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WELL-POSEDNESS OF HIGHER DIMENSIONAL CAMASSA-HOLM EQUATIONS

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Abstract

We formulate the n-dimensional Camassa-Holm (CH) equation on an arbitrary compact Riemannian manifold with boundary, and show that these equations are well-posed with respect to Dirichlet or Navier-slip boundary conditions. The method of proof consists in showing that the physically relevant H^1-like Riemannian metrics admit a smooth geodesic spray on the diffeomorphism groups associated to the above boundary conditions.

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1 Introduction

Like the KdV equation, the Camassa-Holm (CH) equation describes the unidirectional propagation of waves at the free surface of shallow water under the influence of gravity, [1]. In the dispersionless limit, the CH equation is given by

$$u_t - u_{xxt} = -3uu_x + 2u_x u_{xx} + uu_{xxx},$$

where the fluid velocity vector u is a function of position $x \in \mathbb{R}$ (or $x \in S^1$ in the case of periodic boundary conditions) and time $t \in \mathbb{R}$. The main results for this equation are its complete integrability, which is guaranteed by its Hamiltonian structure, and the spontaneous emergence of singular solutions, [1]. Another important property, on which we will focus in this note, is that this fluid model describes a geodesic flow on the group of diffeomorphisms of $\mathbb{R}$ (or S^1) relative to a H^1 metric.

The n-dimensional version of the dispersionless CH equation, is obtained by generalizing the above geodesic property to n-dimensional manifolds. The resulting equation is sometimes called EPDiff, which is short for *Euler-Poincaré equation on diffeomorphism group*, see [4].

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By extending the results of [6] and [3], we formulate the n-dimensional CH equation on an arbitrary compact Riemannian manifold with boundary, and show that these equations are well-posed with respect to Dirichlet or Navier-slip boundary conditions. The method of proof consists in showing that the physically relevant H^1 Riemannian metrics admit a smooth geodesic spray. This idea goes back to [2] and was applied there to prove that the incompressible Euler equations are well-posed on a compact Riemannian manifold (M, g) with boundary.

2 The geometry of the n-dimensional CH equation

We consider a compact and oriented n-dimensional Riemannian manifold M with smooth boundary ∂M. We denote by $\mu \in \Omega^n(M)$ the Riemannian volume and by $\mu_\partial \in \Omega^{n-1}(\partial M)$ the naturally induced boundary volume form.

Diffeomorphism groups. For $s > 1 + n/2$ the group $\mathcal{D}^s := \mathcal{D}^s(M)$ of all Sobolev H^s diffeomorphisms of M can be endowed with a smooth infinite dimensional Hilbert manifold structure modeled on the space $\mathfrak{X}^s_\parallel$ of all Sobolev H^s vector field on M parallel to the boundary.

We consider two subgroups of $\mathcal{D}^s$ which correspond to Dirichlet or Navier-slip boundary conditions. The *Dirichlet diffeomorphism group* is defined by $\mathcal{D}^s_D := \{\eta \in \mathcal{D}^s \mid \eta_{|\partial M} = id_{\partial M}\}$. The *Navier diffeomorphism group* is defined by $\mathcal{D}^s_N := \{\eta \in \mathcal{D}^s \mid (T\eta_{|\partial M} \circ n)^{tan} = 0 \text{ on } \partial M\}$, where n denotes the outward-pointing unit normal vector field along ∂M and $(\cdot)^{tan}$ denotes the tangential part to the boundary of a vector in $TM|\partial M$.

The groups $\mathcal{D}^s_D$ and $\mathcal{D}^s_N$ are smooth Hilbert submanifolds and subgroups of $\mathcal{D}^s$ (see [6]). The corresponding tangent spaces at the identity id_M are

$$\begin{aligned}
\mathcal{V}^s_D &:= T_{id_M}\mathcal{D}^s_D = \left\{ u \in \mathfrak{X}^s_\parallel \,\middle|\, u_{|\partial M} = 0 \right\} = H^s \cap H^1_0, \\
\mathcal{V}^s_N &:= T_{id_M}\mathcal{D}^s_N = \left\{ u \in \mathfrak{X}^s_\parallel \,\middle|\, (\nabla_n u_{|\partial M})^{tan} + S_n(u) = 0 \text{ on } \partial M \right\} \\
&= \left\{ u \in \mathfrak{X}^s_\parallel \,\middle|\, [\mathrm{Def}(u)\cdot u]^{tan} = 0 \text{ on } \partial M \right\},
\end{aligned}$$

where $S_n : T\partial M \to T\partial M$ is the Weingarten map defined by $S_n(u) := -\nabla_u n$ and Def is the deformation tensor given by $\mathrm{Def}\, u := \frac{1}{2}\left(\nabla u + \nabla u^T\right)$.

Weak Riemannian metrics. We consider two inner products associated to the following elliptic differential operators

$$\Delta_R := -\nabla^*\nabla = \Delta + \mathrm{Ric} \quad \text{and} \quad \mathcal{L} := -2\,\mathrm{Def}^*\,\mathrm{Def} = \Delta + \mathrm{grad}\,\mathrm{div} + 2\,\mathrm{Ric},$$

where ∇ is the Levi-Civita covariant derivative, $\Delta = -\delta\mathbf{d} - \mathbf{d}\delta$ is the positive definite Hodge Laplacian, Ric is the Ricci operator of the metric, and the star denotes the formal L^2-adjoint differential operator.

Denote by $\langle\cdot,\cdot\rangle_0$ the L^2-inner product on arbitrary tensors of the same type relative to the given Riemannian metric. We recall the Green formulas

$$\langle\Delta_R u, v\rangle_0 = -\langle\nabla u, \nabla v\rangle_0 + \int_{\partial M} g\left(\nabla_n u, v\right)\mu_\partial, \tag{1}$$

$$\langle\mathcal{L}u, v\rangle_0 = -2\langle\operatorname{Def} u, \operatorname{Def} v\rangle_0 + \int_{\partial M} g\left((\nabla_n u)^{tan} + S_n(u), v\right)\mu_\partial. \tag{2}$$

Thus, for all $\alpha > 0$, we can consider the following weak inner products

$$\langle u, v\rangle_{\Delta_R} := \langle(1-\alpha^2\Delta_R)u, v\rangle_0 \quad \text{and} \quad \langle u, v\rangle_{\mathcal{L}} := \langle(1-\alpha^2\mathcal{L})u, v\rangle_0$$

on $\mathcal{V}^s_D$ and $\mathcal{V}^s_N$ respectively. We denote by $\mathcal{G}_{\Delta_R}$ and $\mathcal{G}_{\mathcal{L}}$ the associated right-invariant weak Riemannian metrics on $\mathcal{D}^s_D$ and $\mathcal{D}^s_N$, respectively. Using the general Euler-Poincaré reduction theorem (see [5]), we conclude that a curve $\eta(t) \in \mathcal{D}^s_D$ is a geodesic with respect to the weak Riemannian metric $\mathcal{G}_{\Delta_R}$ if and only if the curve $u(t) := \dot\eta(t)\circ\eta(t)^{-1} \in \mathcal{V}^s_D$ is a solution of the n-dimensional Camassa-Holm equations with *Dirichlet boundary conditions*

$$\begin{cases} \partial_t m + \nabla_u m + \nabla u^T\cdot m + m\operatorname{div} u = 0, \quad m = (1-\alpha^2\Delta_R)u, \\ u = 0 \quad \text{on} \quad \partial M. \end{cases} \tag{3}$$

Similarly, a curve $\eta(t) \in \mathcal{D}^s_N$ is a geodesic with respect to $\mathcal{G}_{\mathcal{L}}$ if and only if $u(t) := \dot\eta(t)\circ\eta(t)^{-1} \in \mathcal{V}^s_N$ is a solution of the n-dimensional Camassa-Holm equations with *Navier-slip boundary conditions*

$$\begin{cases} \partial_t m + \nabla_u m + \nabla u^T\cdot m + m\operatorname{div} u = 0, \quad m = (1-\alpha^2\mathcal{L})u, \\ g(u,n) = 0, \quad [\operatorname{Def}(u)\cdot n]^{tan} = 0 \quad \text{on} \quad \partial M. \end{cases} \tag{4}$$

Thus, in order to prove the well-posedness of these PDE's, it suffices to show the local existence of the geodesics associated to the weak Riemannian metrics $\mathcal{G}_{\Delta_R}$ and $\mathcal{G}_{\mathcal{L}}$ on the corresponding diffeomorphism groups. This approach is due to [2] and used there to show the well-posedness of the incompressible Euler equations. The same method is used in [6] and [3], in the case of the averaged Euler equations.

3 Well-posedness

Theorem 1. *The weak Riemannian Hilbert manifolds $(\mathcal{D}^s_D, \mathcal{G}_{\Delta_R})$ and $(\mathcal{D}^s_N, \mathcal{G}_{\mathcal{L}})$ admit smooth geodesic sprays, denoted by $\mathcal{S}_D$ and $\mathcal{S}_N$, respectively.*

The following corollary is obtained by solving the ordinary differential equations $\ddot\eta = \mathcal{S}_D(\dot\eta)$ and $\ddot\eta = \mathcal{S}_N(\dot\eta)$ on $\mathcal{D}^s_D$ and $\mathcal{D}^s_N$.

Corollary 1. *Consider a compact manifold M with a smooth boundary (possibly empty) and fix $s > 1+n/2$. Then for all $u_0 \in \mathcal{V}^s_D$, the n-dimensional Camassa-Holm with Dirichlet* *boundary conditions* (3) *admits a unique local in time solution* u_t *with initial condition* u_0. *Moreover we have*

$$t \mapsto u_t \in C^0\left(\,]-\epsilon,\epsilon[\,,\mathcal{V}_D^s\right)\cap C^1\left(\,]-\epsilon,\epsilon[\,,\mathcal{V}_D^{s-1}\right),$$

and for all $t\in\,]-\epsilon,\epsilon[$, *the map* $u_0\in\mathcal{V}_D^s\mapsto u(t)\in\mathcal{V}_D^s$ *is continuous. The same result holds for Navier-slip boundary conditions.*

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