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THE MINIMUM COST FLOW PROBLEM WITH SURPLUS

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Abstract

In this paper we focus on the minimum cost flow problem with surplus because it has many applications in economy, manufacturing, transportation and distribution. In a minimum cost flow problem the sum of supplies equals the sum of demands, but in a minimum cost flow problem with surplus the supplies exceed the demands.

We will solve the minimum cost flow problem with surplus by reducing it to a standard minimum cost flow problem in a transformed network.

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1 The minimum cost flow problem

The minimum cost flow problem, as well as one of its special cases which is the maximum flow problem, is one of the fundamental problems in network flow theory and it was studied extensively in the last five decades. The importance of the minimum cost flow problem is also due to the fact that it arises in almost all industries, including agriculture, communications, defense, education, energy, health care, medicine, manufacturing, retailing and transportation. Indeed, the minimum cost flow problem is pervasive in practice.

Let $G = (N, A)$ be a directed graph, defined by a set N of n nodes and a set A of m arcs. Each arc $(x, y) \in A$ has a capacity $c(x, y)$ and a cost $b(x, y)$. We associate with each node $x \in N$ a number $v(x)$ which indicates its supply or demand depending on whether $v(x) > 0$ or $v(x) < 0$. In the directed network $G = (N, A, c, b, v)$, the minimum cost flow problem is to determine the flow $f(x, y)$ on each arc $(x, y) \in A$ which

$$minimize \sum_{(x,y)\in A} b(x,y)f(x,y) \tag{1}$$

subject to

$$\sum_{y|(x,y)\in A} f(x,y) - \sum_{y|(y,x)\in A} f(y,x) = v(x), \qquad \forall x \in N \tag{2}$$

$$0 \le f(x,y) \le c(x,y), \qquad \forall (x,y) \in A. \tag{3}$$

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A flow f satisfying the last two conditions is a feasible flow.

Let $n = |N|$ and $m = |A|$.

Let C denote the largest magnitude of any supply/demand or finite arc capacity, that is

$$C = \max(\max\{v(x)|x \in N\}, \max\{c(x,y)|(x,y) \in A, c(x,y) < \infty\}).$$

Let B denote the largest magnitude of any arc cost, that is

$$B = \max\{b(x,y)|(x,y) \in A\}.$$

The residual network $G(f) = (N, A(f))$ corresponding to a flow f is defined as follows. We replace each arc $(x,y) \in A$ by two arcs (x,y) and (y,x). The arc (x,y) has cost $b(x,y)$ and residual capacity $r(x,y) = c(x,y) - f(x,y)$ and the arc (y,x) has cost $b(y,x) = -b(x,y)$ and residual capacity $r(y,x) = f(x,y)$. The residual network consists only of arcs with positive residual capacity.

We shall assume that the minimum cost flow problem satisfies the following assumptions:

1. All data (cost, supply/demand and capacity) are integral.
2. The network contains no directed negative cost cycle of infinite capacity.
3. All arc costs are nonnegative.
4. The supplies/demands at the nodes satisfy the condition $\sum_{x \in N} v(x) = 0$ and the minimum cost flow problem has a feasible solution.
5. The network contains an uncapacitated directed path (i.e. each arc in the path has infinite capacity) between every pair of nodes.

All these assumptions can be made without any loss of generality (for details see [1]).

We associate a real number $\pi(x)$ with each node $x \in N$. We refer to $\pi(x)$ as the potential of node x. For a given set of node potentials π, we define the reduced cost of an arc (x,y) as

$$b^{\pi}(x,y) = b(x,y) - \pi(x) + \pi(y).$$

The reduced costs are applicable to the residual network as well as to the original network.

Theorem 1. *[1](a) For any directed path P from node w to node z we have*
$\sum_{(x,y) \in P} b^{\pi}(x,y) = \sum_{(x,y) \in P} b(x,y) - \pi(w) + \pi(z)$
(b) For any directed cycle W we have
$\sum_{(x,y) \in W} b^{\pi}(x,y) = \sum_{(x,y) \in W} b(x,y)$

Theorem 2. *(Negative Cycle Optimality Conditions) [1] A feasible solution f is an optimal solution of the minimum cost flow problem if and only if it satisfies the following negative cycle optimality conditions:*

the residual network $G(f)$ contains no negative cost directed cycle.

Theorem 3. *(Reduced Costs Optimality Conditions) [1] A feasible solution f is an optimal solution of the minimum cost flow problem if and only if some set of node potentials π satisfy the following reduced cost optimality conditions:*

$b^{\pi}\,(x,y) \geq 0$ *for every arc (x,y) in $G(f)$*

Theorem 4. *(Complementary Slackness Optimality Conditions) [1] A feasible solution f is an optimal solution of the minimum cost flow problem if and only if for some set of node potentials π, the reduced costs and flow values satisfy the following complementary slackness optimality conditions for every arc $(x,y) \in A$:*

$$\textit{If } b^{\pi}(x,y) > 0, \textit{then } f(x,y) = 0 \tag{4}$$

$$\textit{If } 0 < f(x,y) < c(x,y), \textit{then } b^{\pi}(x,y) = 0 \tag{5}$$

$$\textit{If } b^{\pi}(x,y) < 0, \textit{then } f(x,y) = c(x,y) \tag{6}$$

A pseudoflow is a function $f : A \to \mathbb{R}^{+}$ satisfying only conditions (3). For any pseudoflow f, we define the imbalance of node x as

$e(x) = v(x) + f(N,x) - f(x,N)$, for all $x \in N$.

If $e(x) > 0$ for some node x, we refer to $e(x)$ as the excess of node x; if $e(x) < 0$, we refer to $-e(x)$ as the deficit of node x. If $e(x) = 0$ for some node x, we refer to node x as the balanced.

The residual network corresponding to a pseudoflow is defined in the same way as we define the residual network for a flow.

The optimality conditions can be extended for pseudoflows.

We refer to a flow or a pseudoflow f as ϵ-optimal for some $\epsilon > 0$ if for some node potentials π , the pair (f,π) satisfies the following ϵ-optimality conditions:

$$\text{If } b^{\pi}(x,y) > \epsilon, \text{then } f(x,y) = 0 \tag{7}$$

$$\text{If } -\epsilon \leq b^{\pi}(x,y) \leq \epsilon, \text{then } 0 \leq f(x,y) \leq c(x,y) \tag{8}$$

$$\text{If } b^{\pi}(x,y) < -\epsilon, \text{then } f(x,y) = c(x,y) \tag{9}$$

These conditions are relaxations of the (exact) complementary slackness optimality conditions (4) - (6) and they reduce to complementary slackness optimality conditions when $\epsilon = 0$.

For solving a minimum cost flow problem, based on these optimality conditions, several algorithms were developed from the primal-dual algorithm proposed by Ford and Fulkerson in 1962 to the polynomial-time cycle-canceling algorithms described by Sokkalingam, Ahuja and Orlin in 2001.

The basic algorithms for minimum cost flow can be divided into two classes: those that maintain feasible solutions and strive toward optimality and those that maintain infeasible solutions that satisfy optimality conditions and strive for feasibility. Algorithms from the first class are: the cycle-canceling algorithm and the out-of-kilter algorithm. The cycle-canceling algorithm maintains a feasible flow at every iteration, augments flow along negative cycle in the residual network and terminates when there is no more negative cycle in the residual network, which means (from Theorem 2) that the flow is a minimum cost flow. The out-of-kilter algorithm maintains a feasible flow at every iteration and augments flow along the shortest path in order to satisfy the optimality conditions. Algorithms from the second class are: the successive shortest path algorithm and primal-dual algorithm. The successive shortest path algorithm maintains a pseudoflow that satisfies the optimality conditions and augments flow along the shortest path from excess nodes to deficit nodes in the residual network in order to convert the pseudoflow into an optimal flow. The primal-dual algorithm also maintains a pseudoflow that satisfies the optimality conditions and solves maximum flow problems in order to convert the pseudoflow into an optimal flow.

Starting from the basic algorithms for minimum cost flow, several polynomial-time algorithms were developed. Most of them were obtained by using the scaling technique. By capacity scaling, by cost scaling or by capacity and cost scaling, the following polynomial-time algorithms were developed: capacity scaling algorithm, cost scaling algorithm, double scaling algorithm, repeated capacity scaling algorithm and enhanced capacity scaling algorithm.

Another approach for obtaining polynomial-time algorithms is to select carefully the negative cycles in the cycle-canceling algorithm.

2 The minimum cost flow problem with surplus

Applications from several different domains can be modelled and solved as minimum flow cost problem in which the sum of the supplies exceeds the sum of the demands. For instance, problems from economy, manufacturing, transportation and distribution. In these problems, assumption 4 is not satisfied, because the supplies/demands at the nodes do not satisfy the condition $\sum_{x\in N} v(x) = 0$; more precisely the supplies exceed the demands.

In the directed network $G = (N, A, c, b, v)$, the minimum cost flow problem with surplus is to determine the flow $f(x, y)$ on each arc $(x, y) \in A$ which

$$minimize \sum_{(x,y)\in A} b(x, y)f(x, y) \tag{10}$$

subject to

$$\sum_{y|(x,y)\in A} f(x,y) - \sum_{y|(y,x)\in A} f(y,x) \le v(x), \qquad \forall x \in S \tag{11}$$

$$\sum_{y|(x,y)\in A} f(x,y) - \sum_{y|(y,x)\in A} f(y,x) = v(x), \qquad \forall x \in N - S \tag{12}$$

$$0 \le f(x,y) \le c(x,y), \qquad \forall (x,y) \in A. \tag{13}$$

where $S = \{x \in N | v(x) > 0\}$ and $T = \{x \in N | v(x) < 0\}$.

In a minimum cost flow problem with surplus, $\sum_{x\in S} v(x) > |\sum_{x\in T} v(x)|$

We can transform this problem into a standard minimum cost flow problem in the transformed network $G' = (N', A', c', b', v')$, where

$N' = N \cup \{t'\}$
$A' = A \cup A_{t'}$, $A_{t'} = \{(x,t') \in A | x \in S\}$
$c'(x,y) = c(x,y), \forall (x,y) \in A$, $c'(x,y) = \infty, \forall (x,y) \in A_{t'}$
$b'(x,y) = b(x,y), \forall (x,y) \in A$, $b'(x,y) = 0, \forall (x,y) \in A_{t'}$
$v'(x) = v(x), \forall x \in N$, $v'(t') = -\sum_{x\in N} v(x)$

Theorem 5. *There is a feasible solution of the minimum cost flow problem with surplus in the network $G = (N, A, c, b, v)$ if and only if there is a feasible solution of the standard minimum cost flow problem in the transformed network $G' = (N', A', c', b', v')$.*

Proof. Let f be a feasible solution of the minimum cost flow problem with surplus in the network $G = (N, A, c, b, v)$. We can determine a feasible flow f' in the transformed network $G' = (N', A', c', b', v')$ in the following manner:

$f'(x,y) = f(x,y), \forall (x,y) \in A$
$f'(x,t') = v(x) - \sum_{y|(x,y)\in A} f'(x,y) + \sum_{y|(y,x)\in A} f'(y,x), \forall (x,t') \in A_{t'}$

We have

$\sum_{x|(t',x)\in A'} f'(t',x) - \sum_{x|(x,t')\in A'} f(x,t') =$
$= -\sum_{x|(x,t')\in A'} f(x,t') =$
$= -(\sum_{x\in N} v(x) - \sum_{x\in N}\sum_{y|(x,y)\in A} f'(x,y) + \sum_{x\in N}\sum_{y|(y,x)\in A} f'(y,x)) =$
$= -\sum_{x\in N} v(x) =$
$= v'(t)$

For any $x \in S$ we have

$\sum_{y|(x,y)\in A'} f'(x,y) - \sum_{y|(y,x)\in A'} f'(y,x) =$
$= \sum_{y|(x,y)\in A} f'(x,y) + f(x,t') - \sum_{y|(y,x)\in A} f'(y,x) =$
$= \sum_{y|(x,y)\in A} f'(x,y) + v(x) - \sum_{y|(x,y)\in A} f'(x,y) + \sum_{y|(y,x)\in A} f'(y,x) - \sum_{y|(y,x)\in A} f'(y,x) =$
$= v(x) = v'(x)$

For any $x \in N - S$ we have

$\sum_{y|(x,y)\in A'} f'(x,y) - \sum_{y|(y,x)\in A'} f'(y,x) =$
$= \sum_{y|(x,y)\in A} f'(x,y) - \sum_{y|(y,x)\in A} f'(y,x) =$
$= v(x) = v'(x)$

Thus, $\sum_{y|(x,y)\in A'} f'(x,y) - \sum_{y|(y,x)\in A'} f'(y,x) = v'(x), \forall x \in N'$.

Obviously, $0 \le f'(x,y) \le c'(x,y), \forall (x,y) \in A'$.

Consequently, f' is a feasible flow in the transformed network $G' = (N', A', c', b', v')$.

Reciprocaly, let f' be a feasible flow in the transformed network $G' = (N', A', c', b', v')$. Then $f = f'|_A$ is obviously a feasible solution of the minimum cost flow problem with surplus in the network $G = (N, A, c, b, v)$.

□

Theorem 6. *There is an optimal solution of the minimum cost flow problem with surplus in network $G = (N, A, c, b, v)$ if and only if there is an optimal solution of the standard minimum cost flow problem in the transformed network $G' = (N', A', c', b', v')$.*

Proof. It follows directly from Theorem 5 and from the fact that the flow f in G and the flow f' in G' have the same cost, because all the additional arcs from $A_{t'}$ have zero cost. □

Consequently, we can solve the minimum cost flow problem with surplus in the network $G = (N, A, c, b, v)$ by applying any minimim cost flow algorithm in the transformed network $G' = (N', A', c', b', v')$. Let f' be a minimum cost flow in the transformed network G'. Then $f = f'|_A$ is obviously an optimal solution of the minimum cost flow problem with surplus in the network G.

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