Bulletin of the *Transilvania* University of Braşov • Vol 3(52) - 2010
Series III: Mathematics, Informatics, Physics, 53-66

GEODESICS ON THE INDICATRIX OF A COMPLEX FINSLER MANIFOLD

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Abstract

In this note the geometry of the indicatrix $(\mathbf{I}, \tilde{L})$ is studied as a hypersurface of a complex Finsler space (M, L). The induced Chern-Finsler and Berwald connections are defined and studied. The induced Berwald connection coincides with the intrinsic Berwald connection of the indicatrix bundle.

We considered a special projection of a geodesic curve on a complex Finsler space (M, L), called the induced complex geodesic, and a complex geodesic curve on the indicatrix $(\mathbf{I}, \tilde{L})$ obtained by using the variational problem for their horizontal lift to $T_C(T'M)$. Then we determined the circumstances in which the induced geodesic coincides with the complex geodesic on the indicatrix. If (M, L) is a weakly Kähler Finsler space, then one condition for these curves to coincide is that the weakly Kähler character conveys to the indicatrix via the induced Chern-Finsler connection.

2000 *Mathematics Subject Classification:* 53B40.
Key words: complex Finsler space, indicatrix, geodesics.

1 Preliminaries and settings

Let us begin our study with a short survey of complex Finsler geometry and with a set up of the basic notions and terminology. For more, see [A-P, Mu].

Let M be a complex manifold, (z^k) complex coordinates in a local chart.

The complexified of the real tangent bundle $T_C M$ splits into the sum of holomorphic tangent bundle $T'M$ and its conjugate $T''M$. The bundle $T'M$ is in its turn a complex manifold, the local coordinates in a chart being denoted by (z^k, η^k).

A *complex Finsler space* is a pair $(M, \mathbf{F})$, where $\mathbf{F} : T'M \to \mathbb{R}^+$ is a continuous function satisfying the conditions:

i) $L := \mathbf{F}^2$ is smooth on $\widetilde{T'M} := T'M \backslash \{0\}$;

ii) $\mathbf{F}(z, \eta) \geq 0$, the equality holds if and only if $\eta = 0$;

iii) $\mathbf{F}(z, \lambda\eta) = |\lambda|\mathbf{F}(z, \eta)$ for $\forall\lambda \in \mathbb{C}$, the homogeneity condition;

iv) the Hermitian matrix $\left(g_{i\bar{j}}(z, \eta)\right)$, with $g_{i\bar{j}} = \frac{\partial^2 L}{\partial\eta^i\partial\bar{\eta}^j}$ the fundamental metric tensor, is positively definite. Equivalently, it means that the *indicatrix* $\mathbf{I}_z = \{\eta \ / \ g_{i\bar{j}}(z, \eta)\eta^i\bar{\eta}^j = 1\}$ is strongly pseudoconvex for any $z \in M$.

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Consequently, from *iii*) we have:

$$\frac{\partial L}{\partial \eta^k}\eta^k = \frac{\partial L}{\partial \bar{\eta}^k}\bar{\eta}^k = L \; ; \; \frac{\partial g_{i\bar{j}}}{\partial \eta^k}\eta^k = \frac{\partial g_{i\bar{j}}}{\partial \bar{\eta}^k}\bar{\eta}^k = 0 \tag{1.1}$$

and $L = g_{i\bar{j}}\eta^i\bar{\eta}^j$.

Roughly speaking, in complex Finsler geometry we want to study the geometric objects on the complex manifold $T'M$ endowed with a Hermitian metric structure defined by $g_{i\bar{j}}$.

In this sense, the first step is the study of sections of the complexified tangent bundle of $T'M$ which is decomposed into the sum $T_C(T'M) = T'(T'M) \oplus T''(T'M)$. Let $V(T'M) \subset T'(T'M)$ be the vertical bundle, locally spanned by $\{\frac{\partial}{\partial \eta^k}\}$ and let $V(T''M)$ be its conjugate.

At this point the idea of complex nonlinear connection, briefly (c.n.c.), is instrumental in 'linearizing' this geometry. A (c.n.c.) is a supplementary complex subbundle to $V(T'M)$ in $T'(T'M)$, i.e. $T'(T'M) = H(T'M) \oplus V(T'M)$. The horizontal distribution $H_u(T'M)$ is locally spanned by $\{\frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N_k^j \frac{\partial}{\partial \eta^j}\}$, where $N_k^j(z,\eta)$ are the coefficients of the (c.n.c.), which obey a certain rule of change at the charts changes such that $\frac{\delta}{\delta z^k} = \frac{\partial z'^j}{\partial z^k}\frac{\delta}{\delta z'^j}$ holds true. Obviously, we also have that $\frac{\partial}{\partial \eta^k} = \frac{\partial z'^j}{\partial z^k}\frac{\partial}{\partial \eta'^j}$. The pair $\{\dot{\partial}_k := \frac{\partial}{\partial \eta^k}, \delta_k := \frac{\delta}{\delta z^k}\}$ will be called the adapted frame of the (c.n.c.). By conjugation everywhere an adapted frame $\{\dot{\partial}_{\bar{k}}, \delta_{\bar{k}}\}$ is obtained on $T''_u(T'M)$. The dual adapted bases are $\{dz^k, \delta\eta^k\}$ and $\{d\bar{z}^k, \delta\bar{\eta}^k\}$.

Let us consider the Sasaki type lift of the metric tensor $g_{i\bar{j}}$,

$$G = g_{i\bar{j}}dz^i \otimes d\bar{z}^j + g_{i\bar{j}}\delta\eta^i \otimes \delta\bar{\eta}^j. \tag{1.2}$$

Certainly, one main problem in this geometry is to determine a (c.n.c.) related only to the fundamental function of the complex Finsler space (M, L). One (c.n.c.) which has been extensively used is the Chern-Finsler (c.n.c.) ([A-P, Mu]):

$$\overset{CF}{N_j^k} = g^{\bar{m}k}\frac{\partial g_{l\bar{m}}}{\partial z^j}\eta^l. \tag{1.3}$$

The next step is to specify the action of a derivative law D on the sections of $T_C(T'M)$.

A Hermitian connection D, of $(1,0)-$type, which satisfies in addition $D_{JX}Y = JD_XY$, for all horizontal vectors X and J the natural complex structure of the manifold, is the so called Chern-Finsler linear connection, in brief C-F, which is locally given by the following set of coefficients (cf. [Mu]):

$$\overset{CF}{L^i_{jk}} = g^{\bar{l}i}\delta_k(g_{j\bar{l}}) \;\; ; \; \overset{CF}{C^i_{jk}} = g^{\bar{l}i}\dot{\partial}_k(g_{j\bar{l}}) \;\; ; \; \overset{CF}{L^{\bar{i}}_{\bar{j}k}} = 0 \;\; ; \; \overset{CF}{C^{\bar{i}}_{\bar{j}k}} = 0, \tag{1.4}$$

where $D_{\delta_k}\delta_j = L^i_{jk}\delta_i$ $, D_{\delta_k}\dot{\partial}_j = L^i_{jk}\dot{\partial}_j$, $D_{\dot{\partial}_k}\dot{\partial}_j = C^i_{jk}\dot{\partial}_i, D_{\dot{\partial}_k}\delta_j = C^i_{jk}\delta_i$ etc. Of course, $\overline{D_XY} = D_{\bar{X}}\bar{Y}$ is performed.

On $T_C(T'M)$ the following $1-$form is well defined:

$$\omega = \omega' + \omega'' := \eta_k dz^k + \bar{\eta}_k d\bar{z}^k, \tag{1.5}$$

where $\eta_k := g_{k\bar{j}}\bar{\eta}^j = \frac{\partial L}{\partial \eta^k}$. Also, from (1.1) it follows that $C^i_{jk}\eta^j = C^i_{jk}\eta^k = 0$.

In [A-P], a complex Finsler space is said to be weakly Kähler iff $g_{i\bar{l}}(L^i_{jk} - L^i_{kj})\eta^k\bar{\eta}^l = 0$, Kähler iff $L^i_{jk}\eta^j = L^i_{kj}\eta^j$ and strongly Kähler iff $L^i_{jk} = L^i_{kj}$. Actualy, the Kähler condition coincides with that of strongly Kähler, [C-S]

Further on, an index will have a superscript bar to denote the conjugate object, i.e. $\bar{\eta}^j := \eta^{\bar{j}}$.

2 The intrinsic geometry of complex indicatrix

The study of the indicatrix of a real Finsler space is one of interest, first, because it is a compact and strictly convex set surrounding the origin ([B-C-S], p. 84). It is the motive that, for instance, the indicatrix plays a special role in the definition of the volume on a Finsler space. The geometry of the indicatrix as a hypersurface of the total space is studied in [AZ], p. 147, and it is proved that it plays a special role in obtaining necessary and sufficient conditions for an isotropic Finsler manifold to be of constant sectional curvature. A smooth compact and connected manifold with the properties of an indicatrix was called by R. Bryant ([Br]) with generalized Finsler structure. In this last study the Kähler structure of a generalized Finsler space with constant positive flag curvature is taken into account. The unit tangent bundle of a Riemannian or Hermitian manifold is a more general position of an indicatrix bundle and here there are some interesting results, see [Bl, Na, Bo, BY], etc.

Let us consider $\dim_C M = n + 1$ and $\pi : T'M \to M$ be its holomorphic bundle, $(z^k, \eta^k)_{k=\overline{1,n+1}}$ complex coordinates on the manifold $T'M$, $dim_C T'M = 2n + 2$.

We consider $\mathbf{I}_z = \{\eta \;/\; g_{i\bar{j}}(z, \eta)\eta^i\bar{\eta}^j = 1\}$ the indicatrix at z of a complex Finsler space (M, L) and $\pi_I : \mathbf{I} \to M$ the indicatrix bundle (or, cf. [Bl] p.142, the holomorphic spheric bundle), $\mathbf{I} = \cup_{z\in M}\mathbf{I}_z$.

$\mathbf{I} \subset T'M$ is a compact and strictly connected hypersurface of $T'M$. Certainly, I is not a complex manifold since it is odd dimensional. In [Mu2] we marked out the existence of an almost contact structure on I intrinsical related to complex Finsler structure (M, L).

Further on we will study the geometry of hypersurface $\mathbf{I}$ of the complex manifold $T'M$. If we consider $(\tilde{z}^k, \theta^\alpha)_{\alpha=\overline{1,n}}$ a parametric representation of the indicatrix hypersurface, then we have the following local representation:

$$\tilde{z}^k = z^k \;\text{ and }\; \eta^k = B^k_\alpha(z)\theta^\alpha \;,\; \text{with }\; rank(B^k_\alpha) = n. \tag{2.1}$$

The tangent vectors are:

$$\frac{\partial}{\partial\tilde{z}^k} = \frac{\partial}{\partial z^k} + B^j_{\alpha k}\theta^\alpha\frac{\partial}{\partial\eta^j} \;\;;\;\; \frac{\partial}{\partial\theta^\alpha} = B^k_\alpha\frac{\partial}{\partial\eta^k} \;, \tag{2.2}$$

where $B^j_{\alpha k} = \frac{\partial B^j_\alpha}{\partial z^k}$. The dual frames are connected by

$$dz^k = d\tilde{z}^k \;\text{ and } d\eta^k = B^k_{\alpha j}\theta^\alpha d\tilde{z}^j + B^k_\alpha d\theta^\alpha. \tag{2.3}$$

By $\frac{\partial}{\partial\bar{\tilde{z}}^k}, \frac{\partial}{\partial\bar{\theta}^\alpha}$ we denote the tangent vectors obtained by conjugation everywere in (2.2).

The distribution $V\mathbf{I}$ spanned by $\{\dot{\partial}_\alpha := \frac{\partial}{\partial\theta^\alpha}\}$, is called vertical, and it is a subdistribution of $V(T'M)$.

On indicatrix $\mathbf{I}$ we have $L(\tilde{z}^k, \eta^k(\theta)) = 1$ and by differentiation with respect to $\dot{\partial}_\alpha$ it results that:
$\frac{\partial g_{i\bar{j}}}{\partial\eta^k} B^k_\alpha \eta^i \bar{\eta}^j + g_{i\bar{j}} B^i_\alpha \bar{\eta}^j = 0.$

On account of (1.1) homogeneity conditions $\frac{\partial g_{i\bar{j}}}{\partial\eta^k}\eta^i = 0$ it follows $g_{i\bar{j}} B^i_\alpha \bar{\eta}^j = 0$, which is to say that the Liouville vector $\mathbf{N} = \eta^k \frac{\partial}{\partial\eta^k}$ is normal to the vertical distribution $V\mathbf{I}$ spanned by the tangential vectors $\dot{\partial}_\alpha$ to the hypersurface $\mathbf{I}$. Moreover, $\mathbf{N}$ is a unitary vector since $\eta_k\eta^k = 1$.

Let us consider the frame $\mathcal{R} = \{\dot{\partial}_\alpha = B^k_\alpha \frac{\partial}{\partial\eta^k}, \mathbf{N} = \eta^k \frac{\partial}{\partial\eta^k}\}$ along $V\mathbf{I}$ and $\mathcal{R}^{-1} = \{B^\alpha_k, \eta_k\}$ the inverse matrices of this frame, that is:

$$B^\alpha_k B^k_\beta = \delta^\alpha_\beta \ ; \ B^\alpha_k \eta^k = 0 \ ; \ B^k_\alpha \eta_k = 0 \ ; \ B^k_\alpha B^\alpha_j + \eta^k\eta_j = \delta^k_j \ ; \ \eta_k\eta^k = 1. \tag{2.4}$$

The fundamental function $\tilde{L}(\tilde{z}, \theta) = L(z, \eta(\theta))$ of the complex Finsler space defines a metric tensor on indicatrix $\mathbf{I}$, $g_{\alpha\bar{\beta}} = B^j_\alpha B^{\bar{k}}_{\bar{\beta}} g_{j\bar{k}}$, where $B^{\bar{k}}_{\bar{\beta}} = \overline{B^k_\beta}$. It is easy to check that $g^{\bar{\beta}\alpha} = g^{\bar{j}i} B^\alpha_i B^{\bar{\beta}}_{\bar{j}}$ is the inverse of $g_{\alpha\bar{\beta}}$ and $g^{\bar{j}i} = B^i_\alpha B^{\bar{j}}_{\bar{\beta}} g^{\bar{\beta}\alpha} + \eta^i\eta^{\bar{j}}$. Also on indicatrix $\mathbf{I}_z$ from $\eta_k\eta^k = 1$ it follows that $\theta_\alpha\theta^\alpha = 1$, where $\theta_\alpha = g_{\alpha\bar{\beta}}\theta^{\bar{\beta}}$.

Let us consider the local frame $\{\tilde{\delta}_k = \frac{\partial}{\partial\tilde{z}^k} - \tilde{N}^\alpha_k \frac{\partial}{\partial\theta^{j\alpha}} \ ; \dot{\partial}_\alpha = \frac{\partial}{\partial\theta^\alpha}\ \}$, which spans the horizonthal distibution of $T'(T'M)$, and its dual frame $\{d\tilde{z}^k \ ; \delta\theta^\alpha = d\theta^\alpha + \tilde{N}^\alpha_j d\tilde{z}^j\}$, where $\tilde{N}^\alpha_k$ will be called, by abuse of terminology,the coefficients of the *induced* (*c.n.c.*) iff $\delta\theta^\alpha = B^\alpha_k\delta\eta^k$, that is $d\theta^\alpha + \tilde{N}^\alpha_j d\tilde{z}^j = B^\alpha_k(d\eta^k + N^k_j dz^j)$, and therefore in view of (2.3) it satisfies

$$\tilde{N}^\alpha_j = B^\alpha_k(B^k_{\beta j}\theta^\beta + N^k_j). \tag{2.5}$$

Let $\overset{CF}{N^k_j}$ be the (1.3) Chern-Finsler (c.n.c.) and $\overset{CF}{N^\alpha_j} = g^{\bar{\beta}\alpha}\frac{\partial^2\tilde{L}}{\partial\tilde{z}^j\partial\theta^{\bar{\beta}}} = g^{\bar{\beta}\alpha}\frac{\partial g_{\gamma\bar{\beta}}}{\partial\tilde{z}^j}\theta^\gamma$. Then we have:

Proposition 1. *The induced (c.n.c.) by the Chern-Finsler (c.n.c.)* $\overset{CF}{N^k_j}$ *coincides with* $\overset{CF}{N^\alpha_j}$.

Proof. A straightforward computation using (2.2) and (2.4) implies:
$\overset{CF}{N^\alpha_j} = g^{\bar{\beta}\alpha}\frac{\partial^2\tilde{L}}{\partial\tilde{z}^j\partial\theta^{\bar{\beta}}} = g^{\bar{\beta}\alpha}\frac{\partial}{\partial\tilde{z}^j}(B^{\bar{k}}_{\bar{\beta}}\frac{\partial\tilde{L}}{\partial\eta^{\bar{k}}}) = g^{\bar{\beta}\alpha}B^{\bar{k}}_{\bar{\beta}}(\frac{\partial^2 L}{\partial z^j\partial\eta^{\bar{k}}} + B^l_{\gamma j}\theta^\gamma g_{l\bar{k}})$
$= g^{\bar{m}p}B^\alpha_p B^{\bar{\beta}}_{\bar{m}} B^{\bar{k}}_{\bar{\beta}}(\frac{\partial^2 L}{\partial z^j\partial\eta^{\bar{k}}} + B^l_{\gamma j}\theta^\gamma g_{l\bar{k}}) = g^{\bar{m}p}B^\alpha_p(\delta^{\bar{k}}_{\bar{m}} - \eta_{\bar{m}}\eta^{\bar{k}})(\frac{\partial^2 L}{\partial z^j\partial\eta^{\bar{k}}} + B^l_{\gamma j}\theta^\gamma g_{l\bar{k}})$
$= B^\alpha_p(g^{\bar{m}p}\frac{\partial^2 L}{\partial z^j\partial\eta^{\bar{m}}} + B^p_{\gamma j}\theta^\gamma - \eta^p\eta^{\bar{k}}\frac{\partial^2 L}{\partial z^j\partial\eta^{\bar{k}}} - \eta^p\eta_l B^l_{\gamma j}\theta^\gamma)$
$= B^\alpha_p(B^p_{\gamma j}\theta^\gamma + \overset{CF}{N^k_j}) = \overset{CF}{\tilde{N}^\alpha_j}$.
We take in reductions from the last part the fact that $B^\alpha_p\eta^p = 0$. □

We note that in general $\{\tilde{\delta}_k\}$ are not $d-$tensor fields, i.e. they do not change like vectors on the manifold. Also by inclusion tangent map, $\mathbf{i}_*(\tilde{\delta}_k)$, which for convenience will be often identified with $\tilde{\delta}_k$ on $T'M$, is written as:

$\tilde{\delta}_k = \frac{\partial}{\partial \tilde{z}^k} - \tilde{N}_k^\alpha B_\alpha^j \frac{\partial}{\partial \eta^j} = \frac{\partial}{\partial z^k} + (B_{\alpha k}^j \theta^\alpha - \tilde{N}_k^\alpha B_\alpha^j)\frac{\partial}{\partial \eta^j}$ and, by using (2.4) and (2.5), we have:

$$\tilde{\delta}_k = \delta_k + H_k^0 \mathbf{N} \text{ and } \dot{\partial}_k = B_k^\alpha \dot{\partial}_\alpha - \eta_k \mathbf{N} \tag{2.6}$$

where $H_k^0 = (B_{\alpha k}^j \theta^\alpha + N_k^j)\eta_j$.

Further, let us consider the dual induced coframe $\tilde{d}z^k = d\tilde{z}^k$ and $\delta\theta^\alpha = d\theta^\alpha + N_j^\alpha d\tilde{z}^j$. The induced frame and coframe on the whole $T_C\mathbf{I}$ and the induced metric structure are obtained by conjugation everywhere:

$$\tilde{G} = g_{i\bar{j}} \tilde{d}z^i \otimes \tilde{d}\bar{z}^j + g_{\alpha\bar{\beta}} \delta\theta^\alpha \otimes \delta\bar{\theta}^\beta, \tag{2.7}$$

where $g_{i\bar{j}}(\tilde{z}, \eta(\theta))$ is the metric tensor of the space along the points of the indicatrix.

In [Mu1] we pointed out that another (c.n.c.) has a special meaning in the study of geodesics of a complex Finsler space, namely the *canonical (c.n.c.)*, given by $\overset{c}{N_j^i} := \frac{1}{2}\dot{\partial}_j(\overset{CF}{N_k^i}\eta^k) = \frac{1}{2}(\overset{CF}{L_{jk}^i}\eta^k + \overset{CF}{N_j^i})$. This connection will play an essential role in the construction of the complex Berwald connection and it comes from a spray (see [Mu]). Let us study the same problem of its induced canonical (c.n.c.),

$\overset{c}{\tilde{N}_j^\alpha} = B_k^\alpha(B_{\beta j}^k \theta^\beta + \overset{c}{N_j^k}) = B_k^\alpha\{B_{\beta j}^k \theta^\beta + \frac{1}{2}(\overset{CF}{L_{ji}^k}\eta^i + \overset{CF}{N_j^k})\}$

$\quad = \frac{1}{2}B_k^\alpha(B_{\beta j}^k + B_\beta^i \overset{CF}{L_{ji}^k})\theta^\beta + \frac{1}{2}\overset{CF}{\tilde{N}_j^\alpha}$.

Indeed, an intrinsic canonical (c.n.c.) on $T'\mathbf{I}$ is unnecessary because the problem of a complex spray is undesirable on $T'\mathbf{I}$.

Now, let us proceed to find the induced C-F linear connection. For this we consider the Gauss-Weingarten equations of the hypersurface $\mathbf{I}$ with respect to the C-F complex linear connection. Consider for any $X \in \Gamma(T_C\mathbf{I})$ and $Y \in \Gamma(V_C\mathbf{I})$ the decomposition:

$$D_X Y = \tilde{D}_X Y + H(X, Y) \tag{2.8}$$

where $\tilde{D}_X Y \in \Gamma(V_C\mathbf{I})$ is the tangential component and $H(X,Y)$ is the normal component.

Denote $\overset{CF}{\tilde{D}}_{\tilde{\delta}_k} \dot{\partial}_\beta = \overset{CF}{\tilde{L}_{\beta k}^\alpha} \dot{\partial}_\alpha$ and $\overset{CF}{\tilde{D}}_{\dot{\partial}_\gamma} \dot{\partial}_\beta = \overset{CF}{\tilde{C}_{\beta\gamma}^\alpha} \dot{\partial}_\alpha$. Since D preserves the distributions and $V\mathbf{I}$ is spanned by $\dot{\partial}_\alpha = B_\alpha^j \dot{\partial}_j$ and $V^\perp\mathbf{I}$ is generated by $\mathbf{N}$, we have

$\overset{CF}{\tilde{D}}_{\tilde{\delta}_k} \dot{\partial}_\beta = \overset{CF}{D}_{\tilde{\delta}_k} B_\beta^j \dot{\partial}_j = \tilde{\delta}_k(B_\beta^j)\dot{\partial}_j + B_\beta^j \overset{CF}{D}_{\delta_k + H_k^0\mathbf{N}} \dot{\partial}_j = \{B_{\beta k}^i + B_\beta^j \overset{CF}{L_{jk}^i} + B_\beta^j H_k^0 \eta^p \overset{CF}{C_{jp}^i}\}$ $\dot{\partial}_i$. But in view of the homogeneity of Finsler metrics $\eta^p C_{jip}^i = 0$, we have:

$$\overset{CF}{\tilde{L}_{\beta k}^\alpha} = B_i^\alpha(B_{\beta k}^i + B_\beta^j \overset{CF}{L_{jk}^i}). \tag{2.9}$$

Similarly we find that $\overset{CF}{\tilde{C}_{\beta\gamma}^\alpha} = B_i^\alpha B_\beta^j B_\gamma^k \overset{CF}{C_{jk}^i}$ and $\overset{CF}{\tilde{L}_{\beta\bar{k}}^\alpha} = \overset{CF}{\tilde{C}_{\beta\bar{\gamma}}^\alpha} = 0$.

For the normal component we have $G(\overset{CF}{D}_X \dot{\partial}_\alpha, \bar{\mathbf{N}}) = G(H(X, \dot{\partial}_\alpha), \bar{\mathbf{N}})$ and furthermore if we take X to be a vector of the adapted frames $\tilde{\delta}_k, \tilde{\delta}_{\bar{k}}$, we easily obtain that

$$H_{\alpha k} = B^j_{\alpha k}\eta_j + B^j_\alpha \overset{CF}{L^i_{jk}} \eta_i \text{ and } H_{\alpha\bar{k}} = 0.$$

If X is $\dot{\partial}_\beta$ or $\dot{\partial}_{\bar{\beta}}$, then the fundamental form will be $H_{\alpha\beta} = B^j_\alpha B^k_\beta \overset{CF}{C^i_{jk}} \eta_i$ and $H_{\alpha\bar{\beta}} = 0$.

Further, if $\overset{CF}{\tilde{D}}_{\tilde{\delta}_k} \tilde{\delta}_j = \overset{CF}{\tilde{L}^i_{jk}} \tilde{\delta}_i$, then direct computation of $\overset{CF}{\tilde{D}}_{\tilde{\delta}_k} \tilde{\delta}_j = \overset{CF}{D}_{\delta_k + H^0_k\mathbf{N}} (\delta_j + H^0_j\mathbf{N})$, by using the homogeneity conditions, finally get that $\overset{CF}{\tilde{L}^i_{jk}} = \overset{CF}{L^i_{jk}}$, because of $\overset{CF}{D}_{\mathbf{N}} \delta_j = 0$, and the normal part is $H_{kj} = \delta_k(H^0_j) + N^0_k\mathbf{N}(H^0_j) - H^0_i \overset{CF}{L^i_{jk}}$.

It is obvious that even if the C-F connection is a normal one, that is $D_{\delta_k}\delta_j = L^i_{jk}\delta_i$, $D_{\delta_k}\dot{\partial}_j = L^i_{jk}\dot{\partial}_j$, the induced connection is not a normal one. Another property of interest which is not preserved by this projection is that of Berwald type connection, namely it is well known that $\overset{CF}{L^i_{jk}} = \dot{\partial}_j \overset{CF}{N^i_k}$, while for the induced connection we have: $\dot{\partial}_\beta \overset{CF}{\tilde{N}^\alpha_k} = \dot{\partial}_\beta\{B^\alpha_j(B^j_{\gamma k}\theta^\gamma + \overset{CF}{N^j_k})\} = B^\alpha_j(B^j_{\beta k} + B^i_\beta \overset{CF}{L^j_{ik}}) = \overset{CF}{\tilde{L}^\alpha_{\beta k}} + B^\alpha_j B^k_\beta(\overset{CF}{L^j_{ki}} - \overset{CF}{L^j_{ik}})$, and hence it reduces to $\overset{CF}{\tilde{L}^\alpha_{\beta k}}$ if the space is Kähler for instance.

Now let us introduce another useful linear connection for our study, the complex Berwald connection. We proved in [Mu1] that it plays an interesting role in the characterization of totally geodesic curves on a holomorphic subspace. Let us consider $\overset{c}{N^k_j}$ the canonical (c.n.c.) introduced above and we define the *complex Berwald connection* as being

$$\overset{B}{D}\Gamma = (\overset{c}{N^i_k}, \overset{B}{L^i_{jk}} = \frac{\partial \overset{c}{N^i_k}}{\partial\eta^j}, \ \overset{B}{L^{\bar{i}}_{\bar{j}k}} = 0, \ \overset{B}{C^i_{jk}} = 0, \ \overset{B}{C^{\bar{i}}_{\bar{j}k}} = 0). \tag{2.10}$$

We see that the Berwald connection is in the vertical bundle $V(T'M)$ and it is easy to check that

$$\overset{c}{N^i_k} = \frac{1}{2}(\overset{CF}{L^i_{k0}} + \overset{CF}{N^i_k}) \quad\text{and}\quad \overset{B}{L^i_{jk}} = \frac{1}{2}(\overset{CF}{L^i_{jk}} + \overset{CF}{L^i_{kj}}) + \frac{1}{2}\dot{\partial}_j(\overset{CF}{L^i_{km}})\eta^m.$$

By using again the G-W decomposition, since $\overset{B}{C^i_{jk}} = 0$, it is easy to obtain that the induced Berwald connection is $\overset{B}{\tilde{L}^\alpha_{\beta k}} = B^\alpha_i(B^i_{\beta k} + B^j_\beta \overset{B}{L^i_{jk}})$ and $\overset{B}{\tilde{C}^\alpha_{\beta k}} = \overset{B}{\tilde{L}^\alpha_{\beta\bar{k}}} = \overset{B}{\tilde{C}^\alpha_{\beta\bar{k}}} = 0$. Of course, we can not talk about horizontal components $\overset{B}{\tilde{L}^i_{jk}}$.

But the induced Berwald connection has the property that it is of Berwald type. Indeed, starting from:

$\dot{\partial}_\beta \overset{c}{\tilde{N}^\alpha_j} = \frac{1}{2}\dot{\partial}_\beta\{B^\alpha_i(B^i_{\gamma k}\theta^\gamma + \overset{CF}{L^i_{jk}} \eta^k) + \overset{CF}{\tilde{N}^\alpha_j}\}$

$$= \tfrac{1}{2}B_i^{\alpha}\{(B^i_{\beta j} + B^k_{\beta}\overset{CF}{L^i_{jk}} + \dot{\partial}_{\beta}(\overset{CF}{L^i_{jk}})\eta^k\} + \tfrac{1}{2}\,\dot{\partial}_{\beta}\overset{CF}{\tilde{N}^{\alpha}_j}$$

and replacing the expression of $\dot{\partial}_{\beta}\overset{CF}{\tilde{N}^{\alpha}_k}$, after reducing the same terms, it results:

$\dot{\partial}_{\beta}\overset{c}{\tilde{N}^{\alpha}_j} = \tfrac{1}{2}B_i^{\alpha}\{(B^i_{\beta j} + B^k_{\beta}\overset{CF}{L^i_{kj}} + B^m_{\beta}\dot{\partial}_m(\overset{CF}{L^i_{jk}})\eta^k\} + \tfrac{1}{2}\,\overset{CF}{\tilde{L}^{\alpha}_{\beta j}}$.

On the other hand from (2.9),

$\overset{B}{\tilde{L}^{\alpha}_{\beta j}} = B_i^{\alpha}(B^i_{\beta j} + B^k_{\beta}\overset{B}{L^i_{jk}}) = B_i^{\alpha}\{B^i_{\beta j} + \tfrac{1}{2}B^k_{\beta}(\overset{CF}{L^i_{jk}} + \overset{CF}{L^i_{kj}}) + \tfrac{1}{2}B^k_{\beta}\dot{\partial}_j(\overset{CF}{L^i_{km}})\eta^m\}$

$= \tfrac{1}{2}B_i^{\alpha}\{(B^i_{\beta j} + B^k_{\beta}\overset{CF}{L^i_{kj}} + B^m_{\beta}\dot{\partial}_m(\overset{CF}{L^i_{jk}})\eta^k\} + \tfrac{1}{2}\,\overset{CF}{\tilde{L}^{\alpha}_{\beta k}}$.

Comparing the last lines, we get

Proposition 2. *The induced Berwald connection coincides with the intrinsic Berwald connection of the indicatrix bundle, that is:*

$$\overset{B}{\tilde{L}^{\alpha}_{\beta j}} = \frac{\partial\,\overset{c}{\tilde{N}^{\alpha}_j}}{\partial\theta^{\beta}}. \tag{2.11}$$

Similarly, following the general settings from the geometry of hypersurfaces ([Mu1]), a linear connection D induces a normal connection $D^{\perp}$ and let $A_{\mathbf{N}}X := A(X) \in V\mathbf{I}$ be the shape operator. Then we have:

$$D_X\mathbf{N} = -A_{\mathbf{N}}X + D^{\perp}_X\mathbf{N}.$$

From $G(D_X\mathbf{N}, \dot{\partial}_{\bar{k}}) = -G(A(X), \dot{\partial}_{\bar{k}})$, where X is taken to be one of the vectors $\tilde{\delta}_k, \tilde{\delta}_{\bar{k}}, \dot{\partial}_{\beta}$ or $\dot{\partial}_{\bar{\beta}}$, we obtain that:

$A^{\alpha}_k = B^{\alpha}_l(N^l_k - \eta^j L^l_{jk})$; $A^{\alpha}_{\bar{k}} = 0$; $A^{\alpha}_{\beta} = -B^{\alpha}_k$ and $A^{\alpha}_{\bar{\beta}} = 0$. Let us remark that both C-F and canonical (c.n.c.) are homogeneous and hence $N^l_k = \eta^j L^l_{jk}$. Consequently for the C-F or Berwald connections it results that $A^{\alpha}_k = 0$.

We compute below the normal component of the Berwald connection. For $X = \overset{c}{\tilde{\delta}_k}$, we have

$\overset{B}{D}_{\overset{c}{\tilde{\delta}_k}}\mathbf{N} = \{\overset{c}{\tilde{\delta}_k}(\eta^i) + \overset{c}{N^l_k}\,\eta_l\eta^i\}\dot{\partial}_i + \eta^j\,\overset{CF}{L^i_{jk}}\,\dot{\partial}_i = (\eta^j\,\overset{CF}{L^i_{jk}} - \overset{c}{N^i_k})\dot{\partial}_i + \overset{c}{N^l_k}\,\eta_l\mathbf{N} = \overset{c}{N^l_k}\,\eta_l\mathbf{N}$

Thus we proved that

$$A_{\mathbf{N}}\,\overset{c}{\tilde{\delta}_k} = 0 \quad\text{and}\quad \overset{B}{D}{}^{\perp}_{\overset{c}{\tilde{\delta}_k}}\,\mathbf{N} = \overset{c}{N^l_k}\,\eta_l\mathbf{N}.$$

For $X = \overset{c}{\tilde{\delta}_{\bar{k}}}$, we deduce that $A_{\mathbf{N}}\,\overset{c}{\tilde{\delta}_{\bar{k}}} = 0$ and $\overset{B}{D}{}^{\perp}_{\overset{c}{\tilde{\delta}_k}}\,\mathbf{N} = \mathbf{0}$.

Similar computations get :

$A_{\mathbf{N}}\dot{\partial}_{\gamma} = -\dot{\partial}_{\gamma}$ and $\overset{B}{D}{}^{\perp}_{\dot{\partial}_{\gamma}}\,\mathbf{N} = -\,B^l_{\gamma}\eta_l\mathbf{N}$; $A_{\mathbf{N}}\dot{\partial}_{\bar{\gamma}} = 0$ and $\overset{B}{D}{}^{\perp}_{\dot{\partial}_{\bar{\gamma}}}\,\mathbf{N} = 0$.

For the Berwald connection the torsion and curvature tensors are denoted by:

$$\begin{aligned} &vT(\overset{c}{\delta}_k,\overset{c}{\delta}_j)=\Omega^i_{jk}\dot{\partial}_i \text{ where } &&\Omega^i_{jk}=\overset{c}{\delta}_j\overset{c}{N}{}^i_k-\overset{c}{\delta}_k\overset{c}{N}{}^i_j\\ &vT(\overset{c}{\delta}_{\bar{k}},\overset{c}{\delta}_j)=\Theta^i_{j\bar{k}}\dot{\partial}_i &&\Theta^i_{j\bar{k}}=\overset{c}{\delta}_{\bar{k}}\overset{c}{N}{}^i_j\\ &vT(\dot{\partial}_{\bar{k}},\overset{c}{\delta}_j)=\rho^i_{j\bar{k}}\dot{\partial}_i &&\rho^i_{j\bar{k}}=\dot{\partial}_{\bar{k}}\overset{c}{N}{}^i_j \end{aligned} \tag{2.12}$$

and the nonzero curvatures:

$$\begin{aligned} R^i_{jkh}X_i &= R(\overset{c}{\delta}_h,\overset{c}{\delta}_k)X_j & R^i_{j\bar{k}h}X_i &= R(\overset{c}{\delta}_h,\overset{c}{\delta}_{\bar{k}})X_j\\ P^i_{jkh}X_i &= R(\dot{\partial}_h,\overset{c}{\delta}_k)X_j & P^i_{j\bar{k}h}X_i &= R(\overset{c}{\delta}_h,\dot{\partial}_{\bar{k}})X_j \end{aligned}$$

where X_i is first $\overset{c}{\delta}_i$ and then $\dot{\partial}_i$.

The computation of these curvatures leads to

$$\begin{aligned} R^i_{jkh} &= \overset{c}{\delta}_h\overset{B}{L}{}^i_{jk}-\overset{c}{\delta}_k\overset{B}{L}{}^i_{jh}+\overset{B}{L}{}^l_{jk}\overset{B}{L}{}^i_{lh}-\overset{B}{L}{}^l_{jh}\overset{B}{L}{}^i_{lk}\ ;\\ R^i_{j\bar{k}h} &= -\overset{c}{\delta}_{\bar{k}}\overset{B}{L}{}^i_{jh}\ ;\ \ P^i_{jkh}=\dot{\partial}_h\overset{B}{L}{}^i_{jk}\ ;\ \ P^i_{j\bar{k}h}=-\dot{\partial}_{\bar{k}}\overset{B}{L}{}^i_{jh}\ . \end{aligned} \tag{2.13}$$

Proposition 3. *The induced torsions and curvature with respect to Berwald connection are given by:*

i) $\tilde{\Omega}^\alpha_{jk}=B^\alpha_i\Omega^i_{jk}\ ;\ \tilde{\Theta}^\alpha_{j\bar{k}}=B^\alpha_i\Theta^i_{j\bar{k}}\ ;\ \tilde{\rho}^\alpha_{j\bar{\beta}}=B^{\bar{k}}_{\bar{\beta}}B^\alpha_i\rho^i_{j\bar{k}}$.

ii)

$$\begin{aligned} \tilde{R}^\alpha_{\beta kh} &= B^\alpha_i\{B^j_\beta R^i_{jkh}+\overset{c}{\delta}_h(B^j_\beta)\overset{B}{L}{}^i_{jk}-\overset{c}{\delta}_k(B^j_\beta)\overset{B}{L}{}^i_{jh}+B^j_\beta(\overset{c}{N}{}^0_h\overset{B}{L}{}^i_{lk}-\overset{c}{N}{}^0_k\overset{B}{L}{}^i_{lh})\}\ ;\\ \tilde{R}^\alpha_{\beta\bar{k}h} &= -B^\alpha_i\{B^j_\beta R^i_{j\bar{k}h}+B^j_\beta\eta_{\bar{k}}\eta^{\bar{l}}P^i_{j\bar{l}h}\}\ ;\\ \tilde{P}^\alpha_{\beta k\gamma} &= B^h_\gamma\{B^i_{\beta k}+B^j_\beta\overset{B}{L}{}^i_{jk})(\dot{\partial}_hB^\alpha_i+\eta_h\mathbf{N}(B^\alpha_i)+B^\alpha_iB^j_\beta P^i_{jkh}+\eta_hB^\alpha_iB^j_\beta\overset{B}{L}{}^i_{jk}\ ;\\ \tilde{P}^\alpha_{\beta\bar{k}\gamma} &= -B^\alpha_iB^j_\beta P^i_{j\bar{k}h}-B^h_\gamma B^i_{\beta\bar{k}}(\dot{\partial}_hB^\alpha_i+\eta_h\mathbf{N}(B^\alpha_i). \end{aligned}$$

With these computations the *Gauss*, respectively *H−Codazzi equations* of the indicatrix subspace $(\mathbf{I},\tilde{L})$ are deduced from:

$$\mathbf{G}\left(R(X,Y)vZ,\bar{v}U\right)=\tilde{\mathbf{G}}(\tilde{R}(X,Y)\tilde{v}Z,\bar{\tilde{v}}U)+\tilde{\mathbf{G}}\left(A_{H(X,\tilde{v}Z)}Y-A_{H(Y,\tilde{v}Z)}X,\bar{\tilde{v}}U\right) \tag{2.14}$$

and

$$\begin{aligned} \mathbf{G}\left(R(X,Y)vZ,W\right) &= \tilde{\mathbf{G}}\left((D_XH)(Y,vZ)-(D_YH)(X,vZ),W\right)\\ &\quad+\tilde{\mathbf{G}}(H(\tilde{T}(X,Y),Z),W) \end{aligned} \tag{2.15}$$

where $W\in\Gamma(\overline{V^\perp_C T'\mathbf{I}})$ and $v,\bar{v},\ \tilde{v},\bar{\tilde{v}}$ are the projectors on the corresponding vertical distributions.

Similarly, equating the components from $V_C T'\mathbf{I}$ and $V_C T'\ \mathbf{I}^{\perp}$ of the normal curvatures we obtain the following $A-$*Codazzi*, respectively *Ricci equations*:

$$\mathbf{G}\left(R(X,Y)W,\bar{v}Z\right) = \tilde{\mathbf{G}}\left((\tilde{D}_Y A)(W,X) - (\tilde{D}_X A)(W,Y), \bar{\tilde{v}}Z\right) - \tilde{\mathbf{G}}\left(A_W(T(X,Y), \bar{\tilde{v}}Z\right) \tag{2.16}$$

and

$$\mathbf{G}\left(R(X,Y)W,\bar{v}N\right) = \tilde{\mathbf{G}}\left(\tilde{R}^{\perp}(X,Y)W, \bar{\tilde{v}}N\right) + \tilde{\mathbf{G}}\left(H(Y,A_W X) - H(X,A_W Y), \bar{\tilde{v}}N\right) \tag{2.17}$$

for $\forall X,Y \in \Gamma(T_C T'\ \mathbf{I})$ and $W,N \in \Gamma(V_C T'\ \mathbf{I}^{\perp})$.

An expansive writting for these formulas can be obtained by replacing the vectors with the adapted frames of the Berwald (c.n.c.).

3 Complex geodesics on $(\mathbf{I}, \tilde{L})$

Let us consider $\sigma : t \to \left(z^k(t), \eta^k(t) = \frac{dz^k}{dt}\right)$ a *complex geodesic* curve on the complex Finsler space (M, L). According to [A-P], p. 101, it satisfies the system

$$\frac{d^2 z^i}{dt^2} + \overset{CF}{N^i_k}(z(t),\eta(t))\frac{dz^k}{dt} = \mathbf{\Theta}^{*i}\ ,\ i = \overline{1,n+1}, \tag{3.1}$$

where $\overset{CF}{N^i_k}$ are the coefficients of the Chern-Finsler (c.n.c.), and

$$\mathbf{\Theta}^{*i} = g^{\bar{m}i} g_{j\bar{l}} \{ \overset{CF}{L^{\bar{l}}_{\bar{n}\bar{m}}} - \overset{CF}{L^{\bar{l}}_{\bar{m}\bar{n}}} \} \eta^j \bar{\eta}^n. \tag{3.2}$$

In [Mu] we proved that an equivalent expression for $\mathbf{\Theta}^{*i}$ is $\mathbf{\Theta}^{*i} = g^{\bar{m}i} \overset{c}{\delta_{\bar{m}}} L$.

If $\mathbf{\Theta}^{*i} = 0$ then the space is weakly Kähler, therefore we call the form $\mathbf{\Theta}^{*} = g^{\bar{m}i} g_{j\bar{l}} \{ \overset{CF}{L^{\bar{l}}_{\bar{n}\bar{m}}} - \overset{CF}{L^{\bar{l}}_{\bar{m}\bar{n}}} \} dz^j \wedge d\bar{z}^n \otimes \delta_i$ the weakly Kähler form. If in addition the torsion of the C-F linear connection satisfies a supplementary condition, a special geodesic is obtained, named $c-$*geodesic complex curve*, which is the correspondent via a holomorphic map of a geodesic from the unit disc with the Poincaré metric. We point out that there is a sensitive difference between these geodesics. Taking into account the $1-$homogeneity of the coefficients of C-F (c.n.c.), we readily check that $\overset{c}{N^i_0} = \overset{CF}{N^i_0}$. Thus, the equations (3.1) will be rewritten equivalently as

$$\frac{d^2 z^i}{dt^2} + \overset{c}{N^i_k}(z(t),\eta(t))\frac{dz^k}{dt} = \mathbf{\Theta}^{*i}\ ,\ i = \overline{1,n+1}. \tag{3.3}$$

From $\overset{B}{L}{}^i_{jk}\ \eta^j\eta^k = \frac{\partial \overset{c}{N}{}^i_k}{\partial \eta^j}\eta^j\eta^k = \overset{c}{N}{}^i_k\ \eta^k = N^i_0$, it results that (3.3) becomes

$$\frac{d^2z^i}{dt^2} + \overset{B}{L}{}^i_{jk}\ (z(t),\eta(t))\frac{dz^j}{dt}\frac{dz^k}{dt} = \mathbf{\Theta}^{*i}\ \ ,\ i = \overline{1, n+1},$$

which implies that if the space is weakly Kähler, then a complex geodesic is an horizontal curve with respect to the Berwald connection.

On the other hand, since $\eta^k = \frac{dz^k}{dt}$ and $\overset{c}{\delta}\ \eta^i = d\eta^i + \overset{c}{N}{}^i_k\ dz^k$, along a complex geodesic curve we have

$$\frac{\overset{c}{\delta}\ \eta^i(t)}{dt} = \mathbf{\Theta}^{*i}\ \text{ and } \eta^i(t) = \frac{dz^i}{dt}. \tag{3.4}$$

Now, contracting in (3.4) with B^α_i and taking into account the request of induced (c.n.c.) $\overset{c}{\delta}\ \theta^\alpha = B^\alpha_i\ \overset{c}{\delta}\ \eta^i$, we obtain *an induced curve* $\tilde{\sigma} : t \to \left(\tilde{z}^i(t), \theta^\alpha(t)\right)$ on $(\mathbf{I}, \tilde{L})$ by the complex geodesic curve σ from (M, L). It satisfies the equations:

$$\frac{\overset{c}{\delta}\ \theta^\alpha(t)}{dt} = \mathbf{\Theta}^{*\alpha}\ , \tag{3.5}$$

where $\frac{\overset{c}{\delta}\theta^\alpha(t)}{dt} = \frac{d\theta^\alpha}{dt} + \overset{c}{N}{}^\alpha_i\ \frac{dz^i}{dt}$ and $\mathbf{\Theta}^{*\alpha}(t) = B^\alpha_i \mathbf{\Theta}^{*i}(t) = B^\alpha_i g^{\bar{m}i}\ \overset{c}{\delta}_{\bar{m}}\ L$ along the points of $\tilde{\sigma}$.

Let $\dot{\sigma}(t) = \frac{dz^i}{dt}\frac{\partial}{\partial z^i}$ be the tangent vector to σ and $l(\sigma) = \int_a^b \mathbf{F}(z^i(t), \dot{\sigma}^i(t))dt$ the length arc, $t \in [a, b]$.

Since $\eta^i = \frac{dz^i}{dt} = \frac{d\tilde{z}^i}{dt}$ and $\eta^i = B^i_\alpha\theta^\alpha$ along the induced curve $\tilde{\sigma}$, then its tangent vector will be $\dot{\tilde{\sigma}}\ (t) = \frac{d\tilde{z}^i}{dt}\frac{\partial}{\partial \tilde{z}^i} = B^i_\alpha\theta^\alpha(t)\frac{\partial}{\partial \tilde{z}^i}$ and hence, using the homogeneity of the Finsler function the length arc of the induced geodesic curve will be $l(\tilde{\sigma}) = \int_a^b \left|B^i_\alpha\right| \tilde{\mathbf{F}}(\tilde{z}^i(t), \theta^\alpha(t))dt$. We note that from $\eta_i\eta^i = 1$ it follows that $\dot{\tilde{\sigma}}\ (t)$ is a unitary tangent vector to the indicatrix.

Thus, any complex geodesic curve on (M, L) induces a curve on $(\mathbf{I}, \tilde{L})$ with unit tangent vector. Conversely, let $\tilde{\sigma} : t \to \left(\tilde{z}^i(t), \theta^\alpha(t)\right)$ be a curve on $(\mathbf{I}, \tilde{L})$, with $\theta^\alpha\theta_\alpha = 1$, that is the tangent vector to $\tilde{\sigma}$ is unitary. It can be lifted to a curve $\sigma : t \to \left(z^i(t), \frac{dz^i}{dt} = B^i_\alpha\theta^\alpha(t)\right)$ on (M, L). Certainly, its induced curve on the indicatrix by the above procedure is just $\tilde{\sigma}$.

The frames $\{\tilde{\delta}_i\}$ are linearly independent and span a distribution $\tilde{H}\mathbf{I}$ which is not diffeomorphic to $T'M$ because $\tilde{\delta}_i$ are not $d-$tensor fields. We know that $T'M$ is diffeomorphic with $H(T'M)$ via the horizontal lift $\frac{\partial}{\partial z^i} \overset{l^h}{\to} \frac{\delta}{\delta z^i}$. For an appropriate study with that made in [A-P] for the first variation, along the curve $\tilde{\sigma}$ we consider the following lift of the tangent vector

$$\dot{\tilde{\sigma}}\ (t) = \frac{d\tilde{z}^i}{dt}\frac{\partial}{\partial \tilde{z}^i} \overset{l^i}{\to} \tilde{T}^h = \frac{d\tilde{z}^i}{dt}\frac{\overset{CF}{\delta}}{\delta z^i} = B^i_\alpha\theta^\alpha(t)\frac{\overset{CF}{\delta}}{\delta z^i} \tag{3.6}$$

along the curve $\tilde{\sigma}$. Of course since $G(\overset{CF}{\delta}{}_i, \overset{CF}{\delta}{}_{\bar{j}}) = g_{i\bar{j}}$ we deduce that $\parallel \tilde{T}^h \parallel = 1$. Let us consider $s \in (-\varepsilon, \varepsilon)$ and $\tilde{\Sigma}_s$ a variation of the curve $\tilde{\sigma}$ with fixed points, $(\tilde{z}^i(t, s), \theta^\alpha(t, s))$, and for the tangent vector $\tilde{U} = \frac{d\tilde{\Sigma}^i}{ds}\frac{\partial}{\partial \bar{z}^i}$ consider their lift to $T'M$ denoted by $\tilde{U}^h = \frac{d\tilde{\Sigma}^i}{ds}\frac{\overset{CF}{\delta}}{\delta z^i}$. We assume here that the variation of $\tilde{\sigma}$ is such that $\| \tilde{U}^h \| = 1$, which is not a strong restriction.

By this, the same construction from Theorem 2.4.1 from [A-P], for the first variation, we get that $\tilde{\sigma}$ is a complex geodesic of $(\mathbf{I}, \tilde{L})$ space iff

$$\overset{CF}{D}_{\tilde{T}^h + \overline{\tilde{T}^h}}\, \tilde{T}^h = \Xi^*(\tilde{T}^h, \overline{\tilde{T}^h}) \tag{3.7}$$

where Ξ^* convey the weakly Kähler form on $\tilde{\sigma}$ from $(\mathbf{I}, \tilde{L})$, with respect to induced C-F linear connection, that is $\Xi^{*i} = g^{\bar{m}i} g_{j\bar{l}} \{ \overset{CF}{\tilde{L}}{}^{\bar{l}}_{\bar{n}\bar{m}} - \overset{CF}{\tilde{L}}{}^{\bar{l}}_{\bar{m}\bar{n}} \} \dot{\tilde{\sigma}}^j \overline{\dot{\tilde{\sigma}}^n}$ along the curve $\tilde{\sigma}$.

The (3.7) equation of the geodesic says:
$\left(\dot{\tilde{\sigma}}^j \overset{CF}{\delta}_j (\dot{\tilde{\sigma}}^i) + \overset{CF}{\tilde{L}}{}^i_{jk} \dot{\tilde{\sigma}}^j \dot{\tilde{\sigma}}^k \right) \overset{CF}{\delta}_i + \left(\overline{\dot{\tilde{\sigma}}^j} \overset{CF}{\delta}_{\bar{j}} (\dot{\tilde{\sigma}}^i) \right) \overset{CF}{\delta}_i = \Xi^{*i} \overset{CF}{\delta}_i$ and because $\frac{d}{dt} = \frac{\overset{CF}{\delta} \dot{\tilde{\sigma}}^j}{dt} \overset{CF}{\delta}_j + \frac{\overset{CF}{\delta} \dot{\tilde{\sigma}}^j}{dt} \overset{CF}{\delta}_{\bar{j}}$, it follows that :

$$\frac{d}{dt}(\dot{\tilde{\sigma}}^i) + \overset{CF}{\tilde{L}}{}^i_{jk} \dot{\tilde{\sigma}}^j \dot{\tilde{\sigma}}^k = \Xi^{*i} \tag{3.8}$$

is the equation of a geodesic on $(\mathbf{I}, \tilde{L})$.

Now by taking into account that along the $\tilde{\sigma}$ curve we have $\dot{\tilde{\sigma}}^i = B^i_\alpha(t)\theta^\alpha(t)$ and $\overset{CF}{\tilde{L}}{}^i_{jk}$ coincides with $\overset{CF}{L}{}^i_{jk}$ it results:

$$\frac{d}{dt}(B^i_\alpha \theta^\alpha) + \overset{CF}{L}{}^i_{jk}\, B^j_\beta B^k_\gamma \theta^\beta \theta^\gamma = \Xi^{*i}$$

that is,

$$\frac{d\theta^\alpha}{dt} + B^\alpha_i \left(\frac{d}{dt}(B^i_\beta \theta^\beta + \overset{CF}{L}{}^i_{jk}\, B^j_\beta B^k_\gamma \theta^\beta \theta^\gamma \right) = \Xi^{*\alpha}, \tag{3.9}$$

where $\Xi^{*\alpha} = B^\alpha_i \Xi^{*i}$.

But C-F is a normal connection, i.e. $D_{\delta_k}\delta_j = \overset{CF}{L}{}^i_{jk}\, \delta_i$ and $D_{\delta_k}\dot{\partial}_j = \overset{CF}{L}{}^i_{jk}\, \dot{\partial}_j$, for which we can consider its induced vertical connection (2.9), and hence for (3.9) we get

$$\frac{d\theta^\alpha}{dt} + B^\alpha_i \left(\frac{d}{dt}(B^i_\beta)\theta^\beta - B^i_{\beta k} B^k_\gamma \theta^\beta \theta^\gamma \right) + \overset{CF}{\tilde{L}}{}^\alpha_{\beta k}\, B^k_\gamma \theta^\beta \theta^\gamma = \Xi^{*\alpha}.$$

We remark that B^i_β depends on t only by means of $z(t)$ and therefore $\frac{dB^i_\beta}{dt} = \frac{dB^i_\beta}{dz^k}\frac{dz^k}{dt} = B^i_{\beta k}\eta^k$ along the curve σ. Accordingly the last bracket became:
$\frac{d}{dt}(B^i_\beta)\theta^\beta - B^i_{\beta k} B^k_\gamma \theta^\beta \theta^\gamma = B^i_{\beta k}\theta^\beta \eta^k - B^i_{\beta k}\theta^\beta \eta^k = 0$, and thus the equation of the complex geodesic on $(\mathbf{I}, \tilde{L})$ reduces to: $\frac{d\theta^\alpha}{dt} + \overset{CF}{\tilde{L}}{}^\alpha_{\beta k}\, B^k_\gamma \theta^\beta \theta^\gamma = \Xi^{*\alpha}$, or else written:

$$\frac{d\theta^\alpha}{dt} + \overset{CF}{\tilde{L}}{}^\alpha_{\beta k}\, \theta^\beta \frac{d\tilde{z}^k}{dt} = \Xi^{*\alpha}. \tag{3.10}$$

Further, let see the circumstances in which it coincides to the induces geodesic on (M, L) given by (3.5).

We proved that $\dot{\partial}_\beta \overset{CF}{\tilde{N}^\alpha_k} = \overset{CF}{\tilde{L}^\alpha_{\beta k}} + B^\alpha_j B^k_\beta (\overset{CF}{L^j_{ki}} - \overset{CF}{L^j_{ik}})$ and $d\theta^\alpha = \overset{c}{\delta}\theta^\alpha - \overset{c}{\tilde{N}^\alpha_k} d\tilde{z}^k$, which replaced in (3.10), forasmuch $(\overset{CF}{L^j_{ki}} - \overset{CF}{L^j_{ik}})\eta^i\eta^k = 0$, it results

$$\frac{\overset{c}{\delta}\theta^\alpha}{dt} + \left(\dot{\partial}_\beta(\overset{CF}{\tilde{N}^\alpha_k})\theta^\beta - \overset{c}{\tilde{N}^\alpha_k}\right)\frac{d\tilde{z}^k}{dt} = \Xi^{*\alpha}.$$

But $\dot{\partial}_\beta(\overset{CF}{\tilde{N}^\alpha_k})\theta^\beta = B^\alpha_i B^i_{\beta k}\theta^\beta + B^j_\beta \overset{CF}{L^i_{jk}}\theta^\beta = B^\alpha_i(B^i_{\beta k}\theta^\beta + \overset{CF}{N^i_k}) = \overset{CF}{\tilde{N}^\alpha_k}$. Hence the above equations become

$$\frac{\overset{c}{\delta}\theta^\alpha}{dt} + (\overset{CF}{\tilde{N}^\alpha_k} - \overset{c}{\tilde{N}^\alpha_k})\frac{d\tilde{z}^k}{dt} = \Xi^{*\alpha}.$$

Now, $\overset{CF}{\tilde{N}^\alpha_k}$ and $\overset{c}{\tilde{N}^\alpha_k}$ are induced (c.n.c.) and hence they satisfy formula (2.5). By reducing the same terms, we have:

$$\frac{\overset{c}{\delta}\theta^\alpha}{dt} + B^\alpha_i(\overset{CF}{N^i_k} - \overset{c}{N^i_k})\frac{d\tilde{z}^k}{dt} = \Xi^{*\alpha}.$$

Since along the geodesic curve on $(\mathbf{I}, \tilde{L})$, $\overset{CF}{N^i_k}\eta^k = \overset{c}{N^i_k}\eta^k$ (by using the homogeneity condition in their definitions), with $\eta^k = \frac{d\tilde{z}^k}{dt}$, it results that the equations of complex geodesic curve $\tilde{\sigma}$ reduce to

$$\frac{\overset{c}{\delta}\theta^\alpha}{dt} = \Xi^{*\alpha}. \tag{3.11}$$

In conclusion

Theorem 1. *The induced geodesic from (M, L) complex Finsler space coincides with the defined complex geodesic on the indicatrix space $(\mathbf{I}, \tilde{L})$, if and only if the induced weakly Kähler form Θ^* coincides with Ξ^*.*

In particular,

Corollary 1. *If (M, L) complex Finsler space with weakly Kähler Finsler metrics, then the induced geodesics will be a complex geodesic for $(\mathbf{I}, \tilde{L})$ indicatrix space if and only if the weakly Kähler character conveys to the indicatrix via the induced Chern-Finsler induced connection.*

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