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REPRESENTATION OF THE K - FUNCTIONAL $K(f, C[a,b], C^1[a,b], \cdot)$ - A NEW APPROACH

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Abstract

We give a new proof of the equality $K(f, C[a,b], C^1[a,b], t) = \frac{1}{2} \cdot \tilde{\omega}(f, 2t)$, for $f \in C[a,b]$, $0 < t \leq (b-a)/2$.

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1 Introduction. Main result

The moduli of continuity and the K-functionals are crucial tools in study of the degree of approximation by using positive linear operators. There is a strong relationship between them. An extensive approach on these relations can be found in Peetre [5], Ditzian and Totik [2], Lorentz and DeVore [1].

Recall that if $(X, \|\cdot\|_X)$ is a normed space of functions and $Y \subset X$ is a subspace endowed with a seminorm $|\cdot|_Y$, we can associate to the pair (X, Y) the following K functional:

$$K(f, X, Y, t) = \inf\{\|f - g\|_X + t|g|_Y\}, \quad f \in X,\ t > 0. \tag{1}$$

Generally speaking the K functionals are equivalent with the suitable moduli of continuity or smoothness. The K functional $K(f, X, Y, t)$ is said to be equivalent to a certain modulus $\Omega(f, t)$, if there are two constants $C_1 > 0$, $C_2 > 0$, such that

$$C_1\Omega(f, t) \leq K(f, X, Y, t) \leq C_2\Omega(f, t), \quad \text{for all } f \in X,\ t > 0. \tag{2}$$

However a more precise representation of the K functional $K(f, X, Y, t)$ exists in the simple case where $X = C[a,b]$, endowed with the sup-norm $\|f\|_X := \|f\|$, where $\|f\| = \max_{x\in[a,b]} |f(x)|$, $f \in C[a,b]$ and $Y = Lip1$, endowed with the seminorm $|g|_Y = |g|_{Lip1}$, where $|g|_{Lip1} = \inf\{M,\ |g(x) - g(y)| \leq M,\ \forall x, y \in [a,b]\}$. Then the K functional $K(f, C[a,b], Lip1, t)$ can be expressed with an equality in terms of the least concave majorant of the modulus of continuity.

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In order to define this modulus, recall that the least concave majorant of a function $f : [a,b] \to \mathbb{R}$ is the function $\tilde{f} : [a,b] \to \mathbb{R}$, defined by

$$\tilde{f}(x) = \inf\{g(x),\ g \geq f,\ g \text{ concave}\}. \tag{3}$$

From this definition it follows the formulae:

$$\tilde{f}(x) = \inf\{l(x),\ l \geq f,\ l \text{ linear}\}. \tag{4}$$

and

$$\tilde{f}(x) = \sup\{\frac{(t-x)f(s) + (x-s)f(t)}{t-s},\ a \leq s \leq x \leq t \leq b,\ s < t\}. \tag{5}$$

The first modulus of continuity of a bounded function $f : [a,b] \to \mathbb{R}$ is defined by:

$$\omega(f,h) = \sup\{|f(u) - f(v)|,\ u,v \in [a,b],\ |u-v| \leq h\}, \quad \text{for } h > 0. \tag{6}$$

If, for a given function f we construct the least concave majorant of the map $t \mapsto \omega(f,t)$, $t \in [0, b-a]$, we obtain the least concave majorant of the first modulus of f, which is denoted by $\tilde{\omega}(f,\cdot)$. This modulus was used intensively in various approximation problems.

With these data we have the following result of Kornechuk [4].

Theorem 1. *For any $f \in C[a,b]$ and any $0 < t \leq (b-a)/2$ we have*

$$K(f, C[a,b], Lip1, t) = \frac{1}{2} \cdot \tilde{\omega}(f, 2t). \tag{7}$$

For the proof see Mitjagin and Semenov [6], or De Vore and Lorentz [1].

Note that if we consider the subspace $C^1[a,b \subset C[a,b]$, endowed with the seminorm $|g'| = \max_{x\in[a,b]} |g;(x)|$, for $g \in C^1[a,b]$, we have

$$K(f, C[a,b], Lip1, t) = K(f, C[a,b], C^1[a,b], t). \tag{8}$$

Consequently we also have,

$$K(f, C[a,b], C^1[a,b], t) = \frac{1}{2} \cdot \tilde{\omega}(f, 2t). \tag{9}$$

Mentiom that an analougus theorem was proved by Petree [5], in the case of periodic functions.

The aim of this note is to give a new proof for Theorem 1.

2 Results

Our proof is based essentially on the following lemma.

Lemma 1. *Let the points $a = x_0 < x_1 < \ldots < x_n = b$ and let $y_i \in \mathbf{R}$, $0 \le i \le n$. Suppose that there are the numbers $m \ge 0$ and $q \ge 0$, such that*

$$|y_i - y_j| \le m|x_i - x_j| + 2q, \quad \forall i, j.$$

Then, there are the numbers z_i, $0 \le i \le n$, such that

$$|z_i - y_i| \le q, \ \forall i \quad \textit{and} \quad |z_i - z_j| \le m|x_j - x_i|, \forall i, j.$$

Proof. Let

$$D = \prod_{0 \le i \le n} [y_i - q, y_i + q]$$

and the function $\Theta : D \to \mathbf{R}$,

$$\Theta(z_0, \ldots, z_n) = \max_{i \ne j} \frac{|z_i - z_j|}{|x_i - x_j|}.$$

Since Θ is continuous on a compact, it admits a minimum μ. Let ad absurdum that $\mu > m$.

Denote the elements of D, by $\bar{z} = (z_0, \ldots, z_n)$. Put

$$D_0 = \{\bar{z} \in D, \ \Theta(\bar{z}) = \mu\}.$$

For $\bar{z} \in D_0$, denote

$$p(\bar{z}) = \text{cardinal}\{(i,j), \ 0 \le i < j \le n, \ \frac{|z_j - z_i|}{x_j - x_i} = \mu\}.$$

Then define

$$p_0 = \min_{\bar{z} \in D_0} p(\bar{z}).$$

Choose $\bar{z} \in D_0$, such that $p(\bar{z}) = p_0$. Then choose the pair (i,j), $0 \le i < j \le n$ such that $x_j - x_i$ is the greatest difference of two knots $x_k - x_l$, $0 \le l < k \le n$ for which $\frac{|z_k - z_l|}{x_k - x_l} = \mu$. We can suppose, without any less of generality, that $z_j - z_i > 0$. So, we have

$$\frac{z_j - z_i}{x_j - x_i} = \mu. \tag{10}$$

We have $z_i < y_i + q$ or $z_j > y_j - q$, since otherwise we have: $z_i = y_i + q$ and $z_j = y_j - q$ and then

$$y_j - y_i = z_j - z_i + 2q = \mu(x_j - x_i) + 2q > m(x_j - x_i) + 2q.$$

Contradiction.

Suppose, for a choice, that

$$z_i < y_i + q. \tag{11}$$

Now, suppose, ad absurdum that there is $0 \le k \le n$, $k \ne i, j$, such that

$$\frac{z_i - z_k}{|x_i - x_k|} = \mu.$$

We have three cases.

Case 1. $k < i < j$. We have $z_i - z_k = \mu(x_i - x_k)$ and $z_j - z_i = \mu(x_j - x_i)$. Consequently, $z_j - z_k = \mu(x_j - x_k)$. But $x_j - x_k > x_j - x_i$ and this contradicts the choice of the pair (i, j).

Case 2. $i < k < j$. We have $z_i - z_k = \mu(x_k - x_i)$ and $z_j - z_i = \mu(x_j - x_i)$. Consequently,

$$z_j - z_k = \mu(x_k + x_j - 2x_i) = \mu(x_j - x_k + 2(x_k - x_i)) > \mu(x_j - x_k).$$

Contradiction.

Case 3. $i < j < k$. We have $z_i - z_k = \mu(x_k - x_i)$ and $x_k - x_i > x_j - x_i$. This contradicts the choice of the pair (i, j).

Therefore we proved that

$$\frac{z_i - z_k}{|x_i - x_k|} < \mu, \quad 0 \le k \le n \quad k \ne i, j. \tag{12}$$

From relations (10), (11), (12) it follows that we can choose a number $0 < \rho$, sufficiently small, such that

$$\frac{z_j - z_i - \rho}{x_j - x_i} \ge 0, \tag{13}$$

$$z_i + \rho < y_i + q, \tag{14}$$

$$\frac{z_i + \rho - z_k}{|x_i - x_k|} < \mu, \quad 0 \le k \le n \quad k \ne i, j. \tag{15}$$

Replace component z_i in vector $\bar{z}$ by $z_i + \rho$ and denote $\bar{u}$ the vector which is obtained. Then $\bar{u} \in D$. We have

$$\frac{|u_i - u_k|}{|x_i - x_k|} = \frac{|z_i + \rho - z_k|}{|x_i - x_k|} < \mu, \quad 0 \le k \le n \quad k \ne i$$

and

$$\frac{|u_k - u_l|}{x_k - x_l} = \frac{|z_k - z_l|}{x_k - x_l}, \quad 0 \le l < k \le n \quad k, l \ne i.$$

If $p(\bar{z}) = 1$, then $\Theta(\bar{u}) < \mu$, which contradicts the definition of μ. If $p(\bar{z}) > 1$ we find $p(\bar{u}) = p(\bar{z}) - 1 < p_0$, which contradicts the definition of p_0. Hence we obtained contradiction in both the cases. It follows that the supposition $\mu > m$ is wrong. Then $\mu \le m$. Finally we can chose $\bar{z} \in D$, such that $\Theta(\bar{z}) = \mu$. A such vector $\bar{z}$ satisfies the conditions in the lemma.

□

Proof of Theorem 1

Let $f \in C[a,b]$ be fixed. We denote, for simplicity, $K(t) = K(f, C[a,b], C^1[a,b], t)$ and $\omega(t) = \omega(f,t)$, for $t > 0$. First we prove the inequality:

$$K(t) \geq \frac{1}{2} \cdot \tilde{\omega}(f, 2t),\ t \in [0, (b-a)/2]. \tag{16}$$

The proof is reduced to the following simple facts: i) K is concave and ii) $2K(t) \geq \omega_1(f, 2t)$.

Indeed,let $t_1,\ t_2 > 0$ and $\lambda \in (0,1)$. Let $g \in C^1[a,b]$. We have

$$\begin{aligned} & \lambda K(t_1) + (1-\lambda)K(t_2) \leq \\ \leq\ & \lambda[\|f-g\| + t_1\|g'\|] + (1-\lambda)[\|f-g\| + t_2\|g'\|] = \\ =\ & \|f-g\| + (\lambda t_1 + (1-\lambda)t_2). \end{aligned}$$

Since $g \in C^1[a,b]$ was arbitrary taken we have

$$\lambda K(t_1) + (1-\lambda)K(t_2) \leq K(\lambda t_1 + (1-\lambda)t_2),$$

i.e. the function $K(t)$ is concave.

Also, let $0 < t \leq \frac{1}{2}[b-a]$. Chose $\varepsilon > 0$ arbitrary. We can find a function $g \in C^1[a,b]$ such that $\|f-g\| + t\|g'\| < K(t) + \varepsilon$. Let $u, v \in [a,b]$, such that $|v-u| \leq 2t$. Using Lagrange theorem we have:

$$\begin{aligned} |f(v) - f(u)| &\leq |f(u) - g(u)| + |f(v) - g(v)| + |g(v - g(u)| \leq \\ &\leq 2\|f-g\| + \|g'\| \cdot |v-u| \leq \\ &\leq 2\|f-g\| + 2t\|g'\| \leq \\ &\leq 2(K(t) + \varepsilon). \end{aligned}$$

Since points $u, v \in [a,b]$, $|v-u| \leq 2t$ were taken arbitrarily, it follows that $\omega(f, 2t) \leq 2(K(t) + \varepsilon)$. Since $\varepsilon > 0$ was taken arbitrarily we have $\omega(f, 2t) \leq 2K(t)$.

Now, sice $2K$ is concave and $2K(t) \geq \omega(f, 2t)$, it follows $2K(t) \geq \tilde{\omega}(f, 2t)$, for $t \in [0, (b-a)/2]$. Relation (16) is proved.

We pass now to the converse inequality,

$$K(t) \leq \frac{1}{2} \cdot \tilde{\omega}(f, 2t),\ t \in [0, (b-a)/2], \tag{17}$$

which is the main part of the proof.

For the proof, fix $t \in [0, (b-a)/2]$. Suffice it to show that for any polynomial l, of degree 1 such that $\omega \leq l$, on $[0, b-a]$ and any $\varepsilon > 0$, there is $g \in C^1[a,b]$, such that

$$\|f-g\| + t\|g'\| \leq \frac{1}{2} l(2t) + \varepsilon. \tag{18}$$

Indeed, from (4) we have

$$\tilde{\omega}(2t) = \inf\{l(2t) |\ l \in \Pi_1,\ \omega \leq l\}.$$

Then, for an arbitrary $\varepsilon > 0$ we can choose $l \in \Pi_1$, such that $\omega \le l$ and $l(2t) < \tilde{\omega}(2t)+\varepsilon$. If there exists a function $g \in C^1[a,b]$ with property (18), then we obtain

$$2K(t) \le 2(\|f-g\| + t\|g'\|) \le l(2t) + 2\varepsilon \le \tilde{\omega}(2t) + 3\varepsilon.$$

Since, $\varepsilon > 0$ was arbitrary, we arrive at relation (17).

In what follows we prove the existence of function $g \in C^1[a,b]$ satisfying (18). Let $l \in \Pi_1$, such that $\omega \le l$, on $[0, b-a]$ and let $\varepsilon > 0$. Write l in the form $l(t) = mt + 2q$, $t \in [0, b-a]$.

We can consider only the case $m \ge 0$. Indeed, if we suppose that a function g, satisfying (18) could be chosen for all linear functions $l(t) = mt + 2q$, with $m \ge 0$, then g could be chosen in the particular case when l is a constant function. Let now $l(x) = mx + 2q$, with $m < 0$, such that $\omega \le l$ and let $\varepsilon > 0$. Let l_0 be constant function $l_0(x) = \omega(b-a)$, $x \in [0, b-a]$. We have $\omega \le l_0$, on $[0, b-a]$. From the above it follows that we can choose a function $g \in C^1[a,b]$, such that

$$\|f-g\| + t\|g'\| \le \frac{1}{2} l_0(2t) + \varepsilon.$$

Then we have

$$\|f-g\| + t\|g'\| \le \frac{1}{2} l_0(2t) + \varepsilon = \frac{1}{2}\omega(b-a) + \varepsilon \le \frac{1}{2} l(b-a) + \varepsilon \le \frac{1}{2} l(2t) + \varepsilon.$$

So, consider $m \ge 0$. Clear $q \ge 0$, since $l(0) \le \omega(0)$.

Since f is uniformly continuous, we can find a number $n \in \mathbf{N}$, such that $m\frac{b-a}{n} < \frac{\varepsilon}{4}$ and $|f(u) - f(v)| < \frac{\varepsilon}{4}$, if $|u - v| < \frac{b-a}{n}$. Next consider the equidistant knots $a = x_0 < \ldots < x_n = b$. Denote $y_i = f(x_i)$, $0 \le i \le n$. Note that, for any i, j, we have

$$|y_i - y_j| \le \omega(|x_i - x_j|) \le l(|x_i - x_j|) = m|x_i - x_j| + 2q.$$

Apply Lemma and find points $z_0 < \ldots < z_n$, which satisfy the given properties. Then let $h : [a,b] \to \mathbf{R}$ be the linear piecewise function which take the values $h(x_i) = z_i$, $0 \le i \le n$ and is linear on intervals $[x_i, x_{i+1}]$.

Let $u \in [a,b]$. Let i such that $u \in [x_i, x_{i+1}]$. We have

$$|h(u)-f(u)| \le |h(u)-h(x_i)|+|h(x_i)-f(x_i)|+|f(x_i)-f(u)| \le m\frac{b-a}{n}+|z_i-y_i|+\frac{\varepsilon}{4} \le q+\frac{\varepsilon}{2}.$$

So we obtained

$$\|h-f\| \le q + \frac{\varepsilon}{2} \quad \text{and} \quad |h'(x)| \le m, \quad x \in (x_{i-1}, x_i),\ 1 \le i \le n. \tag{19}$$

Finally, we can find a function $g \in C^1[a,b]$, such that

$$\|g-h\| < \frac{\varepsilon}{2} \text{ and } \|g'\| \le m. \tag{20}$$

Indeed, for $1 \le i \le n$, and $x \in (x_{i-1}, x_i)$ let write $h(x) = \beta_i x + \gamma_i$. Hence $|\beta_i| \le m$ and from the continuity of h, we have $\beta_i x_i + \gamma_i = \beta_{i+1} x_i + \gamma_{i+1}$, if $1 \le i \le n-1$. Choose

$0 < \rho < \frac{b-a}{2n}$. Define the function $g : [a,b] \to \mathbb{R}$, such that $g(x) = \alpha_i(x - x_i + \rho)^2 + \beta_i x + \gamma_i$, where $\alpha_i = \frac{\beta_{i+1} - \beta_i}{4\rho}$, if $x \in (x_i - \rho, x_i + \rho)$, $1 \leq i \leq n-1$ and $g(x) = h(x)$, if $x \in [a,b]$ does not belong to an interval $(x_i - \rho, x + \rho)$. We can verify immediately, that $g \in C^1[a,b]$. Moreover, $g'(x) = \frac{1}{2\rho}[\beta_{i+1}(x - x_i + \rho) - \beta_i(x - x_i - \rho)]$, if $x \in (x_i - \rho, x_i + \rho)$. From this it follows that $g'(x)$ is between β_i and β_{i+1} on this interval. Consequently $\|g'\| \leq m$, for any $0 < \rho < \frac{b-a}{2n}$. Moreover, for $1 \leq i \leq n-1$, and $x \in (x_i - \rho, x_i + \rho)$, we have $|h(x) - g(x)| \leq |h(x_i) - g(x_i)| = \frac{1}{4}(\beta_{i+1} - \beta_i)\rho$. So, if we choose a sufficient small $\rho > 0$ we obtain $\|g - h\| < \frac{\varepsilon}{2}$.

From relations (19) and (20) it follows $\|g - f\| \leq q + \varepsilon$ and hence

$$\|f - g\| + t\|g'\| \leq q + tm + \varepsilon = \frac{1}{2}l(2t) + \varepsilon.$$

Hence relation (18) is proved.

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