

THE INVERSE MAXIMUM FLOW PROBLEM UNDER WEIGHTED l_∞ NORM

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Abstract

The problem introduced in this paper (denoted IMFW_∞) is to modify the capacities on arcs from a network so that a given feasible flow becomes a maximum flow and the maximum cost of change of the capacities is minimum. This problem is a generalization of the inverse maximum flow problem under l_∞ norm (denoted IMF_∞ , where the per unit cost of modification is equal to 1 on all arcs), which was previously studied and solved in polynomial time. In this paper, the algorithm for IMF_∞ is adapted to solve IMFW_∞ .

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1 Introduction

An inverse combinatorial optimization problem consists in modifying some parameters of a network such as capacities or costs so that a given feasible solution of the direct optimization problem becomes an optimal solution and the distance between the initial vector and the modified vector of parameters is minimum. Different norms such as l_1 , l_∞ and even l_2 are considered to measure this distance. In the last years many papers were published in the field of inverse combinatorial optimization [5]. Almost every inverse problem was studied considering l_1 and l_∞ norms, resulting in different problems with completely different solution methods. Strongly polynomial time algorithms to solve the inverse maximum flow problem under l_1 norm (denoted IMF) were presented in [9] and, then in [3]. IMF is reduced to a minimum cut problem in an auxiliary network. Four inverse maximum flow problems are also studied by Liu and Zhang [6] under the sum-type and bottleneck-type weighted Hamming distances. Strongly polynomial algorithms for these problems are proposed.

In this paper, the inverse maximum flow considering weighted l_∞ norm (denoted IMFW_∞) is studied. In IMFW_∞ , the maximum cost of change of the capacities on arcs is minimized. The inverse maximum flow problem under l_∞ norm (denoted IMF_∞) was studied and solved in [4]. In this paper we show how the algorithm for IMF_∞ can be adapted to solve the more general problem: IMFW_∞ .

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2 THE IMFW_∞ PROBLEM

Let $G = (N, A, c, s, t)$ be an s - t network, where N is the set of nodes, A is the set of directed arcs, c is the capacity vector, s is the source and t is the sink node.

If a network has more than a source or/and more than a sink node, it can be transformed into a s - t network (introducing a super-source and a super-sink node) [1].

Let f be a given feasible flow in the network G . It means that f has to satisfy the flow balance condition and the capacity restrictions. The balance condition for the flow f is:

$$(1) \quad \sum_{y \in N, (x,y) \in A} f(x,y) - \sum_{y \in N, (y,x) \in A} f(y,x) = \begin{cases} v(f), & x = s \\ -v(f), & x = t \\ 0, & x \in N - \{s, t\} \end{cases}, \forall x \in N,$$

where $v(f)$ is the value of the flow f from s to t .

The capacity restrictions are:

$$(2) \quad 0 \leq f(x,y) \leq c(x,y), \forall (x,y) \in A.$$

The maximum flow problem is:

$$(3) \quad \begin{cases} \max v(f) \\ f \text{ is a feasible flow in } G \end{cases}.$$

The residual network attached to the network G for the flow f is $G_f = (N, A_f, r, s, t)$, where for each pair of nodes (x,y) the value of $r(x,y)$ is defined as follows:

$$(4) \quad r(x,y) = \begin{cases} c(x,y) - f(x,y) + f(y,x), & \text{if } (x,y) \in A \text{ and } (y,x) \in A \\ c(x,y) - f(x,y), & \text{if } (x,y) \in A \text{ and } (y,x) \notin A \\ f(y,x), & \text{if } (x,y) \notin A \text{ and } (y,x) \in A \\ 0, & \text{otherwise} \end{cases}.$$

The set A_f contains as arcs of the residual network only the pairs of nodes $(x,y) \in N \times N$ for which the residual capacity is positive, i.e., $r(x,y) > 0$.

The inverse maximum flow problem under weighted l_∞ norm is to change the capacity vector c so that the given feasible flow f becomes a maximum flow in G and the maximum cost of change of the capacities on arc is minimum.

IMFW_∞ can be formulated using the following mathematical model:

$$(5) \quad \begin{cases} \min \|c - \bar{c}\|_\infty^w \\ f \text{ is a maximum flow in } \bar{G} = \{N, A, \bar{c}, s, t\} \\ c(x,y) - \delta(x,y) \leq \bar{c}(x,y) \leq c(x,y) + \alpha(x,y), \forall (x,y) \in A \end{cases},$$

where $\|c - \bar{c}\|_{\infty}^w = \max_{(x,y) \in A} \{w(x,y) \cdot |c(x,y) - \bar{c}(x,y)|\}$ and $w(x,y)$ is the per unit cost of change of the capacity on the arc $(x,y) \in A$. Of course, $w(x,y) > 0, \forall (x,y) \in A$.

The values $\alpha(x,y)$ and $\delta(x,y)$ are given non-negative numbers and $\delta(x,y) \leq c(x,y)$, for each arc $(x,y) \in A$. These values show how much the capacities of the arcs can vary.

In order to make the flow f become a maximum flow in the network G , the capacities of some arcs from A must be decreased. So, the conditions $\bar{c}(x,y) \leq c(x,y) + \alpha(x,y)$, for each arc $(x,y) \in A$ have no effect and, instead of (5), the following mathematical model is considered:

$$(5') \quad \left\{ \begin{array}{l} \min \|c - \bar{c}\|_{\infty}^w \\ f \text{ is a maximum flow in } \tilde{G} = \{N, A, \bar{c}, s, t\} \cdot \\ c(x,y) - \delta(x,y) \leq \bar{c}(x,y), \forall (x,y) \in A \end{array} \right.$$

A graph denoted $\tilde{G} = (N, \tilde{A})$ can be constructed to verify the feasibility of IMFW $_{\infty}$ (see [4]), where:

$$(6) \quad \tilde{A} = \{(x,y) \in A \mid f(x,y) + \delta(x,y) < c(x,y)\} \cup \\ \cup \{(x,y) \in N \times N \mid (y,x) \in A \text{ and } f(y,x) > 0\}.$$

We have the following theorem:

Theorem 1. *In the network G , IMFW $_{\infty}$ has optimal solution for the given flow f , if and only if there is no directed path in the graph \tilde{G} from the node s to the node t .*

Proof. see [4]. □

The verification of IMFW $_{\infty}$ being feasible can be done in $O(p)$ time complexity, using a graph search algorithm in \tilde{G} , where p is the number of arcs in the set \tilde{A} with $p \leq m$. Moreover, this test of feasibility can be applied to any inverse maximum flow problem (under any norm).

We define the weighted residual capacity r_w for an arc (x,y) of the residual network G_f as follows:

$$(7) \quad r_w(x,y) = \begin{cases} w(x,y) \cdot (c(x,y) - f(x,y) + f(y,x)), & \text{if } (x,y) \in A \text{ and } (y,x) \in A \\ w(x,y) \cdot (c(x,y) - f(x,y)), & \text{if } (x,y) \in A \text{ and } (y,x) \notin A \\ 0, & \text{otherwise} \end{cases}.$$

It is easy to see from (6) and (7) that for any arc $(x,y) \in A_f - \tilde{A}$ we have:

$$(8) \quad r_w(x,y) = w(x,y) \cdot (c(x,y) - f(x,y)) = w(x,y) \cdot r(x,y).$$

Transforming the flow f into a maximum flow in the network G is equivalent to eliminate arcs of the residual network G_f so that the source node s will no longer communicate

with the sink node t (there is no directed path in G_f from s to t). Setting the residual capacity on these arcs to 0 does the elimination of arcs. It is easy to see that in the algorithm for solving IMFW_∞ the arcs from \tilde{A} in the residual network will not be eliminated. If IMFW_∞ is feasible (see theorem 1) and all the arcs in the set $A_f - \tilde{A}$ are eliminated, then the flow f becomes a maximum flow in the network G with the modified capacity vector. The arcs from $A_f - \tilde{A}$ must be determined so that if they are eliminated from A_f , the flow f becomes a maximum flow in G and the corresponding modified capacity vector c^* is an optimal solution for the problem (5'), i.e., $\|c - c^*\|_\infty^w$ is minimum.

From (8) and because only the biggest cost of change matters (see (5')) it results that the arcs from $A_f - \tilde{A}$ with the lowest weighted residual capacities will be eliminated from the residual network G_f .

3 ALGORITHM FOR IMFW_∞

The algorithm for IMFW_∞ starts with the set $H = A_f - \tilde{A}$. If IMFW_∞ is feasible, then in the residual network the arcs of H are sorted descending by their weighted residual capacities. They are eliminated sequentially from H (from the greatest weighted residual capacity to the lowest) till the eliminated arcs and the arcs from \tilde{A} forms a graph in which there is a directed path from s to t . The remaining arcs in the set H are the arcs that will be eliminated from the residual network G_f . The flow f will become a maximum flow in G , after the modification is done to the capacities.

The algorithm for IMFW_∞ is:

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Construct the graph  $\tilde{G} = (N, \tilde{A})$ ;
B :=  $\tilde{A}$ ;
Find the set  $W$  of the nodes accessible from the source node  $s$ 
  in the graph  $G' = (N, B)$ , eliminating the visited arcs of  $B$ ;
If  $t \in W$  then
  The  $\text{IMFW}_\infty$  problem has no solution;
  STOP.
endif;
Construct the residual network  $G_f = (N, A_f)$ ;
 $H := A_f - \tilde{A}$ ;
Sort the arcs from  $H$  descending by their weighted residual capacities;
For  $(x, y) \in H$  do
  If  $y \notin W$  then
     $B := B \cup \{(x, y)\}$ ;
    If  $x \in W$  then
      Find the nodes  $W'$  accessible from the nodes of  $W$  in the
        graph  $G' = (N, B)$ , eliminating the visited arcs of  $B$ ;
       $W := W \cup W'$ ;
    endif;
  endif;
endif;
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    If  $t \in W$  then
         $d := r_w(x, y)$ ;
        BREAK;
    else  $H := H - \{(x, y)\}$ ;
    endif;
endfor;
For  $(x, y) \in A$  do
    If  $(x, y) \in H$  then  $c^*(x, y) := f(x, y)$ ;
    else  $c^*(x, y) := c(x, y)$ ;
    endif;
endfor;
 $c^*$  is the solution of the IMFW $\infty$  problem.

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The correctness of the algorithm is given by the following theorem:

Theorem 2. *The vector c^* found by the algorithm is the solution of IMFW ∞ and we have $\|c - c^*\|_\infty^w = d$.*

Proof. The proof is similar to the proof of correctness of the algorithm for IMF ∞ (see [4]). □

The time complexity of the algorithm is given by the following theorem:

Theorem 3. *The algorithm for IMFW ∞ has a time complexity of $O(m \cdot \log(n))$.*

Proof. The proof is similar to the proof of time complexity of the algorithm for IMF ∞ (see [4]). □

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