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SOME PROPERTIES OF COMPLEX BERWALD SPACES

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Abstract

The notions of complex Berwald-Finsler spaces and complex Berwald-Cartan spaces are considered. Some new properties of this spaces concerning to the holonomy group of complex Berwald connections in relation with similar properties from the real case, see [6, 7, 8], are studied in the paper.

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1 Preliminaries

The geometry of the real Berwald-Cartan spaces can be found in the monograph [13] Chs. 6-7, and in some papers of M. Anastasiei [7, 8], where new properties of this spaces concerning to the holonomy group of Berwald connections are studied. In the framework of Finsler (Lagrange) vector bundles, similar results can be found in the paper [6]. For the geometry of real Berwald-Finsler spaces, see for instance [9, 11, 12, 17]. In the complex setting, the study of complex Berwald spaces was initiated by T. Aikou in [2, 3]. Recently N. Aldea and G. Munteanu [5], make an exhaustive study of complex Berwald and Landsberg spaces and obtain a classification of two dimensional complex Berwald spaces. Here we resume our study just to emphasize the holonomy group of the complex Berwald connections and to prove some new properties of complex Berwald spaces in relation with similar properties from the real case, see [6, 7, 8].

Let us begin our study with a short review of complex Finsler and Cartan geometry and set up the basic notions and terminology. For more, see Ch. 4 and Ch. 6 from [15].

Let us consider V to be a finite dimensional vector spaces over $\mathbb{C}$. Let $\{e_1, ..., e_n\}$ be a basis of V and $(v^1, ..., v^n)$ be complex coordinates of a vector v.

Definition 1. *We say that a function $f : V \to \mathbb{R}$ is a complex Minkowski norm on V if it has the following properties:*

(i) $f(v) \geq 0$ *for any* $v \in V$ *and* $f(v) = 0$ *if and only if* $v = 0$*;*

(ii) $f(\lambda v) = |\lambda|^2 f(v)$ *for any* $\lambda \in \mathbb{C}$*;*

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(iii) f *is* C^∞ *on* $V-\{0\}$*;*

(iv) the hermitian matrix $(\partial^2 f/\partial v^i\partial\overline{v}^j)$ *is positively definite at any* $v\neq 0$.

For more details on complex Minkowski metrics on the space $V=\mathbb{C}^n$, see [16].

Let M be a complex n dimensional complex manifold and $(z^k), k=1,...,n$ the complex coordinates in a local chart U. The complexified of the real tangent bundle $T_{\mathbb{R}}M$, denoted by $T_{\mathbb{C}}M$, splits into the direct sum of holomorphic tangent bundle $T^{'}M$ and antiholomorphic tangent bundle $T^{''}M$, namely $T_{\mathbb{C}}M=T^{'}M\oplus T^{''}M$. The total space of holomorphic tangent bundle $\pi:T^{'}M\rightarrow M$ is in turn a $2n$ dimensional complex manifold with $u=(z^k,\eta^k), k=1,...,n$, the induced complex coordinates in the local chart $\pi^{-1}(U)$, where $\eta=\eta^k\frac{\partial}{\partial z^k}\in T^{'}_zM$ is a directional section.

Accordind to [1, 2, 15], a *complex Finsler metric* on M is given by a complex Minkowski norm $L_z=L(z,\cdot)$ on $T^{'}_zM$, for any $z\in M$. Consequently, from the homogeneity conditions we have

$$\frac{\partial L}{\partial\eta^k}\eta^k=\frac{\partial L}{\partial\overline{\eta}^k}\overline{\eta}^k=L\,;\ \frac{\partial g_{i\overline{j}}}{\partial\eta^k}\eta^k=\frac{\partial g_{i\overline{j}}}{\partial\overline{\eta}^k}\overline{\eta}^k=0\,;\ g_{i\overline{j}}\eta^i\overline{\eta}^j=L, \tag{1}$$

where $g_{i\overline{j}}=\partial^2L/\partial\eta^i\partial\overline{\eta}^j$.

Roughly speaking, the geometry of a complex Finsler space consist in the study of the geometric objects of complex manifold $T^{'}M$ endowed with a hermitian metric structure defined by $g_{i\overline{j}}$.

Let $V^{'}(T^{'}_0M)\subset T^{'}(T^{'}_0M)$ be the holomorphic vertical bundle, locally spanned by $\{\frac{\partial}{\partial\eta^k}\}$ and $V^{''}(T^{'}_0M)$ be its conjugate, locally spanned by $\{\frac{\partial}{\partial\overline{\eta}^k}\}$. Here, $T^{'}_0M=T^{'}M-\{0\}$. A *complex nonlinear connection*, briefly c.n.c., on $T^{'}_0M$ is a supplementary complex subbundle to $V^{'}(T^{'}_0M)$ in $T^{'}(T^{'}_0M)$, namely $T^{'}(T^{'}_0M)=H^{'}(T^{'}_0M)\oplus V^{'}(T^{'}_0M)$. The horizontal subbundle $H^{'}(T^{'}_0M)$ is locally spanned by $\{\frac{\delta}{\delta z^k}=\frac{\partial}{\partial z^k}-N^j_k\frac{\partial}{\partial\eta^j}\}$, where $N^j_k(z,\eta)$ are the coefficients of the c.n.c., which obey a certain rule of change at the local charts change such that $\frac{\delta}{\delta z^k}=\frac{\partial z^{'j}}{\partial z^k}\frac{\delta}{\delta z^{'j}}$ performs. Obviously, we also have that $\frac{\partial}{\partial\eta^k}=\frac{\partial z^{'j}}{\partial z^k}\frac{\partial}{\partial\eta^{'j}}$. The pair $\{\delta_k:=\frac{\delta}{\delta z^k}\,;\,\dot{\partial}_k:=\frac{\partial}{\partial\eta^k}\}$, $k=1,...,n$ will be called the adapted frames of the c.n.c. By conjugation, an adapted frame $\{\delta_{\overline{k}}\,;\,\dot{\partial}_{\overline{k}}\}$ is obtained on $T^{''}(T^{'}_0M)$. The dual adapted bases are $\{dz^k,\ \delta\eta^k=d\eta^k+N^k_jdz^j,\ d\overline{z}^k,\ \delta\overline{\eta}^k=d\overline{\eta}^k+N^{\overline{k}}_{\overline{j}}d\overline{z}^j\}$. A c.n.c. related only to the fundamental function of the complex Finsler space (M,L) is almost classical now, the Chern-Finsler c.n.c., locally given by $\overset{CF}{N^j_k}=g^{\overline{m}j}\frac{\partial g_{l\overline{m}}}{\partial z^k}\eta^l$, where $(g^{\overline{m}j})$ denotes the inverse of $(g_{j\overline{m}})$.

As we already know, the complex Finsler structure L on M determines a hermitian metric on the holomorphic vertical bundle $V^{'}(T^{'}_0M)$ by

$$\mathcal{G}^v(X,\overline{Y})(z,\eta)=g_{i\overline{j}}(z,\eta)\ \dot{X^i}\ (z,\eta)\overline{\dot{Y^j}\ (z,\eta)} \tag{2}$$

for all $X=\dot{X^i}\ (z,\eta)\ \dot{\partial}_i$, $Y=\dot{Y^j}\ (z,\eta)\ \dot{\partial}_j\in\Gamma(V^{'}(T^{'}_0M))$.

The unique hermitian connection ∇ of the hermitian bundle $(V'(T'_0M), \mathcal{G}^v)$ is the so-called Chern-Finsler linear connection of (M, L), see [1]. With respect to adapted frames and coframes of $\overset{CF}{N^j_k}$, the $(1,0)$ connection forms of ∇ are given by

$$\omega^i_j = L^i_{jk} dz^k + C^i_{jk} \delta\eta^k \tag{3}$$

with $L^i_{jk} = g^{\overline{m}i}\delta_k(g_{j\overline{m}})$ and $C^i_{jk} = g^{\overline{m}i}\,\dot{\partial}_k\,(g_{j\overline{m}})$. We notice that, we have $C^i_{jk} = C^i_{kj}$ and $L^i_{jk} = \dot{\partial}_k\,(\overset{CF}{N^i_j})$.

Let us consider now, the holomorphic cotangent bundle $\pi^* : T'^*M \to M$. As above, T'^*M has a natural structure of $2n$ dimensional complex manifold and a point is denoted by $u^* = (z^k, \zeta_k)$, $k = 1, ..., n$, where (ζ_k) should be regarded as a momentum direction $\zeta \in T'^*_zM$ with respect to the canonical base $\{dz^k|_z\}$.

According to [14], a *complex Cartan metric* on M is given by a complex Minkowski norm $H_z = H(z, \cdot)$ on T'^*_zM, for any $z \in M$. Similarly, we have

$$\frac{\partial H}{\partial \zeta_k}\zeta_k = \frac{\partial H}{\partial \overline{\zeta}_k}\overline{\zeta}_k = H\,;\; \frac{\partial h^{\overline{j}i}}{\partial \zeta_k}\zeta_k = \frac{\partial h^{\overline{j}i}}{\partial \overline{\zeta}_k}\overline{\zeta}_k = 0\,;\; h^{\overline{j}i}\zeta_i\overline{\zeta}_j = H, \tag{4}$$

where $h^{\overline{j}i} = \partial^2 H/\partial\zeta^i\partial\overline{\zeta}_j$.

Now, for T'^*M complex manifold we consider $T'(T'^*_0M)$ the holomorphic tangent bundle of T'^*_0M. Let $V'(T'^*_0M) \subset T'(T'^*_0M)$ be the holomorphic vertical bundle, locally spanned by $\{\frac{\partial}{\partial\zeta_k}\}$ and $V''(T'^*_0M)$ be its conjugate, locally spanned by $\{\frac{\partial}{\partial\overline{\zeta}_k}\}$. A *complex nonlinear connection*, briefly c.n.c., on T'^*_0M is a supplementary complex subbundle to $V'(T'^*_0M)$ in $T'(T'^*_0M)$, namely $T'(T'^*_0M) = H'(T'^*_0M) \oplus V'(T'^*_0M)$. The horizontal subbundle $H'(T'^*_0M)$ is locally spanned by $\{\frac{\delta^*}{\delta z^k} = \frac{\partial}{\partial z^k} + N_{jk}\frac{\partial}{\partial\zeta_j}\}$, where $N_{jk}(z, \eta)$ are the coefficients of this c.n.c. The pair $\{\delta^*_k := \frac{\delta^*}{\delta z^k}\,;\, \dot{\partial}^k := \frac{\partial}{\partial\zeta_k}\}$, $k = 1, ..., n$ will be called the adapted frames of the c.n.c. By conjugation, an adapted frame $\{\delta^*_{\overline{k}}\,;\, \dot{\partial}^{\overline{k}}\}$ is obtained on $T''(T'^*_0M)$. The dual adapted bases are $\{d^*z^k,\ \delta\zeta_k = d\zeta_k - N_{kj}d^*z^j,\ d^*\overline{z}^k,\ \delta\overline{\zeta}^k = d\overline{\zeta}_k - N_{\overline{k}\overline{j}}d^*\overline{z}^j\}$. A c.n.c. related only to the fundamental function of the complex Cartan space (M, H) is the Chern-Cartan c.n.c., locally given by $\overset{CC}{N_{jk}} = -h_{j\overline{m}}\frac{\partial h^{\overline{m}l}}{\partial z^k}\zeta_l$, where $(h_{j\overline{m}})$ denotes the inverse of $(h^{\overline{m}j})$. Similarly, the Chern-Cartan linear connection denoted by ∇^* is defined by the following set of local coefficients $(H^i_{jk}\,, C^{ik}_j)$, where $H^i_{jk} = -h_{j\overline{m}}\delta_k(h^{\overline{m}i})$ and $C^{ik}_j = -h_{j\overline{m}}\,\dot{\partial}^k\,(h^{\overline{m}i})$. We also notice that $C^{ik}_j = C^{ki}_j$ and $H^i_{jk} = \dot{\partial}^i\,(\overset{CC}{N_{jk}})$.

2 Complex Berwald-Finsler spaces

Following a definition given by T. Aikou [2, 3], recently N. Aldea and G. Munteanu [5], make an exhaustive study of complex Berwald and Landsberg spaces and obtain a classification of two dimensional complex Berwald spaces. Here we resume our study just to emphasize the holonomy group of the complex Berwald connections.

Definition 2. *A complex Finsler space (M, L) is said to be a complex Berwald-Finsler space if the horizontal Chern-Finsler connection coefficients in natural coordinates have no η dependence, namely $L^i_{jk}(z, \eta) = L^i_{jk}(z)$, and the space is Kähler, i.e. $L^i_{jk}\eta^j = L^i_{kj}\eta^j$. We will denote such a connection by $\overset{B}{L^i_{jk}}(z)$.*

Let $X = X^i\partial_i$ be a holomorphic vector field on M, where $\partial_i := \frac{\partial}{\partial z^i}$. Using the coefficients $\overset{B}{L^i_{jk}}(z)$ we may define a covariant derivative of X by

$$\overset{B}{\nabla} X = (\partial_k X^i + X^j \overset{B}{L^i_{jk}})\partial_i \otimes dz^k. \tag{5}$$

We restrict X to a complex curve on M, $\gamma : t \mapsto z(t)$, $t \in \mathbb{R}$, and define the covariant derivative of X along γ by

$$\frac{\overset{B}{\nabla} X}{dt} = (\frac{dX^i}{dt} + \overset{B}{L^i_{jk}} X^j \frac{dz^k}{dt})\partial_i,$$

and we say that X *is parallel along* γ if $\frac{\overset{B}{\nabla} X}{dt} = 0$. Then, according to [15], p. 101, γ is the projection on M of a geodesic curve $\tilde{\gamma}$ of the complex Finsler space (M, F), with respect to the Chern-Finsler connection.

According to [2] p. 19, if X is parallel along γ, then the function $A : t \mapsto L(z(t), X(t))$, $t \in \mathbb{R}$, is constant. Indeed,

$$\frac{dA}{dt} = (\partial_k L)\frac{dz^k}{dt} + (\dot{\partial}_k L)\frac{dX^k}{dt} + (\partial_{\overline{k}} L)\frac{d\overline{z}^k}{dt} + (\dot{\partial}_{\overline{k}} L)\frac{d\overline{X}^k}{dt}. \tag{6}$$

Taking into account $\frac{\overset{B}{\nabla} X}{dt} = 0$ we have $\frac{dX^k}{dt} = -\overset{B}{L^k_{ij}} X^i \frac{dz^j}{dt}$ and its conjugate. Now, replacing $\frac{dX^k}{dt}$ and $\frac{d\overline{X}^k}{dt}$ into (6) we obtain

$$\frac{dA}{dt} = (\delta_k L)\frac{dz^k}{dt} + (\delta_{\overline{k}} L)\frac{d\overline{z}^k}{dt} = 0 \tag{7}$$

where we used the fact that along the curve γ we have $\overset{CF}{N^j_k} = \overset{B}{L^j_{ik}} X^i$ and the known result from complex Finsler geometry, see for instance [15] p. 61, that $\delta_k L = \delta_{\overline{k}} L = 0$.

Thus, we get

Proposition 1. *If the holomorphic vector field X is parallel along the complex curve $\gamma : t \mapsto z(t)$, then the function $A(t) := L(z(t), X(t))$ is constant along the curve γ.*

For the complex Berwald-Finsler spaces, we have the following theorem:

Theorem 1. *Let (M, L) be a complex Berwald-Finsler space. Whenever M is connected the complex Minkowski spaces $(T'_z M, L_z)$ are all linearly isometric to each other.*

Proof. Let $\gamma : [0,1] \to M$ be a complex curve on M and z, w two points of M joined by the curve γ such that $\gamma(0) = z$ and $\gamma(1) = w$. Let be $Z \in T_z^{'}M$. We consider the unique solution $X = (X^i)$ of the system of linear equations $\frac{dX^i}{dt} + \overset{B}{L^i_{jk}} X^j \frac{dz^k}{dt} = 0$ with the initial condition $X(0) = Z$ and we associate to Z the element $Z^{'} = X(1)$ of $T_w^{'}M$. The mapping $\varphi : T_z^{'}M \to T_w^{'}M$ given by $\varphi(Z) = Z^{'}$ is a linear isomorphism of complex vector spaces. By Proposition 1, $L(z(t), X(t))$ has the same values at $t = 0$. Hence $L_z(Z) = L_w(Z^{'})$. This means that the complex Minkowski spaces $(T_z^{'}M, L_z)$ and $(T_w^{'}M, L_w)$ are linearly isometric for every $z, w \in M$. □

The application $P_\gamma := \varphi$ constructed in the above theorem is called *parallel translation* along γ. Now, if we consider all loops on M in $z \in M$, the corresponding parallel translations as linear isomorphisms $T_z^{'}M \to T_z^{'}M$ provide a group with respect to their composition, called *the holonomy group* $\phi(z)$ of $\overset{B}{\nabla}$ in $z \in M$. When M is connected, by the above theorem, all these groups are isometric and one speaks about the holonomy group ϕ of $\overset{B}{\nabla}$.

Let us consider $S_z^{'}(M) = \{\eta \in T_z^{'}M \,/\, L_z(\eta) = g_{i\bar{j}}\eta^i\overline{\eta}^j = 1\} \subset T_z^{'}M$ the complex indicatrix of L. If we consider $G(S_z^{'}(M))$ the group of all linear isomorphisms of $T_z^{'}M$ which leave invariant the indicatrix $S_z^{'}(M)$, then by Theorem 1, it follows:

Proposition 2. *The holonomy group* $\phi(z)$ *is a subgroup of* $G(S_z^{'}(M))$.

Let us continue to consider a parallel translation along γ, $\varphi : T_z^{'}M \to T_w^{'}M$. Its differential $\varphi_{*,u}$, $u \in T^{'}M$ is a linear isomorphism $V_u^{'}(T^{'}M) \to V_{\widetilde{u}}^{'}(T^{'}M)$ for $\widetilde{u} = \varphi(u)$ and we denote it by φ^v.

In particular, the differentials of the elements of $\phi(z)$ are linear isomorphisms of $V_u^{'}(T^{'}M)$ with $\pi(u) = z$ and these provide a group $\phi^v(u)$ that is a subgroup of $GL(V_u^{'}(T^{'}M))$. We call $\phi^v(u)$ the vertical lift of $\phi(z)$. On the other hand, by (2), for every $u \in T^{'}M$, we have that $(V_u^{'}(T^{'}M), \mathcal{G}_u^v)$ is a hermitian space.

Theorem 2. *The mappings* $\varphi^v : V_u^{'}(T^{'}M) \to V_{\widetilde{u}}^{'}(T^{'}M)$, $\widetilde{u} = \varphi^v(u)$, *are linear isometries of hermitian spaces. In particular, the group* $\phi^v(u)$ *is a subgroup of the isometries of* $(V_u^{'}(T^{'}M), \mathcal{G}_u^v)$.

Proof. Let us denote by (P_j^i) the matrix of $\varphi : T_z^{'}M \to T_w^{'}M$ in the basis $\{\partial_k|_z\}$ and $\{\partial_k|_w\}$. The matrix of φ^v is the same (P_j^i) in the basis $\{\dot{\partial}_k\,|_u\}$ and $\{\dot{\partial}_k\,|_{\widetilde{u}}\}$. Then by a similar argument from the real case, see [6], we obtain $g_{j\bar{k}}(u) = g_{l\overline{m}}(\widetilde{u})P_j^l P_{\bar{k}}^{\overline{m}}$. This exactly means that φ^v is an isometry of hermitian spaces $(V_u^{'}(T^{'}M), \mathcal{G}_u^v)$ and $(V_{\widetilde{u}}^{'}(T^{'}M), \mathcal{G}_{\widetilde{u}}^v)$. □

3 Complex Berwald-Cartan spaces

In this section we give a dual version of the results from the previous section. The notions are introduced by analogy with similar results from the real case, see [7, 8].

Definition 3. *A complex Cartan space (M,H) is said to be a complex Berwald-Cartan space if the horizontal Chern-Cartan connection coefficients in natural coordinates have no ζ dependence, namely $H^i_{jk}(z,\zeta)=H^i_{jk}(z)$, and the space is Kähler, i.e. $H^i_{jk}\zeta_i=H^i_{kj}\zeta_i$. We will denote such a connection by $\overset{B}{H^i_{jk}}(z)$.*

Let $\omega=\omega_i dz^i$ be a holomorphic 1-form and $X=X^j(z)\partial_j$ a holomorphic vector field on M. Using the coefficients $\overset{B}{H^i_{jk}}(z)$ we may define a covariant derivative of ω in the direction of X by

$$\overset{B}{\nabla^*}_X\,\omega=X^k(\partial_k\omega_j-\overset{B}{H^i_{jk}}\,\omega_i)dz^j. \tag{8}$$

We restrict ω to a complex curve $\gamma:t\mapsto z(t)$, $t\in\mathbb{R}$, on M, define the covariant derivative of ω along γ by

$$\frac{\overset{B}{\nabla^*}\,\omega}{dt}=(\frac{d\omega_j}{dt}-\overset{B}{H^i_{jk}}\,\omega_i\frac{d^*z^k}{dt})dz^j,$$

and we say that ω is parallel along γ if $\frac{\overset{B}{\nabla^*}\omega}{dt}=0$.

We show that if ω is parallel along γ then the function $B:t\mapsto H(z(t),\omega(t))$, $t\in\mathbb{R}$, is constant. Indeed,

$$\frac{dB}{dt}=(\partial_k H)\frac{d^*z^k}{dt}+(\dot{\partial}^k\,H)\frac{d\omega_k}{dt}+(\partial_{\overline{k}}H)\frac{d^*\overline{z}^k}{dt}+(\dot{\partial}^{\overline{k}}\,H)\frac{d\overline{\omega}_k}{dt}. \tag{9}$$

Taking into account $\frac{\overset{B}{\nabla^*}\omega}{dt}=0$ we have $\frac{d\omega_k}{dt}=\overset{B}{H^i_{kj}}\,\omega_i\frac{d^*z^j}{dt}$ and its conjugate. Now, replacing $\frac{d\omega_k}{dt}$ and $\frac{d\overline{\omega}_k}{dt}$ into (9) we obtain

$$\frac{dB}{dt}=(\delta^*_k H)\frac{d^*z^k}{dt}+(\delta^*_{\overline{k}}H)\frac{d^*\overline{z}^k}{dt}=0 \tag{10}$$

because along the curve γ we have $\overset{CC}{N_{jk}}=\overset{B}{H^i_{jk}}\,\omega_i$ and $\delta^*_k H=\delta^*_{\overline{k}}H=0$.

Thus, we get

Proposition 3. *If the holomorphic 1-form ω is parallel along the complex curve $\gamma:t\mapsto z(t)$, then the function $B(t):=H(z(t),\omega(t))$ is constant along the curve γ.*

For the complex Berwald-Cartan spaces, we have the following theorem:

Theorem 3. *Let (M,H) be a complex Berwald-Cartan space. Whenever M is connected the complex Minkowski spaces (T'^*_zM,H_z) are all linearly isometric to each other.*

Proof. Let $\gamma:[0,1]\to M$ be a complex curve on M and z,w two points of M joined by the curve γ such that $\gamma(0)=z$ and $\gamma(1)=w$. Let be $\alpha\in T'^*_zM$. We consider the unique solution $\omega=(\omega_i)$ of the system of linear equations $\frac{d\omega_j}{dt}-\overset{B}{H^i_{jk}}\,\omega_i\frac{d^*z^k}{dt}=0$ with the initial condition $\omega(0) = \alpha$ and we associate to α the element $\alpha^{'} = \omega(1)$ of $T_w^{'*}M$. The mapping $\varphi^* : T_z^{'*}M \to T_w^{'*}M$ given by $\varphi^*(\alpha) = \alpha^{'}$ is a linear isomorphism of complex vector spaces. By Proposition 3, $H(z(t), \omega(t))$ has the same values at $t = 0$. Hence $H_z(\alpha) = H_w(\alpha^{'})$. This means that the complex Minkowski spaces $(T_z^{'*}M, H_z)$ and $(T_w^{'*}M, H_w)$ are linearly isometric for every $z, w \in M$. □

The application $P_\gamma^* := \varphi^*$ constructed in the above theorem is called *parallel translation* along γ. Now, if we consider all loops on M in $z \in M$, the corresponding parallel translations as linear isomorphisms $T_z^{'*}M \to T_z^{'*}M$ provide a group with respect to their composition, called *the holonomy group* $\phi^*(z)$ of $\overset{B}{\nabla}{}^*$ in $z \in M$. When M is connected, by the above theorem, all these groups are isometric and one speaks about the holonomy group ϕ^* of $\overset{B}{\nabla}{}^*$.

As in the previous section let us consider $S_z^{'*}(M) = \{\zeta \in T_z^{'*}M \,/\, H_z(\zeta) = h^{\bar{j}i}\zeta_i\bar{\zeta}_j = 1\} \subset T_z^{'*}M$, the complex indicatrix of H. If we consider $G(S_z^{'*}(M))$ the group of all linear isomorphisms of $T_z^{'*}M$ which leave invariant the indicatrix $S_z^{'*}(M)$, then by Theorem 3, it follows:

Proposition 4. *The holonomy group* $\phi^*(z)$ *is a subgroup of* $G(S_z^{'*}(M))$.

Finally, let us consider $\mathcal{G}^{*v}$ the hermitian metric defined by $h^{\bar{j}i}(z,\zeta)$, on the vertical bundle $V^{'}(T^{'*}M)$, see [14]. If we consider φ^{*v} and $\phi^{*v}(u^*)$ the differential of $\varphi*$ and the vertical lift of $\phi^*(z)$, respectively, one gets

Theorem 4. *The mappings* $\varphi^{*v} : V_{u^*}^{'}(T^{'*}M) \to V_{\widetilde{u}^*}^{'}(T^{'*}M)$, $\widetilde{u}^* = \varphi^{*v}(u^*)$, *are linear isometries of hermitian spaces. In particular, the group* $\phi^{*v}(u^*)$ *is a subgroup of the isometries of* $(V_{u^*}^{'}(T^{'*}M), \mathcal{G}_{u^*}^{*v})$.

Remark 1. *The properties studied in this paper can be also extended in the framework of holomorphic vector bundles endowed with a complex Finsler structure of Berwald type, [4], according to similar results from the real case, see [6].*

References

[1] Abate M., Patrizio G., *Finsler metrics-A Global Approach with Applications to Geometric Function Theory*, Lectures Notes in Math., **1591**, Springer-Verlag, 1994.

[2] Aikou T., *On Complex Finsler Manifolds,* Rep. Kagoshima Univ., **24** (1991), 9-25.

[3] Aikou T., *Some remarks on locally conformal complex Berwald spaces*, Contemp. Math. **196** (1996), 109-120.

[4] Aikou T., *Finsler Geometry on Complex Vector Bundles* , Riemann-Finsler Geometry, MSRI Publ., **50** (2004), 83-105.

[5] Aldea N., Munteanu G., *On complex Landsberg and Berwald spaces*, communicated to "Al. Myller Seminar, Centennial Conference, June 21-26 2010, Iaşi, România, submitted.

[6] Anastasiei M., *Geometry of Berwald vector bundles*, Algebra, Groups and Geometries **21**(3) (2004), 251-262.

[7] Anastasiei M., *Geometry of Berwald-Cartan spaces*, BSG Proceedings **11**, (2004), 1-9.

[8] Anastasiei M., *New properties of Berwald-Cartan spaces*, Libertas Math. **24** (2004), 3-10.

[9] Ichijyõ, Y., *Finsler manifolds modeled on a Minkowski spaces*, J. Math. Kyoto Univ., **16**-3 (1976), 639-652.

[10] Kobayashi S., *Negative Vector Bundles and Complex Finsler Structures,* Nagoya Math. J., **57** (1975), 153-166.

[11] Matsumoto, M., *Foundations of Finsler Geometry and Special Finsler Spaces*, Kaiseisha Press, Otsu, Japan, 1986.

[12] Miron R., Anastasiei M., *The Geometry of Lagrange Spaces. Theory and Applications*, Kluwer Acad. Publ. **59**, 1994.

[13] Miron R., Hrimiuc D., Shimada H., Sabău S., *The geometry of Hamilton and Lagrange spaces*, Kluwer Acad. Publ., **118**, 2001.

[14] Munteanu G., *On complex Cartan spaces*, Balkan J. of Geom. Appl., **8(1)** (2003), 71-78.

[15] Munteanu G., *Complex Spaces in Finsler, Lagrange and Hamilton Geometries,* Kluwer Acad. Publ., **141** FTPH, 2004.

[16] Patrizio G., Wong P.-M., *Stability of the Monge-Ampere foliation*, Math. Ann. **263** (1983), 13-29.

[17] Szabó Z.I., *Positive Definite Berwald Spaces*, Tensor N. S., **35** (1981), 25-39.