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PARALLEL SUBMANIFOLDS OF GENERALIZED SASAKIAN SPACE FORMS

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Abstract

The main purpose of this paper is to investigate the existence of parallel hypersurfaces in a generalized Sasakian space forms.

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1 Introduction

The propose of this paper is to study some submanifolds in the generalized Sasakian forms. Therefore, we obtain a characterization of manifolds whose tangent space are Lie triple systems. As an application, we prove that the structure vector field of the generalized Sasakian space forms is either tangent or normal to a parallel submanifold extending Pitis result for $C(\alpha)$ manifold [7].

Almost contact structures (ϕ, ζ, η, g) is an $C(\alpha)$ manifold if the Riemannian tensor R satisfies the following equality

$$\begin{aligned} R(X,Y,Z,U) &= R(X,Y,\phi Z,\phi U) + \alpha[g(X,U)g(Y,Z) - g(X,Z)g(Y,U) \\ &\quad -g(X,Z)g(Y,U) + g(X,\phi Z)g(Y,\phi U) - g(Y,\phi U)g(Y,\phi Z)] \end{aligned}$$

For an $\alpha \in IR$ and for all $X, Y, Z, U \in H(M)$ the curvature tensor of manifolds is given by:

$$\begin{aligned} R(X,Y) &= \frac{c+3\alpha}{4}[g(Y,Z)X - g(X,Z)Y] + \frac{c-\alpha}{4}[\eta(X)\eta(Z)Y - \\ &\quad -\eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\zeta - g(Y,Z)\eta(X)\zeta + g(Z,\phi Y)\phi X \\ &\quad -g(Z,\phi X)\phi Y + 2g(X,\phi Y)\phi Z] \end{aligned}$$

where c is the ϕ sectional curvature [7]. Finally, we recall the notion of the Lie systems this is a linear subspace S of the tangent space T_xM , such that $R(X,Y)Z \in S$ for all $X, Y, Z \in S$

Finally, we show that there is no parallel hypersurfaces in the Sasakian space forms $M^{2n+1}(c)$ with $n \geq 2$.

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2 Preliminaries

Let (M^{2n+1}, g) be a $2n+1$ dimensional manifold and let (ϕ, ζ, η) be tensor fields of type (1,1), (1,0) and (0,1) respectively on M, such that:

$$\eta(\zeta) = 1 \quad \text{and} \quad \phi^2 = -I + \zeta \otimes \eta$$

This implies

$$\eta \circ \phi = 0 \;\; \eta(\zeta) = 0 \text{ and } \;\; rank(\phi) = 2n.$$

If M admits a Riemannian metric G such that:

$$\begin{aligned} g(\phi X, \phi Y) &= g(X, Y) - \eta(x)\eta(y) \\ \eta(X) &= g(X, \zeta) \end{aligned}$$

Then is called an almost contact metric structure on M. If moreover

$$(\tilde{\nabla}_X \theta)Y = g(X, Y)\zeta - \eta(Y)X$$

Where $\tilde{\nabla}$ denotes the Riemannian connection of G , then $(M, \phi, \zeta, \eta, g)$ is called a Saskian manifold (see [5]). The sectional curvature of the plane section spanned by the unit tangent vector field Xorthogonal to ζ and ϕX is called a ϕ -sectional curvature. If M has a constant -sectional curvature C , then M is called a Sasakian space forms and denoted by $M^{2n+1}(c)$. The Riemannian curvature of a Sasakian forms is given by the following formula:

$$\begin{aligned} R(X, Y, Z) &= \frac{c+3}{4}[g(Y, Z)X - g(X, Z)Y] + \frac{c-1}{4}[\eta(X)\eta(Z)Y \\ &\quad -\eta(Y)\eta(Z)X] + \frac{c-1}{4}[g(X, Z)\eta(Y)\zeta - g(Y, Z)\eta(X)\zeta \\ &\quad +g(Z, \phi Y)\phi X - g(Z, \phi X)\phi Y + 2g(X, \phi Y)\phi Z] \end{aligned}$$

[5, 3].

Example 1. We consider $IR^{2n+1}$with the coordinates $(x^i, y^i, z), i = 1, ..., n$ and its usual contact form $\eta = \frac{1}{2}(dz - \sum_{i=1}^{n} y^i dx^i)$. The characteristic field is given by $\zeta = 2\frac{\partial}{\partial z}$,the tensor field ϕ is given by the matrix $\begin{pmatrix} 0 & \partial_{ij} & 0 \\ -\partial_{ij} & 0 & 0 \\ 0 & y^j & 0 \end{pmatrix}$ and the Riemannian metric $g = \eta \otimes \eta + \frac{1}{4} \sum_{i=1}^{n} (dx^i)^2 + (dy^i)^2$ is an associated metric for η. In this case IR^{2n+1} is a Sasakian space forms with ϕ-sectional curvature $c = -3$, denoted by $IR^{2n+1}(-3)$ [5, 3].

Given an almost contact metric $(M, \phi, \zeta, \eta, g)$, M is called generalized Sasakian space forms if there exists three functions f_1, f_2 and f_3 such that the Riemannian curvature tensor is given by the following formula:

$$\begin{aligned} R(X,Y,Z) &= f_1[g(X,Z)X - g(X,Z)Y] + \\ &\quad f_2[g(Z,\phi Y)\phi X - g(Z,\phi X)\phi Y + 2g(X,\phi Y)\phi Z] + \\ &\quad f_3[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\zeta - g(Y,Z)\eta(X)\zeta] \end{aligned} \tag{2.1}$$

In such case, we will write $M(f_1, f_2, f_3)$. This kind of manifolds appears as nastural generalization of the Sasakian space form by taking:

$$f_1 = \frac{c+3}{4} \quad \text{and} \quad f_2 = f_3 = \frac{c-1}{4}.$$

The ϕ- sectional curvature of generalized Sasakian space forms $M(f_1, f_2, f_3)$ is $f_1 + 3f_2$ [1].

Let N^{2n}be an immersed hypersurface of $M^{2n+1}(f_1, f_2, f_3)$. We denote the Levi Cevita connection of M by and the Levi Civita connection of N by ∇ . Then we have the formulas of Gauss and Weingarten

$$\begin{aligned} \tilde{\nabla}_X Y &= \nabla_X Y + h(X,Y)r \\ \tilde{\nabla}_X r &= -SX \end{aligned}$$

where X and Y are tangent vector fields, r a unit normal vector normal to N, H the second fundamental form and S the shape operator of N. Note that H and S are related by $h(X,Y) = g(SX,Y)$. A hypersurface is called parallel if $\tilde{\nabla} h = 0$ [2].

3 Main results

Theorem 3.1. *[6]: Let N be a connected submanifold* (dim $N \geq 2$) *of generalized Sasakian space forms with $f_3 \neq 0$ and $f_3 + 3f_2 \neq 0$. If all tangent spaces are Lie $M^{2n+1}(f_1, f_2, f_3)$ triple systems then the structure vector field ζ is either tangent or normal to N at every point of N.*

Proof. If $R(X,Y)Z \in T_X N$ for all $X, Y, Z \in T_X N$, then by using (2.1) we have:

$$\begin{aligned} & f_2[g(Z,\phi Y)\phi X - g(Z,\phi X)\phi Y + 2g(X,\phi Y)\phi Z]^{\perp} \\ + \quad & f_3[g(X,Z)\eta(Y) - g(Y,Z)\eta(X)]\zeta^{\perp} = 0 \end{aligned} \tag{3.2}$$

where $\zeta^{\perp}$ is the normal component of the vector ζ and ζ^t is the tangent component of the vector ζ .

<u>Case 1</u>: If $f_2 = 0$ then by using (3.2), we have:

$$[g(X,Z)\eta(Y) - g(Y,Z)\eta(X)]\zeta^{\perp} = g(g(X,Z)Y - g(Y,Z)X, \zeta^t)\zeta^{\perp} = 0$$

This implies $\left\{ \begin{array}{c} \zeta^{\perp} = 0 \\ \text{or} \\ g(g(X,Z)Y - g(Y,Z)X, \zeta^t) = 0 \end{array} \right. \Longrightarrow \left\{ \begin{array}{c} \zeta^{\perp} = 0 \\ \text{or} \\ \zeta^t = 0 \end{array} \right.$

Case 2: If $f_2 \neq 0$, suppose that $\zeta^t \neq 0$ and $\zeta^\perp \neq 0$. Since $rank\phi = 2n$ then $\phi\zeta^t \neq 0$ and $\phi\zeta^\perp \neq 0$.

Now for $Z = Y = \zeta^t$, (3.2) becomes

$$f_3[g(X, \zeta^t)\eta(\zeta^t) - g(\zeta^t, \zeta^t)\eta(X)]\zeta^\perp +$$
$$+f_2[g(X, \phi\zeta^t)\phi\zeta^t - g(\zeta^t, \phi\zeta^t)\phi X + 2g(X, \phi\zeta^t)\phi\zeta^t]^\perp = 0$$

since

$$\eta(X) = g(X, \zeta^t),\ g(\zeta^t, \zeta^t) = \eta(\zeta^t),\ \eta \circ \phi = 0,\ g(X, \phi X) = 0 \text{ and } f_2 \neq 0$$

then

$$g(X, (\phi\zeta^t)^t)(\phi\zeta^t)^\perp = 0$$

We have two cases

Case 2.1:$(\phi\zeta^t)^\perp = 0$

Then $(\phi\zeta^t)$ is tangent to N and by taking and , $Y = Z = (\phi\zeta^t)$ (3.2) becomes:

$$3f_2g(\zeta^t, \phi^2\zeta^t)(\phi^2\zeta^t)^\perp + f_3g(\zeta^t, \phi^2\zeta^t)(\eta(\zeta^t))\zeta^\perp = 0$$

Therefore

$$g(\zeta^t, \phi^2\zeta^t)(\eta(\zeta^t))\zeta^\perp = 0.$$

So

$$g(\phi\zeta^t, \phi\zeta^t) = g(\zeta^t, \zeta^t)\zeta^\perp = 0.$$

However, under the assumption $\zeta^t \neq 0$ and $\zeta^\perp \neq 0$ the last equation is impossible.

Case 2.2: $(\phi\zeta^t)^t = 0$

In this case $\phi\zeta^t$ is normal to N anf if $X = \zeta^t$, from (3.2) we obtain:

$$\begin{aligned} & f_2[g(\zeta^t, \phi Z)\phi Y - g(Y, \phi Z)\phi\zeta^t + 2g(\zeta^t, \phi Y)\phi Z]^\perp \\ + \quad & f_3[g(\zeta^t, Z)\eta(Y) - g(Y, Z)\eta(\zeta^t)]\zeta^\perp = 0 \end{aligned}$$

But

$$g(\zeta^t, \phi Z) = -g(\phi\zeta^t, Z) = 0.$$

Because is normal to N which implies

$$-f_2g(Y, \phi Z)\phi\zeta^t + f_3[g(\zeta^t, Z)\eta(Y) - g(Y, Z)\eta(\zeta^t)]\zeta^\perp = 0.$$

Therefore

$$\begin{aligned} g(\zeta^t, Z)g(\zeta^t, Y) - g(Y, Z)g(\zeta^t, \zeta^t) &= 0 \\ & \text{and} \\ g(Y, \phi Z) &= 0 \end{aligned} \tag{3.3}$$

Because $\phi\zeta^t$ and $\zeta^\perp$ are linearly independents. From the first equality (3.3) we deduce

$$g(\zeta^t, Y)\zeta^t - g(\zeta^t, \zeta^t)Y = 0.$$

This contradicts to the hypothesis $\zeta^t \neq 0$ and (dim $N \geq 2$). $\square$

Remark 3.1. *In the previous paper [6], we did not mention the condition $f_3 + 3f_2 \neq 0$, because for $f_3 + 3f_2 = 0$, the generalize Saskian space forms is a space of constant curvature, so we did not study this case.*

Theorem 3.2. *Let N be a connected parallel submanifold dim $N \geq 2$ of the generalized space forms $M^{2n+1}(f_1, f_2, f_3)$ with $f_3 + 3f_2 \neq 0$ and $f_3 \neq 0$. Then the structure vector-vector field ζ is either tangent or normal to N at every point of N.*

Proof. The Codazzi equation is given by:

$$[R(X,Y)Z]^{\perp} = (\tilde{\nabla}_X h)(Y,Z) - (\tilde{\nabla}_Y h)(X,Z)$$

where H is the second fundamental form. If N is parallel $\tilde{\nabla}_x h = 0$, then

$$[R(X,Y)Z]^{\perp} = 0$$

and by using Theorem 2 we obtain the result. □

By taking

$$f_1 = \frac{c+3\alpha}{4} \text{ and } f_2 = f_3 = \frac{c-\alpha}{4}$$

we get Pitis theorem [7].

And for

$$f_1 = \frac{c+3}{4} \text{ and } f_2 = f_3 = \frac{c-1}{4}$$

we have

Colorallary 3.1. *Let N be a connected parallel submanifold dim $N \geq 2$ of Sasakian space forms $M^{2n+1}(c)$ with $c \neq 1$. Then the structure vector-field ζ is either tangent or normal to N at every point of N.*

Colorallary 3.2. *Let N^m a connected parallel submanifold of the Sasakian space forms $M^{2n+1}(c)$ with $m \rangle n$ and $c \neq 1$. Then the structure vector-field ζ is tangent to N at every point of N.*

4 Parallel hypersurfaces of Sasakian space forms

Theorem 4.1. *In the Sasakian space forms $M^{2n+1}(c)$ with $n \geq 2, c \neq 1$ there is not a parallel connected hypersurfaces.*

Proof. If N is a connected parallel hypersurface of the Sasakian space forms M, then by Corollary 2, ζ is tangent to N. We denote by K the unit normal vector field to N and put

$$\zeta_1 = -\phi k.$$

Since K is orthogonal to N, then

$$g(\zeta_1, \zeta_1) = g(\phi K, \phi K) = 1$$

and

$$g(\zeta_1, K) = g(\zeta_1, \zeta) = 0.$$

Hence $\zeta_1$is tangent to N. For $X \epsilon T_x N$, we set

$$\phi X = fX + w(X)K$$

where w and f are tensors fields on N of type (0,1) and (1,1) respectively, also fX represents the tangent part of ϕX . Moreover, it is easy to verify that:

$$w(X) = g(X, \zeta_1) \text{ and } \phi\zeta_1 = K.$$

By the Codazzi equation and that ζ is tangent (from Theorem (3.2)) we obtain for all X, Y and Z in $T_x N$:

$$\begin{aligned} 0 &= (\tilde{\nabla}_X h)(Y, Z) - (\tilde{\nabla}_Y h)(X, Z) = [R(X, Y)Z]^{\perp} \\ &= \frac{c-1}{4}[g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z]^{\perp} \\ &= \frac{c-1}{4}[g(X, \phi Z)w(Y) - g(Y, \phi Z)w(X) + 2g(X, fY)w(Z)]K = 0 \end{aligned}$$

If we take $Z = \zeta_{1,}$ we have

$$\frac{c-1}{4} g(X, fY) = 0$$

Since $c \neq 1$then $fY = 0$. In this case $\dim \phi(T_x N) = 1$. Since

$$T_x M = T_x N + T_x N^{\perp} \text{ and } rank\phi = 2n$$

So

$$2n - 1 \leq \dim \phi(T_x N)^{\perp} \leq 2n$$

which is impossible because

$$n \geq 2 \text{ and } \dim T_x N^{\perp} = 1 \; (\dim \phi(T_x N^{\perp})) \leq 1.$$

□

We deduce

Colorallary 4.1. *There is not parallel-connected hypersurfaces in* $IR^{2n+1}(-3)$ *with* $n \geq 2$.

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