_δ from blowing up as $\|u_0\|_{L^\infty(\Omega)} \to 1$. Thus, necessary conditions for an asymptotic analysis are proper dissipative estimates which are independent of the L^∞-norm of the initial data. Such estimates will be derived in the next section. However, they will be obtained under assumption (13) on the data in the dynamic boundary conditions.

Remark 5. Arguing as in [3], [4], [10] and [13], Theorem 2.7 ensures (by continuity) the existence, as well as the uniqueness, of solutions with initial data belonging to the closure L of Φ in $L^2(\Omega) \times L^2(\Gamma) \times L^2(\Omega)$, namely, $L = \{(u, \psi, w) \in L^\infty(\Omega) \times L^\infty(\Gamma) \times L^2(\Omega) : \|u\|_{L^\infty(\Omega)} \leq 1,\ \|\psi\|_{L^\infty(\Gamma)} \leq 1\}$. In particular, this allows to consider initial data which contain the pure states (i.e., u_0 can take the values ± 1). However, we have not been able to prove that the solutions mix instantaneously (i.e., $\|u(t)\|_{L^\infty(\Omega)} < 1$ as soon as $t > 0$), as it is the case for classical boundary conditions (see [3] and [10]).

3 Finite dimensional attractors

We assume in this section that (3), (4) and (13) hold. In particular, we deduce, owing to the continuity of g, that there exists $\gamma \in (0, 1)$ such that

$$\max_{s\in[\gamma,1]}(-\lambda s - g(s)) < 0, \quad \max_{s\in[-1,-\gamma]}(\lambda s + g(s)) < 0. \tag{15}$$

We further assume that γ is such that

$$f'(s) \geq 0, \quad \forall s \in [-1, -\gamma] \cup [\gamma, 1]. \tag{16}$$

We can define, owing to Theorem 2.5, the (continuous) semigroup $S(t) : \Phi_M \to \Phi_M,\ z_0 = (u_0, \psi_0, w_0) \mapsto z(t) = (u(t), \psi(t), w(t))$, where $z(t)$ is the solution to (1) with initial datum z_0 (note that we will not be able to study the existence of finite dimensional attractors on the whole space Φ, due to the conservation of I_0).

Let $R_0 > 0$ be given and assume that $D[u_0] + \|z_0\|_\Phi^2 \leq R_0^2$. We then have, owing to Theorem 2.2, the existence of $t_0 = t_0(R_0, M) \geq 0$ such that

$$\|w(t)\|_{L^\infty(\Omega)} \leq c_M, \ \forall t \geq t_0, \tag{17}$$

where c_M is independent of R_0. Furthermore, there holds

$$\|w(t)\|_{L^\infty(\Omega)} \leq \beta_M, \ \forall t \geq 0, \tag{18}$$

where $\beta_M = \beta_M(R_0)$. Here, we can assume without loss of generality that $c_M \leq \beta_M$.

We now choose $\delta_M \in (0,1)$ independent of R_0 and $t_1 \geq t_0$ such that

$$\delta_M \in [\gamma, 1], \ f(\delta_M) \geq c_M + 1 \tag{19}$$

and $\alpha(= \alpha(R_0, M)) = \frac{1-\delta_M}{t_1}$ is small enough so that

$$\alpha \leq 1, \ f(1 - \alpha t_0) \geq \beta_M + 1, \tag{20}$$

$$\max_{s \in [\gamma,1]} (-\lambda s - g(s)) + \alpha \leq 0. \tag{21}$$

In particular, the existence of α satisfying (21) is guaranteed by (15).

We finally set $y_+(t) = \begin{cases} 1 - \alpha t, \ 0 \leq t \leq t_1 \\ \delta_M, \ t \geq t_1 \end{cases}$. We have

$$\delta_M \leq y_+(t) < 1, \ \forall t > 0, \ y_+(0) = 1. \tag{22}$$

We set $v = u - y_+$ and $\varphi = \psi - y_+$. These functions are solutions to

$$\frac{\partial v}{\partial t} - \Delta v + f(u) - f(y_+) = G := w - f(y_+) - \frac{\partial y_+}{\partial t}, \text{ in } \Omega, \ t > 0, \ t \neq t_1, \tag{23}$$

and

$$\frac{\partial \varphi}{\partial t} - \Delta_\Gamma \varphi + \lambda \varphi + g(\psi) - g(y_+) + \frac{\partial v}{\partial \nu} = H := -\lambda y_+ - g(y_+) - \frac{\partial y_+}{\partial t}, \text{ on } \Gamma, \tag{24}$$

$t > 0, \ t \neq t_1$, respectively. We have, owing to (19), (21) and (22),

$$H(t) \leq 0, \ \forall t \geq 0, \ t \neq t_1. \tag{25}$$

Furthermore, there holds, owing to (16) and (19),

$$G(t) \leq \begin{cases} \beta_M + 1 - f(1 - \alpha t_0), \ \ 0 < t \leq t_0, \\ c_M + 1 - f(\delta_M), \ t \geq t_0, \ t \neq t_1, \end{cases}$$

hence, in view of (19) and (20),

$$G(t) \leq 0, \ \forall t > 0, \ t \neq t_1. \tag{26}$$

We set $v^+ = \max\{v, 0\}$ and $\varphi^+ = \max\{\varphi, 0\}$. Multiplying (23) by v^+ and integrating over Ω and by parts, we obtain, in view of (24)-(26), $\frac{d}{dt}(\|v^+\|^2 + \|\varphi^+\|^2_\Gamma) \leq c(\|v^+\|^2 + \|\varphi^+\|^2_\Gamma)$, $t > 0$, $t \neq t_1$. Using Gronwall's lemma and noting that both v^+ and φ^+ are continuous with respect to time (with values in $L^2(\Omega)$ and $L^2(\Gamma)$, respectively) and that $v^+(0) = 0$ and $\varphi^+(0) = 0$, we then deduce that $v^+(t) = 0, \ \varphi^+(t) = 0, \ \forall t \geq 0$. Therefore, $u(t) \leq y_+(t), \ \forall t \geq 0$, and $u(t) \leq \delta_M, \ \forall t \geq t_1$.

Proceeding in a similar way to derive a lower bound, we finally deduce that there exists $\delta_M \in (0, 1)$ independent of R_0 such that

$$\|u(t)\|_{L^\infty(\Omega)} \leq \delta_M, \ \forall t \geq t_1, \ t_1 = t_1(R_0), \tag{27}$$

hence a dissipative L^∞-estimate on u.

The dynamical system $(S(t), \Phi_M)$ is thus dissipative (i.e., it possesses a bounded absorbing set $\mathcal{B}_0$, that is, $\forall B \subset \Phi_M$ bounded, $\exists t_0 = t_0(B)$ such that $t \geq t_0$ implies $S(t)B \subset \mathcal{B}_0$; it is understood here that B bounded means that $\exists R \geq 0$ such that $D[u] + \|z\|^2_\Phi \leq R^2$, $\forall z = (u, \psi, w) \in B$) and we have the

Theorem 3.1. *For each fixed $M > 0$, the dynamical system $(S(t), \Phi_M)$ possesses the global attractor $\mathcal{A}_M \subset H^3(\Omega) \times H^3(\Gamma) \times H^3(\Omega)$ with finite fractal dimension.*

The proof of this result follows closely the one performed in [9] for regular potentials (see also [10]), owing to the strict separation property and the dissipative estimate obtained in Section 2, and we thus omit the details.

Remark 6. We recall that the global attractor is the unique compact set of the phase space which is fully invariant by the flow (i.e., $S(t)\mathcal{A}_M = \mathcal{A}_M, \ \forall t \geq 0$) and attracts all bounded sets of initial data as time goes to infinity in the sense of the Hausdorff semi-distance between sets. Furthermore, the finite (fractal) dimensionality essentially means that, even though the phase space is infinite dimensional, the effective dynamics, reduced to the global attractor, is, in some proper sense, finite dimensional and can be described by a finite number of parameters. We refer the interested reader to, e.g., [14] and [16] for more details and discussions on this.

4 Proof of Theorem 2.4

We construct, in the spirit of [2, Section 3], a supersolution ψ_ε to a proper elliptic problem in the thin domain $\Omega \setminus \overline{\Omega}_\varepsilon = \{ x \in \Omega : \ 0 < d_\varepsilon(x) < \varepsilon \}$, where $\Omega_\varepsilon = \{ x \in \Omega : \ d_\varepsilon(x) > \varepsilon \}$ and d_ε denotes the distance to Γ. We assume that $\varepsilon > 0$ is small enough so that Ω_ε is well defined (see [2]). We set

$$\theta_\varepsilon(s) \ = \ \frac{1}{\varepsilon^{2-r}} s^2 - \frac{2}{\varepsilon^{1-r}} s + 1 - \varepsilon^r, \ \frac{2}{p_+ + 1} < r < 1. \tag{28}$$

We then set $\psi_\varepsilon(x) = \theta_\varepsilon(d_\varepsilon(x)), \ x \in \Omega \setminus \overline{\Omega}_\varepsilon$. We have

$$\psi_\varepsilon \in [1-2\varepsilon^r,\, 1-\varepsilon^r],\ \psi_\varepsilon = 1-\varepsilon^r,\ \text{on } \Gamma,\ \psi_\varepsilon = 1-2\varepsilon^r,\ \text{on } \Gamma_\varepsilon, \tag{29}$$

$$\frac{\partial \psi_\varepsilon}{\partial \nu} = \frac{2}{\varepsilon^{1-r}},\ \text{on } \Gamma,\ \frac{\partial \psi_\varepsilon}{\partial \nu} = 0,\ \text{on } \Gamma_\varepsilon, \tag{30}$$

where $\Gamma_\varepsilon = \{x \in \Omega :\ d_\varepsilon(x) = \varepsilon\}$. Furthermore, there holds (see [2]) $-\Delta\psi_\varepsilon = -\frac{2}{\varepsilon^{2-r}} - (\frac{2d_\varepsilon}{\varepsilon^{2-r}} - \frac{2}{\varepsilon^{1-r}})\,\Delta d_\varepsilon$, where $|\Delta d_\varepsilon|$ is bounded independently of ε. Thus, owing to (29) and noting that it follows from (6) that $f(1-c\varepsilon^r) \sim \frac{c_+}{c^{p_+}\varepsilon^{rp_+}}$, $c>0$, in the neighborhood of 0^+, we deduce from (28) that we can choose ε small enough so that

$$-\Delta\psi_\varepsilon + f(\psi_\varepsilon) \geq \beta,\ \text{in } \Omega \setminus \overline{\Omega}_\varepsilon; \tag{31}$$

in particular, ψ_ε is a supersolution to the elliptic problem

$$\begin{cases} -\Delta v_\varepsilon + f(v_\varepsilon) = \beta,\ \text{in } \Omega \setminus \overline{\Omega}_\varepsilon, \\ v_\varepsilon = 1-\varepsilon^r,\ \text{on } \Gamma, \\ v_\varepsilon = 1-2\varepsilon^r,\ \text{on } \Gamma_\varepsilon. \end{cases}$$

We finally choose ε small enough so that

$$f(1-2\varepsilon^r) \geq \beta \tag{32}$$

and we set $u_\varepsilon = \begin{cases} \psi_\varepsilon,\ \text{in } \Omega \setminus \overline{\Omega}_\varepsilon \\ 1-2\varepsilon^r,\ \text{in } \Omega_\varepsilon \end{cases}$. It thus follows from (29)-(30) that $u_\varepsilon \in H^2(\Omega)$ and we deduce from (29)-(32) that (8)-(10) are satisfied with $\delta(\varepsilon) = \varepsilon^r$ and $\gamma(\varepsilon) = \frac{2}{\varepsilon^{1-r}}$, which finishes the proof of Theorem 2.4.

Remark 7. Assumption (6) may look artificial here and related to the choice of the function θ_ε in (28). It is however, in some sense, sharp in order to have the strict separation property (7). This can already be seen by heuristic arguments. To do so, we consider the solution v to the elliptic problem $\begin{cases} -\Delta v + f(v) = \beta,\ \text{in } \Omega \\ v = 1,\ \text{on } \Gamma \end{cases}$. Then, we can expect, close to the boundary, an inequality of the form

$$v(x) \leq 1 - c d_\varepsilon(x)^{\frac{2}{p_+ + 1}},\ c > 0. \tag{33}$$

Indeed, setting, at first approximation, $v(x) = 1 - c_0 d_\varepsilon(x)^\alpha + \varphi(d_\varepsilon(x))$, $\alpha > 0$ (we can note that, close to the boundary, the tangential derivatives are smoother than the normal ones), we find, at first approximation,

$$\begin{aligned} & c_0\alpha(\alpha-1)d_\varepsilon(x)^{\alpha-2} + c_+ c_0^{-p_+} d_\varepsilon(x)^{-\alpha p_+} - \varphi''(d_\varepsilon(x)) \\ & + p_+ c_+ c_0^{-(p_+ + 1)} d_\varepsilon(x)^{-\alpha(p_+ + 1)} \varphi(d_\varepsilon(x)) \\ & = \beta - c_0 \alpha d_\varepsilon(x)^{\alpha-1} \Delta d_\varepsilon(x),\ \varphi(0) = 0 \end{aligned}$$

(here, we have neglected the term $\varphi'(d_\varepsilon(x))\Delta d_\varepsilon(x)$; note that we expect to have $\varphi(x) = o(x^\alpha)$ in the neighborhood of 0^+). Taking $\alpha = \frac{2}{p_+ + 1}$, $c_0 = [\frac{1}{c_+}\alpha(1-\alpha)]^{-\frac{1}{p_+ + 1}}$, we end up with an Euler-type equation of the form

$$\varphi'' + p_+\alpha(\alpha-1)x^{-2}\varphi = -\beta + x^{\alpha-1}\theta,\ \varphi(0) = 0,$$

where $\theta = \theta(x)$ is bounded, hence

$$\begin{aligned}&\varphi = \delta x^2 + \tilde{\varphi} + \lambda x^{\gamma_1},\ \delta = -\tfrac{\beta}{2+p_+\alpha(\alpha-1)},\ \lambda \in \mathbb{R},\\ &\tilde{\varphi} = \tilde{\varphi}(x) = \tfrac{1}{\gamma_1-\gamma_2}[x^{\gamma_1}\textstyle\int_{\varepsilon_0}^x s^{\alpha-\gamma_1}\theta(s)ds - x^{\gamma_2}\int_0^x s^{\alpha-\gamma_2}\theta(s)ds],\ \varepsilon_0 > 0,\\ &\gamma_1 = \tfrac{1}{2}[1+\sqrt{1-4p_+\alpha(\alpha-1)}],\ \gamma_2 = \tfrac{1}{2}[1-\sqrt{1-4p_+\alpha(\alpha-1)}].\end{aligned}$$

Noting that $|\tilde{\varphi}| \le c(x^{\alpha+1} + x^{\gamma_1})$ and that, for $p_+ > 1$, $0 < \alpha < 1$ and $\gamma_1 > 1$, we finally deduce that $\varphi = \mathrm{o}(x^\alpha)$ in the neighborhood of 0^+, as expected. It now follows from (33) that, for $x_0 \in \Gamma$ and $\xi > 0$ small enough,

$$\frac{v(x_0) - v(x_0 - \xi\nu(x_0))}{\xi} \ge c\xi^{\frac{1-p_+}{1+p_+}}.$$

We thus need the condition $p_+ > 1$ to have $\dfrac{\partial v}{\partial \nu}$ tending to $+\infty$ approaching Γ (such a property is essential in our proof of strict separation). We refer the reader to [15] for a more rigorous justification for the need of conditions of the form (6) to have a strict separation property and the existence of strong solutions; in particular, this is rather general, in the sense that we encounter a similar situation in the study of other models such as, e.g., the Cahn-Hilliard equation.

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