Bulletin of the *Transilvania* University of Braşov • Vol 3(52) - 2010
Series III: Mathematics, Informatics, Physics, 19-26

# APPROXIMATING PHILLIPS OPERATORS BY MODIFIED SZÁSZ - INVERSE BETA OPERATORS

**Cristina S. CISMAŞIU**[1]

**Abstract**

It is known that, the Szász - Durrmeyer operator is the limit, in an appropiate sense [2], of both the Bernstein - Durrmeyer and Baskakov - Durrmeyer operators. In this paper we consider the modified Szász - Durrmeyer operators, which were introduced by Phillips [14] and were studied by several other authors and we want to show that these operators are the limit of the modified Szász - Inverse Beta operators and we provide a rate of convergence.

2000 *Mathematics Subject Classification:* 41A35, 41A36, 41A25, 42A61.
*Key words:* Phillips operator, Szász - Inverse Beta operator, rate of convergence, Poisson process, Gamma process.

## 1 Indroduction

The Phillips operators [14] are defined by

$$\begin{aligned} S_r(f;x) &= e^{-rx}f(0) + r\sum_{k=1}^{\infty} s_{r,k}(x)\int_0^{\infty} s_{r,k-1}(u)f(u)du \\ &= \int_0^{\infty} H_r(u;x)f(u)du,\ r\in\mathbb{N},\ x\geq 0, \end{aligned} \tag{1}$$

for $f:[0,\infty)\longrightarrow\mathbb{R}$ any integrabile function, such that $S_r(|f|;x)<\infty$ for $x\geq 0$, with

$$s_{r,k}(x) = e^{-rx}\frac{(rx)^k}{k!}, \tag{2}$$

$$H_r(u;x) = e^{-rx}\delta(u) + r\sum_{k=1}^{\infty} s_{r,k}(x)s_{r,k-1}(u),\ x\geq 0,\ k\in\mathbb{N}\cup\{0\},\ r\in\mathbb{N} \tag{3}$$

[1]Faculty of Mathematics and Informatics, *Transilvania* University of Braşov, Romania, e-mail: ccismasiu@yahoo.com

and $\delta$ the Dirac's Delta function, for which

$$\int_0^\infty \delta(u)f(u)du = f(0). \tag{4}$$

These operators preserve the linear functions and were obtained by Phillips [14] starting from the Szász - Mirakjan operators and having a similar construction as of the Goodman and Sharma operators [7] . These operators were studied by several authors ( see [5], [8], [9], [11], [12]) and are considered "the genuine Durrmeyer - Szász - Mirakjan operators". Note that, the definition (1) - (4) can be extended replacing $r \in \mathbb{N}$ by $r > 0$ and an interesting probabilistic interpretation of these operators as the mean value of the random variables $f(Y_r^x) = f\left(\frac{U_{N(rx)}}{r}\right)$, $r > 0$, $x \geq 0$ was possible [2], having the probability density $H_r(u;x)$, with $\{N(r) : r \geq 0\}$ a standard Poisson process and $\{U_r : r \geq 0\}$ a Gamma process independent of the former and defined on the same probability space:

$$S_r(f;x) = E\left[f\left(Y_r^x\right)\right] = E\left[f\left(\frac{U_{N(rx)}}{r}\right)\right],\ r > 0,\ x \geq 0. \tag{5}$$

A generalization of these operators, using two continuous parameters was obtained by Păltănea R. [13]. We consider now, other mixed operators, namely Szász - Inverse Beta operators, which were intoduced by Gupta V., Noor M. A., [10] and were studied by Finta Z., Govil N. K., Gupta V. [6] defined by

$$\begin{aligned} L_t(f;x) &= e^{-tx}f(0) + \sum_{k=1}^{\infty} s_{t,k}(x) \int_0^\infty b_{t,k}(u)f(u)du \\ &= \int_0^\infty J_t(u;x)f(u)du, \end{aligned} \tag{6}$$

for $f : [0,\infty) \longrightarrow \mathbb{R}$ any integrabile function, such that $L_t(|f|;x) < \infty$, for $t > 0$, $x \geq 0$ with

$$J_t(u;x) = e^{-tx}\delta(u) + \sum_{k=1}^{\infty} s_{t,k}(x)b_{t,k}(u) \tag{7}$$

$$b_{t,k}(u) = \frac{1}{B(k,t+1)} \cdot \frac{u^{k-1}}{(1+u)^{t+k+1}},\ u \geq 0,\ k \in \mathbb{N} \tag{8}$$

an Inverse - Beta probability density $\delta$ and $s_{t,k}(x)$ are defined above.

The Inverse - Beta probability density function can be represented with a negative binomial probability

$$p_{t,k-1}(u) = \binom{t+k}{k-1}\left(\frac{u}{1+u}\right)^{k-1}\left(\frac{1}{1+u}\right)^{t+2} = \binom{t+k}{k-1} \cdot \frac{u^{k-1}}{(1+u)^{t+k+1}}, \tag{9}$$

$t>0,\ u>0,\ k\in\mathbb{N}$, for which $\int\limits_0^\infty p_{t,k-1}(u)du=\dfrac{1}{t+1}$ and so

$$b_{t,k}(u)=\frac{1}{B(k,t+1)}\cdot\frac{u^{k-1}}{(1+u)^{t+k+1}}=(t+1)p_{t,k-1}(u)=\frac{p_{t,k-1}(u)}{\int\limits_0^\infty p_{t,k-1}(u)du}.$$

So, the probability density function (8) becomes

$$J_t(u;x)=e^{-tx}\delta(u)+\sum_{k=1}^{\infty}s_{t,k}(x)b_{t,k}(u)=e^{-tx}\delta(u)+(t+1)\sum_{k=1}^{\infty}s_{t,k}(x)p_{t,k-1}(u) \qquad (10)$$

and operators (6) have a Durrmeyer - type construction

$$\begin{aligned} L_t(f;x) &= e^{-tx}f(0)+\sum_{k=1}^{\infty}s_{t,k}(x)\frac{\int\limits_0^\infty p_{t,k-1}(u)f(u)du}{\int\limits_0^\infty p_{t,k-1}(u)du} \\ &= e^{-tx}f(0)+(t+1)\sum_{k=1}^{\infty}s_{t,k}(x)\int_0^\infty p_{t,k-1}(u)f(u)du. \qquad (11)\end{aligned}$$

Also, they can be represented [3] as the mean value of a random variable $f(Z_t^x)$

$$L_t(f;x)=E\left[f\left(Z_t^x\right)\right]=E\left[f\left(\frac{U_{N(tx)}}{V_{t+1}}\right)\right],\ t\geq 0,\ x\geq 0. \qquad (12)$$

with $Z_t^x=\dfrac{U_{N(tx)}}{V_{t+1}}$ having the probability density function $J_t(\cdot;x)$ as (10),$\{N(t):t\geq 0\}$ being a standard Poisson process, $\{U_t:t\geq 0\}$, $\{V_t:t\geq 0\}$ being two mutually independent Gamma processes, independent of the former and defined on the same probability space. All of these independent stochastic processes starting at the origin , have stationary independent increments and without loss of generality it can be assumed [15] that $\{U_t:t\geq 0\}$, $\{V_t:t\geq 0\}$ for each $t>0$ have a.s. non - decreasing right continuos paths. On the other hand, the Szász - Inverse Beta operators (6) - (8) can be represented as the composition between the Szász Mirakjan operators $M_t(f;x)=\sum\limits_{k=0}^{\infty}f\left(\dfrac{k}{t}\right)s_{t,k}(x)$ with $s_{t,k}(x)$ as (3) and Inverse - Beta operators $T_t(f;x)$ defined with Inverse - Beta probability density function $b_{tx,t+1}(u)$ as (8):

$$T_t(f;x)=\begin{cases}\dfrac{1}{B(tx,t+1)}\int\limits_0^\infty\dfrac{u^{tx-1}}{(1+u)^{tx+t+1}}f(u)du=\int\limits_0^\infty f(u)b_{tx,t+1}(u)du & ,\ t>0,\ x>0\\ f(0) & ,\ x=0.\end{cases}$$

So, $L_t(f;x)=(M_t\circ T_t)(f;x)$, $t>0$, $x\geq 0$.

## 2 Approximating Phillips operators by modified Szász-Inverse Beta operators

Using the same ideea as De la Cal J. , Luquin F. [4] or as Adell J. A., De la Cal J. [2], we consider a new operator defined as the aid of Szász - Inverse Beta operator (6) - (8):

$$\begin{aligned} B_{r,t}(f;x) &= L_{rt}\left(f(tu);\frac{x}{t}\right) = \int_0^\infty \frac{1}{t} J_{rt}\left(\frac{u}{t};\frac{x}{t}\right) f(u)du \\ &= \int_0^\infty \frac{1}{t}\left[e^{-rx}\delta\left(\frac{u}{t}\right) + \sum_{k=1}^\infty s_{rt,k}\left(\frac{x}{t}\right) b_{rt,k}\left(\frac{u}{t}\right)\right] f(u)du \qquad (13) \\ &= e^{-rx}f(0) + \sum_{k=1}^\infty s_{r,k}(x) \int_0^\infty \frac{1}{t} b_{rt,k}\left(\frac{u}{t}\right) f(u)du \end{aligned}$$

$r > 0$, $t > 0$, $x \geq 0$, where $f$ is any real function defined on $[0,\infty)$ such that $B_{r,t}(|f|);x) < \infty$.

**Remark 1.** *In view of (11) with (13) we obtain a constuction of Durrmeyer - type for this new operator.*

Indeed,

$$s_{rt,k}\left(\frac{x}{t}\right) = e^{-rx}\frac{(rx)^k}{k!} = s_{r,k}(x),$$

$$\int_0^\infty \frac{1}{t}e^{-rx}\delta\left(\frac{u}{t}\right) f(u)du = e^{-rx}f(0),$$

$$\begin{aligned} \frac{1}{t}b_{rt,k}\left(\frac{u}{t}\right) &= \frac{1}{t}\cdot\frac{1}{B(k,rt+1)}\cdot\frac{\left(\frac{u}{t}\right)^{k-1}}{\left(1+\frac{u}{t}\right)^{rt+k+1}} \\ &= \frac{rt+1}{t}\binom{rt+k}{k-1}\left(\frac{u}{u+t}\right)^{k-1}\left(\frac{t}{u+t}\right)^{rt+2} \\ &= \frac{rt+1}{t}p_{rt,k-1}\left(\frac{u}{t}\right) = \frac{p_{rt,k-1}\left(\frac{u}{t}\right)}{\int_0^\infty p_{rt,k-1}\left(\frac{u}{t}\right)du},\ k\in\mathbb{N},\ u\geq 0,\ t>0,\ r>0, \end{aligned}$$

$$\int_0^\infty p_{rt,k-1}\left(\frac{u}{t}\right)du = \binom{rt+k}{k-1}\int_0^\infty \frac{\left(\frac{u}{t}\right)^{k-1}}{\left(1+\frac{u}{t}\right)^{rt+k+1}}du = \frac{t}{rt+1} = \frac{1}{r+\frac{1}{t}}.$$

So,

$$\begin{aligned} B_{r,t}(f;x) &= L_{rt}\left(f(tu);\frac{x}{t}\right) = \int_0^\infty \frac{1}{t} J_{rt}\left(\frac{u}{t};\frac{x}{t}\right) f(u)du \\ &= \int_0^\infty \frac{1}{t}\left[e^{-rx}\delta\left(\frac{u}{t}\right) + \sum_{k=1}^\infty s_{rt,k}\left(\frac{x}{t}\right) b_{rt,k}\left(\frac{u}{t}\right)\right] f(u)du \qquad (14) \\ &= e^{-rx} f(0) + \left(r+\frac{1}{t}\right)\sum_{k=1}^\infty s_{r,k}(x)\int_0^\infty p_{rt,k-1}\left(\frac{u}{t}\right) f(u)du. \end{aligned}$$

On the other hand, we have in view of (12) that

$$B_{r,t}(f;x) = L_{rt}\left(f(tu);\frac{x}{t}\right) = E\left[f\left(tZ_{rt}^{\frac{x}{t}}\right)\right] = E\left[f\left(t\frac{U_{N(rx)}}{V_{rt+1}}\right)\right]$$

and with (5) a bound for the difference $|B_{r,t}(f;x) - S_r(f;x)|$ means a bound for the total variantion distance between the probability distributions of the random variables $tZ_{rt}^{\frac{x}{t}}$ and $Y_r^x$ respectively $t\frac{U_{N(rx)}}{V_{rt+1}}$ and $\frac{U_{N(rx)}}{r}$, with the same mean.

**Theorem 1.** *Let $x \geq 0$, $r,t,u > 0$. If, $f$ is a real bounded function on $[0,\infty)$ then*

$$|B_{r,t}(f;x) - S_r(f;x)| = |L_{rt}\left(f(tu);\frac{x}{t}\right) - S_r(f;x)| \leq \|f\| \cdot \frac{r^2x^2+4rx+2}{rt+1}$$

*and we have uniform convergence as $t \longrightarrow \infty$ on every bounded interval $[0,a]$, $a > 0$.*

*Proof.* Using (1) - (3) and (14) we have

$$\begin{aligned} L_{rt}\left(f(tu);\frac{x}{t}\right) - S_r(f;x) &= \int_0^\infty \left[\frac{1}{t}J_{rt}\left(\frac{u}{t};\frac{x}{t}\right) - H_r(u;x)\right] f(u)du \\ &= \int_0^\infty \left[\sum_{k=1}^\infty s_{r,k}(x)\left(\frac{1}{t}b_{rt,k}\left(\frac{u}{t}\right) - rs_{r,k-1}(u)\right)\right] f(u)du \end{aligned}$$

So,

$$|L_{rt}\left(f(tu);\frac{x}{t}\right) - S_r(f;x)| \leq \|f\| \cdot \sum_{k=1}^\infty s_{r,k}(x)\int_0^\infty |\frac{1}{t}b_{rt,k}\left(\frac{u}{t}\right) - rs_{r,k-1}(u)|du.$$

$$|L_{rt}\left(f(tu);\frac{x}{t}\right) - S_r(f;x)| \leq \|f\| \cdot \sum_{k=1}^\infty s_{r,k}(x)\int_0^\infty |\left(r+\frac{1}{t}\right)p_{rt,k-1}\left(\frac{u}{t}\right) - rs_{r,k-1}(u)|du. \quad (15)$$

With the convention $\binom{t}{k} = \dfrac{t(t-1)(t-2)\cdots(t-k+1)}{k!}$, $t>0$, $k\in\mathbb{N}$, let

$$\begin{aligned}
h_{r,t,k}(u) &= 1-\frac{\left(r+\frac{1}{t}\right)p_{rt,k-1}\left(\frac{u}{t}\right)}{rs_{r,k-1}(u)} = 1-\frac{r\left(1+\frac{1}{rt}\right)\binom{rt+k}{k-1}\dfrac{\left(\dfrac{u}{t}\right)^{k-1}}{\left(1+\dfrac{u}{t}\right)^{rt+k+1}}}{re^{-ru}\dfrac{(ru)^{k-1}}{(k-1)!}}\\
&= 1-e^{ru}\left(1+\frac{u}{t}\right)^{-rt-k-1}\prod_{s=1}^{k}\left(1+\frac{s}{rt}\right)
\end{aligned}$$

be a function with $k>1$, $r,t,u>0$.

Because $h'_{r,t,k}(u) = \dfrac{k+1-ru}{t}e^{ru}\left(1+\dfrac{u}{t}\right)^{-rt-k-2}\prod\limits_{s=1}^{k}\left(1+\dfrac{s}{rt}\right)$ we have for $u=\dfrac{k+1}{r}$,

$$\begin{aligned}
\sup_{u\geq 0} h_{r,t,k}(u) &= h_{r,t,k}\left(\frac{k+1}{r}\right) = 1-e^{k+1}\left(1+\frac{k+1}{rt}\right)^{-rt-k-1}\prod_{s=1}^{k}\left(1+\frac{s}{rt}\right)\\
&= 1-e^{k+1}\left(1+\frac{k+1}{rt}\right)^{-rt}\left(\frac{rt}{rt+k+1}\right)^{k+1}\prod_{s=1}^{k}\left(1+\frac{s}{rt}\right)\\
&= 1-e^{k+1}\left(1+\frac{k+1}{rt}\right)^{-rt}\prod_{s=1}^{k}\frac{rt+s}{rt+k+1}\\
&= 1-e^{k+1}\left(1+\frac{k+1}{rt}\right)^{-rt}\prod_{s=1}^{k+1}\left(1-\frac{s}{rt+k+1}\right)\\
&\leq 1-\prod_{s=1}^{k+1}\left(1-\frac{s}{rt+k+1}\right)\\
&\leq \sum_{s=1}^{k+1}\frac{s}{rt+k+1} \leq \frac{(k+1)(k+2)}{2(rt+1)}.
\end{aligned}$$

Because $\dfrac{1}{t}b_{rt,k}\left(\dfrac{u}{t}\right)$ and $rs_{r,k-1}(u)$ are probability densities on $[0,+\infty)$ and

$$\int_0^\infty rs_{r,k-1}(u)du = \int_0^\infty \frac{1}{t}b_{rt,k}\left(\frac{u}{t}\right)du = 1$$

we have $\int\limits_0^\infty h_{r,t,k}(u)du = 0$ and $\int\limits_0^\infty \left|h_{r,t,k}(u)\right|du = 2\int\limits_{h_{r,t,k}(u)\geq 0,\, u\geq 0} h_{r,t,k}(u)du.$

So,

$$\int_0^\infty \left|\frac{1}{t} b_{rt,k}\left(\frac{u}{t}\right) - rs_{r,k-1}(u)\right| du = \int_0^\infty \left|\left(r+\frac{1}{t}\right) p_{rt,k-1}\left(\frac{u}{t}\right) - rs_{r,k-1}(u)\right| du$$
$$= \int_0^\infty |h_{r,t,k}(u)| rs_{r,k-1}(u) du = 2 \int_{h_{r,t,k}(u)\geq 0,\, u\geq 0} h_{r,t,k}(u) rs_{r,k-1}(u) du$$
$$\leq 2 \sup_{u\geq 0} h_{r,t,k}(u) \int_0^\infty rs_{r,k-1}(u) du \leq \frac{k^2+3k+2}{rt+1} \tag{16}$$

because $\int_0^\infty rs_{r,k-1}(u)du = 1$.

We obtain from (15) and (16) with $\sum_{k=1}^\infty k e^{-rx}\frac{(rx)^k}{k!} = rx$ and $\sum_{k=1}^\infty k^2 e^{-rx}\frac{(rx)^k}{k!} = (rx)^2 + rx$, that

$$\left|L_{rt}\left(f(tu); \frac{x}{t}\right) - S_r(f;x)\right| \leq \|f\| \cdot \sum_{k=1}^\infty s_{r,k}(x) \cdot \left(\frac{k^2+3k+2}{rt+1}\right) \leq \|f\| \cdot \frac{r^2x^2+4rx+2}{rt+1}$$

□

## References


[1] Adell J. , De la Cal J., San Miguel M., *Inverse Beta and Generalized Bleimann Butzer - Hahn Operators*, J. Approx. Theory 76, (1994), 54 - 64.

[2] Adell J. A., De la Cal J., *Bernstein - Durrmeyer Operators*, Computers Math. Applic., Vol. 30, No. 3 - 6, (1995), 3 - 6.

[3] Cismaşiu C., *Szász - Inverse Beta Operators revisited*, (to appear).

[4] De la Cal J., Luquin F., *Approximation Szász and Gamma Operators by Baskakov Operators*, J. Math. Anal. Appl. 184, (1994), 585 - 593.

[5] Finta Z., Gupta, V., *Direct and inverse estimates for Phillips type operators*, J. Math.Anal. Appl., 303 (2005), No. 2, 627 - 642.

[6] Finta Z. Govil N. K., Gupta V., *Some results on modified Szász - Mirakjan operators*, J. Math. Anal. Appl. 327 (2007), 1284 - 1296.

[7] Goodman, T. N. T., Sharma , A., *A modified Bernstein Schoenberg operator*, Proceedings of the Conference on Constructive Theory of Functions, Varna, Bulgaria, May, 24 - 31, (1987), ( Sendov, B. et al., Eds. ), House of the Bulgarian Academy of Sciences, Sofia, (1988), 166 - 173.

[8] Govil, N. K., Gupta, V., Noor, M. A., *Simultaneous approximation for the Phillips operators*, International Journal of Mathematics and Mathematical Sciences, 2006, article ID 49094.

[9] Gupta, V., Srivastava, G. S., *On the rate of convergence of Phillips operators for functions of bounded variation*, Annals Societatis Mathematicae Polonae, Seria I. Commentationes Mathematicae, 36 (1996), 123-130.

[10] Gupta V., Noor M. A., *Convergence of derivatives for certain mixed Szász - Beta operators*, J. Math. Anal. Appl., 321 (1), (2006), 1 - 9.

[11] Mazhar S. M., Totik V., *Approximation by modified Szász operator*, Acta Sci. Math. 49 (1985), 257 - 263.

[12] May, C. P., *On Phillips operators*, J. Approx. Theory, 20, (1977) No. 4, 315 - 332.

[13] Păltănea R., *Modified Szász - Mirakjan operators of integral form*, Carpathian J. Math., 24 (2008), no. 3, 378 - 385.

[14] Phillips, R. S., *An inversion formula for Laplace transforms and semi - groups of operators*, Annals of Mathmatics, Second Series, 59 (1954), 325 - 356.

[15] Skorohod A. V., *Random Processes with Independent Increments*, Kluwer, London, 1986.