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SOME IMPROVEMENTS OF STIRLING'S FORMULA

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Abstract

The aim of this paper is to improve a representation formula for the factorial function stated by Liu in [A new version of the Stirling formula, Tamsui Oxf. J. Math. Sci. 23(4) (2007) 389-392].

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1 Introduction

The Stirling approximation formula for big factorials

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

plays a basic role in science as statistical physiscs and probability theory. As a consequence, many authors tried to find other more accurate formulas, which are also as simple as the initial formula to approximate $n!$.

Recently, Hsu in [3] has given the following version of the Stirling formula

$$n! = \left(\frac{n}{e}\right)^n \sqrt{2\pi n} \cdot \exp\left(\sum_{k=n}^{\infty}\sum_{j=2}^{\infty} \frac{j-1}{2j(j+1)} \left(\frac{-1}{k}\right)^j\right) \tag{1.1}$$

whose proof is elementary, without using the Bernoulli numbers as usually. See for example [2], [5].

Having as a starting point representation (1.1), we will give a different characterization of the sequence $(a_n)_{n\geq 1}$ which satisfies the identity:

$$n! = \left(\frac{n}{e}\right)^n \sqrt{2\pi n} \cdot \exp\left(\sum_{k=n}^{\infty} a_k\right). \tag{1.2}$$

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Liu [4] have replaced the double summation in the right hand of (1.1) with an infinite integral. More precisely, the following integral formula is established in [4]:

$$n! = \left(\frac{n}{e}\right)^n \sqrt{2\pi n} \cdot \exp \int_n^\infty \frac{\frac{1}{2} - \{x\}}{x} dx. \tag{1.3}$$

We establish in this paper the following new approximation formula:

$$n! \approx \sqrt{\frac{2\pi}{e}} \left(\frac{n+1}{e}\right)^{n+\frac{1}{2}} = \mathcal{F}_n, \tag{1.4}$$

which is more accurate than the much celebrated Stirling's formula. Other good fact is that formula (1.4) is as simple as Stirling's formula. We say that because other approximation formulas were stated, for example, the formula given in [6] by Schuster, but they are complicated and hard to use in practical problems.

Then we concentrate on setting a formula which is similar with (1.3). Indeed, we will prove the following interesting representation

$$n! = \sqrt{\frac{2\pi}{e}} \left(\frac{n+1}{e}\right)^{n+\frac{1}{2}} \exp \int_n^\infty \frac{\frac{1}{2} - \{x\}}{x+1} dx.$$

2 The main results

The starting idea is the following

Lemma 1. *There exists a convergent series* $\sum_{n=1}^{\infty} a_n$ with positive terms such that it satisfies, for every integer $n \geq 1$, the relation

$$n! = \left(\frac{n}{e}\right)^n \sqrt{2\pi n} \exp\left(\sum_{k=n}^{\infty} a_k\right). \tag{2.1}$$

Proof. For any integer $n \geq 1$, from Stirling's formula we can write

$$n! = \sqrt{2\pi} \left(\frac{n}{e}\right)^n e^{b_n},$$

where $\frac{1}{12n+1} < b_n < \frac{1}{12n}$. Denote $a_n = b_n - b_{n+1}$. Since $\lim_{n\to\infty} b_n = 0$ it folows $b_n = \sum_{k=n}^{\infty} a_k$.

By dividing the equalities

$$n! = \left(\frac{n}{e}\right)^n \sqrt{2\pi n} \exp\left(\sum_{k=n}^{\infty} a_k\right)$$

and

$$(n-1)! = \left(\frac{n-1}{e}\right)^{n-1} \sqrt{2\pi(n-1)} \exp\left(\sum_{k=n-1}^{\infty} a_k\right),$$

we obtain, for every $n \geq 1$, the general term:

$$a_n = \left(n + \frac{1}{2}\right) \ln\left(1 + \frac{1}{n}\right) - 1. \tag{2.2}$$

Easily one has $a_n > 0$, for $n \geq 1$. The obtained series

$$z = \sum_{n=1}^{\infty} \left(\left(n + \frac{1}{2}\right) \ln\left(1 + \frac{1}{n}\right) - 1\right)$$

is convergent (with sum z) and the order of convergence is given by following comparison test:

$$\lim_{n\to\infty} \frac{\left(n + \frac{1}{2}\right) \ln\left(1 + \frac{1}{n}\right) - 1}{\frac{1}{n^2}} = \frac{1}{12}.$$

□

Now let us separate term a_n from the series to obtain:

$$\begin{aligned} n! &= \left(\frac{n}{e}\right)^n \sqrt{2\pi n} \exp a_n \cdot \exp\left(\sum_{k=n+1}^{\infty} a_k\right) = \\ &= \sqrt{\frac{2\pi}{e}} \left(\frac{n+1}{e}\right)^{n+\frac{1}{2}} \exp\left(\sum_{k=n+1}^{\infty} a_k\right). \end{aligned} \tag{2.3}$$

The remainder of the (convergent) series from the right side of (2.3) tends to zero as n tends to infinity, so we have the following estimation:

$$n! \approx \sqrt{\frac{2\pi}{e}} \left(\frac{n+1}{e}\right)^{n+\frac{1}{2}}.$$

By comparing the remainders of the series (with positive terms) from (2.1) and (2.3), we deduce that

$$\left(\frac{n}{e}\right)^n \sqrt{2\pi n} < \sqrt{\frac{2\pi}{e}} \left(\frac{n+1}{e}\right)^{n+\frac{1}{2}} < n!,$$

which proves the assertion that our new formula (1.4) is substantially stronger than the Stirling formula.

Further, remark that the sequence $(a_n)_{n\geq 1}$ is unique defined by relation (2.1) and if compare (1.1), (2.1) and (2.2), we deduce that

$$a_k = \sum_{j=2}^{\infty} \frac{j-1}{2j(j+1)} \left(\frac{-1}{k}\right)^j = \left(k + \frac{1}{2}\right) \ln\left(1 + \frac{1}{k}\right) - 1. \tag{2.4}$$

By replacing (2.4) in (2.3), we obtain the following result which is analogue with representation (1.1) given by Hsu in [3] for the Stirling formula:

Theorem 1. *For every $n \geq 1$, we have:*

$$n! = \sqrt{\frac{2\pi}{e}} \left(\frac{n+1}{e} \right)^{n+\frac{1}{2}} \exp \left(\sum_{k=n}^{\infty} \sum_{j=2}^{\infty} \frac{j-1}{2j(j+1)} \left(\frac{-1}{k+1} \right)^{j} \right).$$

Our Lemma 2.1 permits us to justify the integral representation (1.3) established by Liu in [4]. Indeed, if for some locally integrable function f, we have:

$$\sum_{k=n}^{\infty} a_k = \int_{n}^{\infty} f(x)dx,$$

then the equality

$$n! = \left(\frac{n}{e} \right)^{n} \sqrt{2\pi n} \cdot \exp \int_{n}^{\infty} f(x)dx$$

holds true if and only if for every $k \geq 1$,

$$\int_{k}^{k+1} f(x)dx = \left(k + \frac{1}{2} \right) \ln \left(1 + \frac{1}{k} \right) - 1.$$

It can be easily verified that this condition is satisfied by $f(x) = (1/2 - \{x\})/x$, so (1.3) is true.

We are now in position to establish the following representation formula, which is analogue with the integral formula (1.3) stated by Liu in [4].

Theorem 2. *For every $n \geq 1$, we have:*

$$n! = \sqrt{\frac{2\pi}{e}} \left(\frac{n+1}{e} \right)^{n+\frac{1}{2}} \exp \int_{n}^{\infty} \frac{\frac{1}{2} - \{x\}}{x+1} dx. \tag{2.5}$$

Proof. From (1.3), we obtain:

$$n! = \left(\frac{n}{e} \right)^{n} \sqrt{2\pi n} \cdot \exp \int_{n}^{n+1} \frac{\frac{1}{2} - \{x\}}{x} dx \cdot \exp \int_{n+1}^{\infty} \frac{\frac{1}{2} - \{x\}}{x} dx. \tag{2.6}$$

By a unitary translation in the second integral, we obtain:

$$\int_{n+1}^{\infty} \frac{\frac{1}{2} - \{x\}}{x} dx = \int_{n}^{\infty} \frac{\frac{1}{2} - \{x\}}{x+1} dx,$$

while the first integral from (2.6) is

$$\int_{n}^{n+1} \frac{\frac{1}{2} - \{x\}}{x} dx = \int_{n}^{n+1} \frac{\frac{1}{2} - (x-n)}{x} dx = \left(n + \frac{1}{2} \right) \ln \left(1 + \frac{1}{n} \right) - 1.$$

By exponetiating the previous integrals and replacing in (2.6), we obtain:

$$n! = \left(\frac{n}{e} \right)^{n} \sqrt{2\pi n} \cdot \frac{1}{e} \left(1 + \frac{1}{n} \right)^{n+\frac{1}{2}} \cdot \exp \int_{n}^{\infty} \frac{\frac{1}{2} - \{x\}}{x+1} dx,$$

which is the requested equality (2.5). □

By using our method, we can obtain more and more accurate estimations, all of them being stronger than the classical Stirling's formula. We mean that instead of separating the term a_n from the remainder of the series, we can also separate terms $a_n + a_{n+1}$, $a_n + a_{n+1} + a_{n+2}$ and so on. More precisely, we can give the following estimations:

Theorem 3. *For every $n \geq 1$, we have:*

$$n! = \sqrt{\frac{2\pi}{e}} \left(\frac{n+2}{e}\right)^{n+\frac{3}{2}} \frac{1}{n+1} \exp\left(\sum_{k=n+2}^{\infty} a_k\right) \tag{2.7}$$

$$n! = \sqrt{\frac{2\pi}{e}} \left(\frac{n+3}{e}\right)^{n+\frac{5}{2}} \frac{1}{(n+1)(n+2)} \exp\left(\sum_{k=n+3}^{\infty} a_k\right). \tag{2.8}$$

As a consequence, the following approximation formulas for big factorials are true:

$$n! \approx \sqrt{\frac{2\pi}{e}} \left(\frac{n+2}{e}\right)^{n+\frac{3}{2}} \frac{1}{n+1}$$

$$n! \approx \sqrt{\frac{2\pi}{e}} \left(\frac{n+3}{e}\right)^{n+\frac{5}{2}} \frac{1}{(n+1)(n+2)}.$$

These approximations together with (1.4) are stronger than the Stirling formula, because of the ordering:

$$\left(\frac{n}{e}\right)^e \sqrt{2\pi n} < \sqrt{\frac{2\pi}{e}} \left(\frac{n+1}{e}\right)^{n+\frac{1}{2}} < \sqrt{\frac{2\pi}{e}} \left(\frac{n+2}{e}\right)^{n+\frac{3}{2}} \frac{1}{n+1} <$$

$$< \sqrt{\frac{2\pi}{e}} \left(\frac{n+3}{e}\right)^{n+\frac{5}{2}} \frac{1}{(n+1)(n+2)} < n!.$$

Finally, we give the corresponding integral representations, which can be obtained from (1.3), by separating the integral on $[n, n+1]$, respectively the integral on $[n, n+2]$.

Theorem 4. *For every $n \geq 1$, we have:*

$$n! = \sqrt{\frac{2\pi}{e}} \left(\frac{n+2}{e}\right)^{n+\frac{3}{2}} \frac{1}{n+1} \exp \int_n^{\infty} \frac{\frac{1}{2} - \{x\}}{x+2} dx$$

$$n! = \sqrt{\frac{2\pi}{e}} \left(\frac{n+3}{e}\right)^{n+\frac{5}{2}} \frac{1}{(n+1)(n+2)} \exp \int_n^{\infty} \frac{\frac{1}{2} - \{x\}}{x+3} dx.$$

3 Conclusions

One of the most performant formula ever known is the following, due to Burnside (*e.g.* [1]):

$$n! \approx \sqrt{2\pi}\left(\frac{n+\frac{1}{2}}{e}\right)^{n+\frac{1}{2}} = \mathcal{B}_n \tag{3.1}$$

and we claim that our formula (1.4) is comparable with (3.1). To see this, we study fractions

$$u_n = \frac{\sqrt{\frac{2\pi}{e}}\left(\frac{n+1}{e}\right)^{n+\frac{1}{2}}}{n!} \quad , \quad v_n = \frac{n!}{\sqrt{2\pi}\left(\frac{n+\frac{1}{2}}{e}\right)^{n+\frac{1}{2}}}.$$

Sequences $(u_n)_{n\geq 1}$ and $(v_n)_{n\geq 1}$ are both convergent to 1 and the first values are written in the following table:

n	u_n	v_n
40	0.99797	0.99897
50	0.99837	0.99918
150	0.99945	0.99972
400	0.99980	0.99989
1500	0.99994	0.99997
4000	0.99998	0.99999
10000	0.99999	0.99999

If we are interested to obtain better approximations than the Burnside formula, then we will replace approximations $\mathcal{F}_n$ and $\mathcal{B}_n$ by a mean of them, denoted $\mathcal{M}(\mathcal{F}_n, \mathcal{B}_n)$. We have the inequalities $\mathcal{F}_n < n! < \mathcal{B}_n$, so it is sufficient to choose a mean such that

$$\mathcal{F}_n < n! < \mathcal{M}(\mathcal{F}_n, \mathcal{B}_n) < \mathcal{B}_n$$

to obtain a stronger formula than the Burnside formula. First we have tried with the classical geometric mean, but after some numerical computations, we realized that $\sqrt{\mathcal{F}_n\mathcal{B}_n} < n!$. Then we used the mean

$$\mathcal{G}_n := \sqrt[3]{\mathcal{F}_n\mathcal{B}_n^2} = \mu \cdot \left(\sqrt[6]{\frac{(n+1)\left(n+\frac{1}{2}\right)^2}{e^3}}\right)^{2n+1},$$

where

$$\mu = \frac{\sqrt{2\pi}}{\sqrt[6]{e}} = 2.1218... \ .$$

The obtained estimation $n! \approx \mathcal{G}_n$ is indeed more accurate than the Burnside formula, as it can be seen from the following table:

n	$\mathcal{G}_n/n!$	$\mathcal{B}_n/n!$
2	1.0018	1.0167
5	1.0004	1.0076
7	1.0002	1.0056
10	1.0001	1.0040
12	1.0001	1.0033
19	1.0000	1.0021

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