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IMPROVED CONVERGENCE RATES FOR TAIL PROBABILITIES

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Abstract

Let $X_1, X_2, ...$ be i.i.d. random variables, and put $S_n = X_1 + ... + X_n$. We find necessary and sufficient moment conditions for $\int_\delta^\infty f(x)dx < \infty$, $\delta > \alpha$, where $\alpha \geq 0$ and $f(x) = \sum_n a_n P(|S_n| > xb_n)$ with $a_n > 0$ and b_n is either $n^{1/p}$, $0 < p < 2$, $\sqrt{n \log n}$ or $\sqrt{n \log \log n}$. The series $f(x)$ we deal with are classical series studied by Hsu and Robbins, Erdős, Spitzer, Baum and Katz, Davis, Lai, Gut, etc.

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1 Introduction

Let $X, X_1, X_2, ...$ be i.i.d. random variables with $P(X \neq 0) > 0$ and $EX = 0$, and consider the random walk $S_0 = 0$, $S_n = X_1 + ... + X_n$, $n \geq 1$. We consider series of the type

$$f(x) = \sum_n a_n P(|S_n| > xb_n), \quad x > 0,$$

where $a_n > 0$ and $\sum_n a_n = \infty$, and for b_n we deal with the next cases:

- $b_n = n^{1/p}$, $0 < p < 2$, and $P(|S_n| > xn^{1/p})$ is called probability of large deviation;
- $b_n = \sqrt{n \log n}$, and $P(|S_n| > x\sqrt{n \log n})$ is called probability of moderate deviation;
- $b_n = \sqrt{n \log \log n}$, and $P(|S_n| > x\sqrt{n \log \log n})$ is called probability of small deviation.

Several authors proved that

$$\begin{aligned} E[\varphi(|X|)] &< \infty \text{ for some function } \varphi \Longleftrightarrow \\ \sum_n a_n P(|S_n| &> xb_n) < \infty \text{ for } x > \text{some } a. \end{aligned} \tag{1}$$

We show that, except for two remarkable cases, the following strengthening of (1) is possible. This is the general form of our main result.

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Theorem A. *The following are equivalent:*

(i) $E[\varphi(|X|)] < \infty$;

(ii) $f(x) = \sum_n a_n P(|S_n| > xb_n),\ x > a$;

(iii) $I_\delta = \int_\delta^\infty f(x)dx < \infty,\ \delta > a$, i.e. $\sum_k \sum_n a_n P(|S_n| > kb_n) < \infty$.

The impetus to study the convergence of the integral I_δ comes from the theory of branching processes. Namely, K. B. Athreya (1988) considered a critical Galton-Watson process $(Z_n)_{n\geq 0}$ with $Z_0 = i$ such that $EZ_1^2 < \infty$ to which he associated a random walk S_n generated by the random variable $X =$ the number of offspring produced by a single parent particle and put $M_n = \max_{0\leq k\leq n} Z_k$.Then he proved that $EM_n/\log n \to i$ as $n \to \infty$. The main step in the proof is to establish the convergence of the series $\sum_n E[\frac{|S_n|}{n} I\{|S_n| > \delta n\}]$ for all $\delta > 0$, which is in fact a result of the form $I_\delta < \infty$.

We present special instances of the general Theorem A, exposing in order relevant results concerning the random walk S_n with boundary $\pm xn^{1/p},\ 0 < p < 2$, $\pm x\sqrt{n \log n}$ and $\pm x\sqrt{n \log\log n}$.

2 The boundary $\pm xn^{1/p},\ 0 < p < 2,$ (Large deviations)

For $x > 0$, set $N_x = \sum_n I\{|S_n| > xn\} =$ the number of exits of the random walk S_n beyond the boundary $\pm xn$ and consider the domain $D_x = \{(n, y) : |y| \leq xn\}$. Then the strong law of large numbers (LLN), due to A. N. Kolmogorov (1930), can be rephrased as follows.

Strong Law of Large Numbers. $E|X| < \infty \Longleftrightarrow N_x < \infty$ *a.s.*, $x > 0 \Longleftrightarrow$ *whatever* $x > 0$, $S_n \in D_x$ *a.s. for all but finitely many* n.

Closely related to the strong LLN is the complete convergence theorem. The sufficiency part of this theorem was proved by P. L. Hsu and H. Robbins (1947), while the converse part was obtained by P. Erdős (1949, 1950).

Complete Convergence Theorem. $EX^2 < \infty \Longleftrightarrow f(x) = EN_x = \sum_n P(|S_n| > xn) < \infty$ *for any* $x > 0$.

The function f is nonincreasing. Since $P(X \neq 0) > 0$, it follows that $\lim_{x\searrow 0} f(x) = \sum_n P(S_n \neq 0) = \infty$ by the Borel-Cantelli lemma. Additional information about this limit is provided by the next theorem, due to C. C. Heyde (1975).

Theorem 1. $EX^2 < \infty \Longrightarrow x^2 EN_x \to EX^2$, *and so* $f(x) \sim x^{-2}EX^2$ *as* $x \searrow 0$.

This means that $\int_0^\infty f(x)dx = \infty$ and raises the question about finiteness of $I_\delta = \int_\delta^\infty f(x)dx$ for $\delta > 0$. The answer is given by the following result proved by A. Spătaru (1990).

Theorem 2. $EX^2 < \infty \Longleftrightarrow I_\delta = \int\limits_\delta^\infty EN_x dx = \int\limits_\delta^\infty (\sum\limits_n P(|S_n| > xn))dx < \infty,\ \delta > 0.$

This is an improvement over the complete convergence theorem which asserts that $\sum\limits_n P(|S_n| > xn) < \infty,\ x > 0 \Longleftrightarrow EX^2 < \infty$, while Theorem 2 shows that the same moment condition $EX^2 < \infty$ is equivalent to the convergence of the double series $\sum\limits_k \sum\limits_n P(|S_n| > kn) < \infty$. Theorem 2 was generalized by D. Li and A. Spătaru (2005) as follows.

Theorem 3. *For $q > 0$,*

$$\int\limits_\delta^\infty f(x^q)dx < \infty,\ \delta > 0 \Longleftrightarrow \begin{cases} E\,|X|^{1/q} < \infty & \text{if } q < 1/2 \\ E[X^2 \log^+ |X|] < \infty & \text{if } q = 1/2 \\ EX^2 < \infty & \text{if } q > 1/2 \end{cases}.$$

More generally, for $x > 0$ and $0 < p < 2$, put $N_x = \sum\limits_n I\{|S_n| > xn^{1/p}\}$ = the number of exits of S_n beyond the boundary $\pm xn^{1/p}$ and consider the domain $D_x = \{(n, y) : |y| \le xn^{1/p}\}$. For $x > 0$ and $r \ge 1$, consider also the random series $M_x = \sum\limits_n n^{r-2} I\{|S_n| > xn^{1/p}\}$. Then the generalization of the strong law of large numbers, due to J. Marcinkiewicz and A. Zygmund (1937), can be rephrased as follows.

Generalized Strong Law of Large Numbers. $E\,|X|^p < \infty \Longleftrightarrow N_x < \infty$ *a.s.*, $x > 0 \Longleftrightarrow$ *whatever* $x > 0$, $S_n \in D_x$ *a.s. for all but finitely many* n.

Closely related to the generalized strong LLN is the generalized complete convergence theorem.

Generalized Complete Convergence Theorem. $E\,|X|^{pr} < \infty \Longleftrightarrow f(x) = EM_x = \sum\limits_n n^{r-2} P(|S_n| > xn^{1/p}) < \infty$ *for any* $x > 0$.

For $p = 1$ and $r = 2$ this theorem reduces to the Hsu-Robbins-Erdős complete convergence theorem. The special case $p = r = 1$ was proved by F. Spitzer (1956), and the result in the general form is due to L. E. Baum and M. Katz (1965). The next strengthening of the generalized complete theorem was obtained by D. Li and A. Spătaru (2005).

Theorem 4. For $q > 0$,

$$\int\limits_\delta^\infty f(x^q)dx < \infty,\ \delta > 0 \Longleftrightarrow \begin{cases} E\,|X|^{1/q} < \infty & \text{if } q < 1/pr \\ E[|X|^{pr} \log^+ |X|] < \infty & \text{if } q = 1/pr \\ E\,|X|^{pr} < \infty & \text{if } q > 1/pr \end{cases}.$$

The first exceptional case alluded to above Theorem A refers to Spitzer's theorem. More precisely the following result holds.

Theorem 5. *For $x > 0$, define the stopping times $T_+(x) = \inf\{n : 1 \le n \le \infty : S_n > xn\}$ and $T_-(x) = \inf\{n : 1 \le n \le \infty : S_n < -xn\}$. Then the following statements are equivalent:*

(i) $E[|X| \log^+ |X|] < \infty$;

(ii) $E[\sup_{n\geq 1} |S_n|] < \infty$;

(iii) $E[\frac{S_{T_+(x)}}{T_+(x)} I\{T_+(x) < \infty\}] < \infty$ and $E[\frac{S_{T_-(x)}}{T_-(x)} I\{T_-(x) < \infty\}] > -\infty$;

(iv) $\int_\delta^\infty (\sum_n \frac{1}{n} P(|S_n| > xn))dx < \infty$. *for any* $\delta > 0$.

J. Marcinkiewicz and A. Zygmund (1937) proved that (i)$\Longrightarrow$(ii), and D. L. Burkholder (1962) showed the converse implication (ii)$\Longrightarrow$(i). Recently, A. Spătaru (2006) established the sequence of implications (ii)$\Longrightarrow$(iii)$\Longrightarrow$(iv)$\Longrightarrow$(i).

Another important result in this area is next stated. L. E. Baum and M. Katz (1965) proved that (i)$\Longleftrightarrow$(ii), and A. Spătaru (2006) showed that (i)$\Longleftrightarrow$(iii).

Theorem 6. *Let* $1 < p < 2$. *The following are equivalent:*

(i) $E[|X|^p \log^+ |X|] < \infty$;

(ii) $f(x) = \sum_n \frac{\log n}{n} P(|S_n| > xn^{1/p}) < \infty$ *for any* $x > 0$;

(iii) $I_\delta = \int_\delta^\infty f(x)dx < \infty$ *for any* $\delta > 0$.

The second exceptional case alluded to above Theorem A is related to this theorem and corresponds to the case $p = 1$. Namely, the following result was obtained by A. Spătaru (2006).

Theorem 7. $E[|X| (\log^+ |X|)^2] < \infty \Longleftrightarrow \int_\delta^\infty (\sum_n \frac{\log n}{n} P(|S_n| > xn))dx < \infty,\ \delta > 0$.

3 The boundary $\pm x\sqrt{n \log n}$, $0 < p < 2$, (Moderate deviations)

For $x > 0$ and $r > 1$, define $M_x = \sum_n n^{r-2} I\{|S_n| > x\sqrt{n \log n}\}$ = the number of exits of S_n over the boundary $\pm x\sqrt{n \log n}$ with the "weights" n^{r-2}. The statement (i) in Theorem 8 below is due to T. L. Lai, and the statement (ii) was proved by A. Spătaru (2006).

Theorem 8. *Writing* $EX^2 = \sigma^2$, *the following hold:*

(i) $f(x) = EM_x = \sum_n n^{r-2} P(|S_n| > x\sqrt{n \log n}) < \infty$ *for any* $x > \sigma\sqrt{2r-2}$;

(ii) $I_\delta = \int_\delta^\infty f(x)dx < \infty$ *for any* $\delta > \sigma\sqrt{2r-2}$.

An interesting result concerning moderate deviations is as follows.

Theorem 9. *The following are equivalent:*

(i) $EX^2 < \infty$;

(ii) $f(x) = \sum_n \frac{\log n}{n} P(|S_n| > x\sqrt{n \log n}) < \infty$ *for any* $x > 0$;

(iii) $I_\delta = \int_\delta^\infty f(x)dx < \infty$ *for any* $\delta > 0$.

J. A. Davis (1968) showed that (i)$\Longleftrightarrow$(ii), and S. H. Siraždinov proved that (i)$\Longleftrightarrow$(iii).

4 The boundary $\pm x\sqrt{n \log\log n}$, (Small deviations)

For $x > 0$, define $M_x = \sum_n \frac{1}{n} I\{|S_n| > x\sqrt{n \log\log n}\}$ = the number of exits of S_n over the boundary $\pm x\sqrt{n \log\log n}$ with the "weights" $\frac{1}{n}$. The statement (i) in Theorem 10 below is due to J. A. Davis (1968a), and the statement (ii) was proved by A. Spătaru (2006).

Theorem 10. *Writing $EX^2 = \sigma^2$, the following hold:*

(i) $f(x) = EM_x = \sum_n \frac{1}{n} P(|S_n| > x\sqrt{n \log\log n}) < \infty$ *for any* $x > \sigma\sqrt{2}$;

(ii) $I_\delta = \int_\delta^\infty f(x)dx < \infty$ *for any* $\delta > \sigma\sqrt{2}$.

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