Bulletin of the *Transilvania* University of Braşov • Vol 3(52) - 2010
Series III: Mathematics, Informatics, Physics, 101-106

THE UNIVALENCE OF AN INTEGRAL OPERATOR

Virgil PESCAR[1]

Abstract

For analytic functions f in the open unit disk $\mathcal{U}$, an integral operator $E_{\alpha,\beta}$ is defined. In this paper we derive univalence conditions of the integral operator $E_{\alpha,\beta}$.

2000 *Mathematics Subject Classification:* 30C45.
Key words: analytic functions, integral operator, univalence.

1 Introduction

Let $\mathcal{A}$ be the class of functions f which are analytic in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$, with $f(0) = f'(0) - 1 = 0$. Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of functions $f \in \mathcal{A}$, which are univalent in $\mathcal{U}$. We denote by $\mathcal{P}$ the class of functions p which are analytic in $\mathcal{U}$, $p(0) = 1$ and $Re\ p(z) > 0$, for all $z \in \mathcal{U}$.

In this work, we define a new integral operator, which is given by

$$E_{\alpha,\beta}(z) = \int_0^z \left(\frac{f(u)}{u}\right)^{\alpha} (g(u))^{\beta}\, du, \tag{1}$$

for $\alpha,\ \beta$ complex numbers, $f \in \mathcal{A}$ and $g \in \mathcal{P}$.

From (1), for $\beta = 0,\ \alpha$ a complex number and $f \in \mathcal{A}$, we have the integral operator Kim-Merkes [2],

$$I_{\alpha}(z) = \int_0^z \left(\frac{f(u)}{u}\right)^{\alpha} du. \tag{2}$$

For $\alpha = 0$ and β a complex number and $g \in \mathcal{P}$, we obtain the integral operator, which is defined by

$$G_{\beta}(z) = \int_0^z (g(u))^{\beta}\, du. \tag{3}$$

[1]Departament of Mathematics, *Transilvania* University of Braşov, 500091 Braşov, Romania, e-mail: virgilpescar@unitbv.ro

2 Preliminary results

Lemma 1. *(Becker [1]). If function f is analytic in $\mathcal{U}$ and*

$$\left(1-|z|^2\right)\left|\frac{zf''(z)}{f'(z)}\right| \leq 1, \tag{1}$$

for all $z \in \mathcal{U}$, then function f is univalent in $\mathcal{U}$.

Lemma 2. *(Schwarz [3]). Let f be the function regular in the disk $\mathcal{U}_R = \{z \in \mathbb{C} : |z| < R\}$ with $|f(z)| < M$, M fixed. If f has in $z = 0$ one zero with multiply $\geq m$, then*

$$|f(z)| \leq \frac{M}{R^m}|z|^m, \ \left(z \in \mathcal{U}_R\right), \tag{2}$$

the equality (in the inequality (2) for $z \neq 0$) can hold only if

$$f(z) = e^{i\theta}\frac{M}{R^m}z^m,$$

where θ is constant.

3 Main results

Theorem 1. *Let α β be complex numbers, $M_1, \ M_2$ positive real numbers and the functions $f \in \mathcal{A}, \ f(z) = z + a_2z^2 + a_3z^3 + \dots$ and $g \in \mathcal{P}$, $g(z) = 1 + b_1z + b_2z^2 + \dots$.*

If

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq M_1, \quad (z \in \mathcal{U}), \tag{1}$$

$$\left|\frac{zg'(z)}{g(z)}\right| \leq M_2, \quad (z \in \mathcal{U}), \tag{2}$$

and

$$|\alpha|M_1 + |\beta|M_2 \leq \frac{3\sqrt{3}}{2}, \tag{3}$$

then the function

$$E_{\alpha,\beta}(z) = \int_0^z \left(\frac{f(u)}{u}\right)^{\alpha} (g(u))^{\beta}\, du, \tag{4}$$

is in class $\mathcal{S}$.

Proof. Function $E_{\alpha,\beta}(z)$ is regular in $\mathcal{U}$ and $E_{\alpha,\beta}(0) = E'_{\alpha,\beta}(0) - 1 = 0$. We have:

$$\frac{zE''_{\alpha,\beta}(z)}{E'_{\alpha,\beta}(z)} = \alpha\left(\frac{zf'(z)}{f(z)} - 1\right) + \beta\frac{zg'(z)}{g(z)}, \tag{5}$$

for all $z \in \mathcal{U}$.

From (5) we obtain:

$$\left(1-|z|^2\right)\left|\frac{zE''_{\alpha,\beta}(z)}{E'_{\alpha,\beta}(z)}\right| \le \left(1-|z|^2\right)\left[|\alpha|\left|\frac{zf'(z)}{f(z)} - 1\right| + |\beta|\left|\frac{zg'(z)}{g(z)}\right|\right], \tag{6}$$

for all $z \in \mathcal{U}$. By Lemma 2, from (1) and (2) we get

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \le M_1|z|, \quad (z \in \mathcal{U}), \tag{7}$$

$$\left|\frac{zg'(z)}{g(z)}\right| \le M_2|z|, \quad (z \in \mathcal{U}) \tag{8}$$

and by (6) we have

$$\left(1-|z|^2\right)\left|\frac{zE''_{\alpha,\beta}(z)}{E'_{\alpha,\beta}(z)}\right| \le \left(1-|z|^2\right)|z|\left(|\alpha|M_1 + |\beta|M_2\right), \tag{9}$$

for all $z \in \mathcal{U}$. Since

$$\max_{|z|\le 1}\left[\left(1-|z|^2\right)|z|\right] = \frac{2}{3\sqrt{3}},$$

by (9) and (3) we obtain

$$\left(1-|z|^2\right)\left|\frac{zE''_{\alpha,\beta}(z)}{E'_{\alpha,\beta}(z)}\right| \le 1, \quad (z \in \mathcal{U}). \tag{10}$$

By Lemma 1, we obtain that integral operator $E_{\alpha,\beta}$ belongs to class $\mathcal{S}$. □

Theorem 2. *Let α, β be complex numbers and the functions $f \in \mathcal{S}$, $g \in \mathcal{P}$, $f(z) = z + a_2z^2 + a_3z^3 + \ldots$, $g(z) = 1 + b_1z + b_2z^2 + \ldots$.*

If

$$2|\alpha| + |\beta| \le \frac{1}{2}, \tag{11}$$

then the integral operator $E_{\alpha,\beta}$, defined by (1), is in class $\mathcal{S}$.

Proof. From (5) we obtain:

$$\left(1-|z|^2\right)\left|\frac{zE''_{\alpha,\beta}(z)}{E'_{\alpha,\beta}(z)}\right| \leq \left(1-|z|^2\right)\left[|\alpha|\left(\left|\frac{zf'(z)}{f(z)}\right|+1\right)+|\beta|\left|\frac{zg'(z)}{g(z)}\right|\right] \tag{12}$$

for all $z \in \mathcal{U}$. Since $f \in \mathcal{S}$, $g \in \mathcal{P}$, we have:

$$\left|\frac{zf'(z)}{f(z)}\right| \leq \frac{1+|z|}{1-|z|}, \quad (z \in \mathcal{U}), \tag{13}$$

$$\left|\frac{zg'(z)}{g(z)}\right| \leq \frac{2|z|}{1-|z|^2}, \quad (z \in \mathcal{U}) \tag{14}$$

and hence, by (12) we get

$$\left(1-|z|^2\right)\left|\frac{zE''_{\alpha,\beta}(z)}{E'_{\alpha,\beta}(z)}\right| \leq 4|\alpha|+2|\beta|, \quad (z \in \mathcal{U}). \tag{15}$$

From (15), (11) we obtain

$$\left(1-|z|^2\right)\left|\frac{zE''_{\alpha,\beta}(z)}{E'_{\alpha,\beta}(z)}\right| \leq 1, \quad (z \in \mathcal{U}), \tag{16}$$

and by Lemma 1, it results that $E_{\alpha,\beta} \in \mathcal{S}$. □

4 Corollaries

Corollary 1. *Let α be a complex number, $\alpha \neq 0$ and $f \in \mathcal{A}$, $f(z) = z + a_2z^2 + a_3z^3 + \ldots$. If*

$$\left|\frac{zf'(z)}{f(z)}-1\right| \leq \frac{3\sqrt{3}}{2|\alpha|}, \quad (z \in \mathcal{U}), \tag{1}$$

then the integral operator I_α, defined by (2), is in class $\mathcal{S}$.

Proof. For $\beta = 0$, from Theorem 1 we obtain Corollary 1. □

Corollary 2. *Let β be a complex number, $\beta \neq 0$ and $g \in \mathcal{P}$,*

$$g(z) = 1 + b_1z + b_2z^2 + \ldots.$$

If

$$\left|\frac{zg'(z)}{g(z)}\right| \leq \frac{3\sqrt{3}}{2|\beta|}, \quad (z \in \mathcal{U}), \tag{2}$$

then the integral operator G_β defined by (3), belongs to class $\mathcal{S}$.

Corollary 3. *Let α be a complex number and function $f \in \mathcal{S}$,*
$f(z) = z + a_2 z^2 + a_3 z^3 + \dots$.

If

$$|\alpha| \leq \frac{1}{4}, \tag{3}$$

then the integral operator I_α defined in (2), is in class $\mathcal{S}$.

Proof. We take $\beta = 0$ in Theorem 2, we obtain the Corollary 3. □

Corollary 4. *Let β be a complex number and function $g \in \mathcal{P}$,*
$g(z) = 1 + b_1 z + b_2 z^2 + \dots$.

If

$$|\beta| \leq \frac{1}{2}, \tag{4}$$

then the integral operator G_β defined in (3), is in class $\mathcal{S}$.

Proof. We take $\alpha = 0$ in Theorem 2. □

5 References

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