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GENERAL CONFORMAL METRICAL N-LINEAR CONNECTIONS ON A GENERALIZED HAMILTON SPACE

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Abstract

In the present paper starting from the notion of conformal metrical d-structure we define the notion of general conformal metrical N-linear connection on a generalized Hamilton space. We determine the set of all general conformal metrical N-linear connections on a generalized Hamilton space, in the case when the nonlinear connection is arbitrary and we find important particular cases.

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1 Indroduction

The differential geometry of the second order cotangent bundle $\left(T^{*2}M, \pi^{*^2}, M\right)$ was introduced and studied by R. Miron in [2] , R. Miron, D. Hrimiuc, H. Shimada, , V.S. Sabău in [4], Gh. Atanasiu and M. Târnoveanu in [1], etc.

In the present section we keep the general setting from R. Miron, D. Hrimiuc, H. Shimada, V.S. Sabău, [4] and subsequently we recall only some needed notions. For more details see [4] .

Let M be a real $n-$dimensional manifold and let $\left(T^{*^2}M, \pi^{*^2}, M\right)$ be the dual of the 2−tangent bundle, or 2−cotangent bundle. A point $u \in T^{*^2}M$ can be written in the form $u = (x, y, p)$, having the local coordinates $\left(x^i, y^i, p_i\right), (i = 1, 2, ..., n)$.

A change of local coordinates on the $3n$ dimensional manifold $T^{*^2}M$ is

$$\begin{cases} \bar{x}^i = \bar{x}^i\left(x^1, ..., x^n\right), \det\left(\frac{\partial \bar{x}^i}{\partial x^j}\right) \neq 0, \\ \bar{y}^i = \frac{\partial \bar{x}^i}{\partial x^j} \cdot y^j, \\ \bar{p}_i = \frac{\partial x^j}{\partial \bar{x}^i} \cdot p_j, (i, j = 1, 2, ..., n) . \end{cases} \tag{1.1}$$

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We denote by $T^{*^2}M = T^{*^2}M \backslash \{0\}$, where $0 : M \longrightarrow T^{*^2}M$ is the null section of projection π^{*^2}.

Let us consider the tangent bundle of the differentiable manifold $T^{*^2}M$, $(TT^{*^2}M, \tau^{*^2}, T^{*^2}M)$, where τ^{*^2} is the canonical projection and the vertical distribution $V : u \in T^{*^2}M \longrightarrow V(u) \subset T_u T^{*^2}M$, locally generated by vector fields $\left\{ \frac{\partial}{\partial y^i}\Big|_u, \frac{\partial}{\partial p_i}\Big|_u \right\}, \forall u \in T^{*^2}M$.

We denote with N a nonlinear connection on the manifold $T^{*2}M$, with the local coefficients $\left(N^j{}_i(x,y,p), N_{ij}(x,y,p)\right), (i,j = 1,2,...,n)$.

Hence, the tangent space of $T^{*2}M$ in point $u \in T^{*2}M$ is given by the direct sum of vector spaces:

$$T_u T^{*2}M = N(u) \oplus W_1(u) \oplus W_2(u), \forall u \in T^{*2}M. \tag{1.2}$$

A local adapted basis to the direct decomposition (1.2) is given by:

$$\left\{ \frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}, \frac{\partial}{\partial p_i} \right\}, (i = 1,2,...,n), \tag{1.3}$$

where:

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j{}_i \frac{\partial}{\partial y^j} + N_{ij} \frac{\partial}{\partial p_j}. \tag{1.4}$$

With respect to the coordinates transformations (1.1), we have the rules:

$$\frac{\delta}{\delta x^i} = \frac{\partial \bar{x}^j}{\partial x^i} \frac{\delta}{\delta \bar{x}^j}; \quad \frac{\partial}{\partial y^i} = \frac{\partial \bar{x}^j}{\partial x^i} \cdot \frac{\partial}{\partial \bar{y}^j}; \quad \frac{\partial}{\partial p_i} = \frac{\partial x^i}{\partial \bar{x}^j} \cdot \frac{\partial}{\partial \bar{p}_j}. \tag{1.4'}$$

The dual basis of the adapted basis (1.3) is given by:

$$\left\{ \delta x^i, \delta y^i, \delta p_i \right\}, \tag{1.5}$$

where:

$$\delta x^i = dx^i, \; \delta y^i = dy^i + N^i{}_j dx^j, \; \delta p_i = dp_i - N_{ji} dx^j. \tag{1.5'}$$

With respect to (1.1), the covector fields (1.5) are transformed by the rules:

$$\delta \bar{x}^i = \frac{\partial \bar{x}^i}{\partial x^j} \delta x^j, \; \delta \bar{y}^i = \frac{\partial \bar{x}^i}{\partial x^j} \delta y^j, \; \delta \bar{p}_i = \frac{\partial x^j}{\partial \bar{x}^i} \delta p_j. \tag{1.5''}$$

Let D be an $N-$linear connection on $T^{*2}M$, with the local coefficients in the adapted basis: $D\Gamma(N) = \left(H^i{}_{jk}, C^i{}_{jk}, C_i{}^{jk}\right)$.

An N-linear connection: $D\Gamma(N) = \left(H^i{}_{jk}, C^i{}_{jk}, C_i{}^{jk}\right)$. determines the $h-, w_1-, w_2-$ covariant derivatives in the tensor algebra of d-tensor fields.

Definition 1.1. ([4]) *A generalized Hamilton space of order two is a pair* $GH^{(2)n} = \left(M, g^{ij}(x,y,p)\right)$, *where:*

1° g^{ij} is a d− tensor field of type $(2,0)$, symmetric and nondegenerate on the manifold $T^{*^2}M$.

2° The quadratic form $g^{ij}X_iX_j$ has a constant signature on $T^{*^2}M$.

g^{ij} is called the fundamental tensor or metric tensor of space $GH^{(2)n}$.

In the case when $T^{*^2}M$ is a paracompact manifold then on $T^{*^2}M$ there exist the metric tensors $g^{ij}(x,y,p)$ positively defined such that $\left(M, g^{ij}\right)$ is a generalized Hamilton space.

The covariant tensor field g_{ij} is obtained from the equations:

$$g_{ij}g^{jk} = \delta_i^k \tag{1.6}$$

g_{ij} is a symmetric, nondegenerate and covariant of order two, $d-$tensor field.

Definition 1.2. ([4]) *An $N-$linear connection $D\Gamma(N)$ is called metrical with respect to $GH-$metric g^{ij} if:*

$$g^{ij}{}_{|k} = 0, \quad g^{ij}|_k = 0, \quad g^{ij}|^k = 0. \tag{1.7}$$

The tensorial equations (1.7) imply:

$$g_{ij|k} = 0, \quad g_{ij}|_k = 0, \quad g_{ij}|^k = 0. \tag{1.8}$$

The operators of Obata's type are given by:

$$\Omega_{hk}^{ij} = \frac{1}{2}\left(\delta_h^i\delta_k^j - g_{hk}g^{ij}\right), \; \Omega^{*ij}_{\ \ hk} = \frac{1}{2}\left(\delta_h^i\delta_k^j + g_{hk}g^{ij}\right). \tag{1.9}$$

Let $\mathcal{S}_2(T^{*^2}M)$ be the set of all symmetric d-tensor fields, of the type $(0,2)$ on $T^{*^2}M$. As it is easily shown, the relation for $a_{ij}, b_{ij} \in \mathcal{S}_2(T^{*^2}M)$ defined by:

$$(a_{ij} \sim b_{ij}) \Leftrightarrow ((\exists)\lambda(x,y,p) \in \mathcal{F}(T^{*^2}M),\; a_{ij}(x,y,p) = e^{2\lambda(x,y,p)}b_{ij}(x,y,p),) \tag{1.10}$$

is an equivalence relation on $\mathcal{S}_2(T^{*^2}M)$.

Definition 1.3. *The equivalent class $\hat{g}$ of $\mathcal{S}_2(T^{*^2}M)/_\sim$ to which the fundamental d-tensor field g_{ij} belongs, is called conformal metrical d-structure on $T^{*^2}M$.*

Thus:

$$\hat{g} = \{g' | g'_{ij}(x,y,p) = e^{2\lambda(x,y,p)}g_{ij}(x,y,p), \lambda(x,y,p) \in \mathcal{F}(T^{*^2}M)\}. \tag{1.11}$$

2 General conformal metrical N-linear connections in a generalized Hamilton space

Definition 2.1. *An N-linear connection, D, with local coefficients: $D\Gamma(N) = \left(H^i{}_{jk}, C^i{}_{jk}, C_i{}^{jk}\right)$, is called general conformal metrical N-linear connection with respect to $\hat{g}$ if:*

$$g_{ij|k} = K_{ijk}, \;\; g_{ij}|_k = Q_{ijk}, \;\; g_{ij}|^k = \dot{Q}_{ij}{}^k, \tag{2.1}$$

where $_{|k}$, $|_k$ and $|^k$ denote the $h-$, w_1- and w_2- covariant derivatives with respect to D and $K_{ijk}, Q_{ijk}, \dot{Q}_{ij}{}^k$ are arbitrary tensor fields on $T^{*^2}M$ of the types (0,3), (0,3) and (2,1) respectively, with the properties:

$$K_{ijk} = K_{jik}, Q_{ijk} = Q_{jik}, \dot{Q}_{ij}{}^k = \dot{Q}_{ji}{}^k. \tag{2.2}$$

Definition 2.2. *An N-linear connection, D, with local coefficients:* $D\Gamma(N) = \left(H^i{}_{jk}, C^i{}_{jk}, C_i{}^{jk}\right)$*, for which there exists the 1-form* ω*,* $\omega = \omega_i dx^i + \dot{\omega}_i \delta y^i + \ddot{\omega}^i \delta p_i$*, such that:*

$$\begin{cases} g_{ij|k} = 2\omega_k g_{ij}, \quad g_{ij}|_k = 2\dot{\omega}_k g_{ij}, \\ g_{ij}|^k = 2\ddot{\omega}^k g_{ij}, \end{cases} \tag{2.3}$$

where $|_k$*,* $|_k$ *and* $|^k$ *denote the* $h-$*,* w_1- *and* w_2- *covariant derivatives with respect to D is called conformal metrical N-linear connection, with respect to the conformal metrical d-structure* $\hat{g}$*, corresponding to the 1-form* ω *and is denoted by:* $D\Gamma(N, \omega)$*.*

We shall determine the set of all general conformal metrical N-linear connections, with respect to $\hat{g}$.

Let $\overset{0}{D}\,\Gamma(\overset{0}{N}) = \left(\overset{0}{H^i}{}_{jk}, \overset{0}{C^i}{}_{jk}, \overset{0}{C_i}{}^{jk}\right)$ be the local coefficients of a fixed $\overset{0}{N}$- linear connection $\overset{0}{D}$, where $(\overset{0}{N^j}{}_i(x,y,p), \overset{0}{N}_{ij}(x,y,p)), (i,j = 1,2,...,n)$ are the local coefficients of the nonlinear connection $\overset{0}{N}$.

Then any N-linear connection D, with the local coefficients $D\Gamma(N) = \left(H^i{}_{jk}, C^i{}_{jk}, C_i^{jk}\right)$, where $(N^j{}_i(x,y,p), N_{ij}(x,y,p)), (i,j = 1,2,...,n)$ are the local coefficients of the nonlinear connection N, can be expressed in the form ([6]):

$$\begin{cases} N^i_j = \overset{0}{N^i}{}_j - A^i_j, \\ N_{ij} = \overset{0}{N}_{ij} - A_{ij}, \\ H^i_{jk} = \overset{0}{H^i}{}_{jk} + A^l_k \overset{0}{C^i}{}_{jl} - A_{kl}\overset{0}{C_j}{}^{il} - B^i_{jk}, \\ C^i_{jk} = \overset{0}{C^i}{}_{jk} - D^i_{jk}, \\ C_i^{jk} = \overset{0}{C_i}{}^{jk} - D_i^{jk}, \ (i,j,k = 1,2,...,n), \end{cases} \tag{2.4}$$

with

$$A^k{}_{\underset{i|j}{0}} = 0, \ A_{\underset{ik|j}{0}} = 0 (i,j,k = 1,2,...,n), \tag{2.5}$$

where $\overset{0}{|}$ denotes the h-covariant derivative with respect to $\overset{0}{D}$ and $(A^i_j, A_{ij}, B^i_{jk}, D^i_{jk}, D_i^{jk})$ are the components of the difference tensor fields of D from $\overset{0}{D}$.

Using relations (2.1), (2.4), (1.4) and Theorem 1 given by R.Miron in ([3]) for the case of Finsler connections we obtain:

Theorem 2.1. *Let* $\overset{0}{D}$ *be a given* $\overset{0}{N}$ *-linear connection, with local coefficients* $\overset{0}{D}\,\Gamma(\overset{0}{N}) = \left(\overset{0}{H^i}{}_{jk}, \overset{0}{C^i}{}_{jk}, \overset{0}{C_i}{}^{jk}\right)$*. The set of all general conformal metrical N-linear connections, with respect to* $\hat{g}$*, with local coefficients* $D\Gamma(N) = \left(H^i{}_{jk}, C^i{}_{jk}, C_i^{jk}\right)$ *is given by:*

$$\begin{cases} N^i_{\ j} = \overset{0}{N}{}^i_{\ j} - X^i_{\ j}, \\ N_{ij} = \overset{0}{N}_{ij} - X_{ij}, \\ H^i_{\ jk} = \overset{0}{H}{}^i_{\ jk} + X^l_{\ k}\overset{0}{C}{}^i_{\ jl} - X_{kl}\overset{0}{C}{}^{il}_j + \frac{1}{2}g^{im}(g_{mj\overset{0}{|}k} - K_{mjk} + g_{mj} \overset{0}{|}_l X^l_{\ k} - \\ -g_{mj} \overset{0}{|}{}^l X_{kl}) + \Omega^{ir}_{sj}X^s_{\ rk}, \\ C^i_{\ jk} = \overset{0}{C}{}^i_{\ jk} + \frac{1}{2}g^{im}(g_{mj}\overset{0}{|}_k - Q_{mjk}) + \Omega^{ir}_{sj}Y^s_{\ rk}, \\ C^{jk}_i = \overset{0}{C}{}^{jk}_i + \frac{1}{2}g^{mj}(g_{mj}\overset{0}{|}{}^k - \dot{Q}_{mi}{}^k) + \Omega^{jr}_{si}Z_r^{\ sk}, \end{cases} \tag{2.6}$$

with:

$$X^k{}_{\overset{0}{i|j}} = 0,\ X_{\overset{0}{ik|j}} = 0,\ (i,j,k = 1,2,...,n), \tag{2.7}$$

where $\overset{0}{|}_k$, $\overset{0}{\Big|}_k$ *and* $\overset{0}{|}{}^k$ *denote the* $h-$, w_1- *and* w_2- *covariant derivatives with respect to* $\overset{0}{D}$, $X^i_{\ j}, X_{ij}, X^i_{\ jk}, Y^i_{jk}, Z^{jk}_i$ *are arbitrary d-tensor fields and* $K_{ijk}, Q_{ijk}, \dot{Q}_{mi}{}^k$ *are arbitrary d-tensor fields of the types (0,3), (0,3) and (2,1) respectively, with the properties (2.2).*

Particular cases:

1. If we take $K_{ijk} = 2\omega_k g_{ij}, Q_{ijk} = 2\dot{\omega}_k g_{ij}, \dot{Q}_{ij}{}^k = 2\ddot{\omega}^k g_{ij}$ in Theorem 2.1 obtain:

Theorem 2.2. *Let* $\overset{0}{D}$ *be a given* $\overset{0}{N}$*-linear connection, with local coefficients* $\overset{0}{D}\ \Gamma(\overset{0}{N}) = \left(\overset{0}{H}{}^i_{\ jk}, \overset{0}{C}{}^i_{\ jk}, \overset{0}{C}{}^{jk}_i\right)$. *The set of all conformal metrical N-linear connections with respect to* $\hat{g}$, *corresponding to the 1-form* ω, *with local coefficients* $D\Gamma(N,\omega) = \left(H^i_{\ jk}, C^i_{\ jk}, C^{jk}_i\right)$ *is given by:*

$$\begin{cases} N^i_{\ j} = \overset{0}{N}{}^i_{\ j} - X^i_{\ j}, N_{ij} = \overset{0}{N}_{ij} - X_{ij}, \\ H^i_{\ jk} = \overset{0}{H}{}^i_{\ jk} + X^l_{\ k}\overset{0}{C}{}^i_{\ jl} - X_{kl}\overset{0}{C}{}^{il}_j + \\ +\frac{1}{2}g^{im}(g_{mj\overset{0}{|}k} - 2\omega_k g_{mj} + g_{mj} \overset{0}{|}_l X^l_{\ k} - \\ -g_{mj} \overset{0}{|}{}^l X_{kl}) + \Omega^{ir}_{sj}X^s_{\ rk}, \\ C^i_{\ jk} = \overset{0}{C}{}^i_{\ jk} + \frac{1}{2}g^{im}(g_{mj}\overset{0}{|}_k - 2\dot{\omega}_k g_{mj}) + \\ +\Omega^{ir}_{sj}Y^s_{\ rk}, \\ C^{jk}_i = \overset{0}{C}{}^{jk}_i + \frac{1}{2}g^{mj}(g_{mi}\overset{0}{|}{}^k - 2\ddot{\omega}^k g_{mi}) + \\ +\Omega^{rj}_{si}Z_r^{\ sk},\ (i,j,k = 1,2,...,n), \end{cases} \tag{2.8}$$

with:

$$X^{k}{}_{\overset{0}{i|j}} = 0,\ X_{\overset{0}{ik|j}} = 0, (i,j,k = 1,2,...,n), \tag{2.9}$$

where $\overset{0}{|}_k$, $\overset{0}{|}_k$ *and* $\overset{0}{|}^k$ *denote the* $h-$, w_1- *and* w_2- *covariant derivatives with respect to* $\overset{0}{D}$, $X^i{}_j, X_{ij}, X^i{}_{jk}, Y^i{}_{jk}, Z_i^{jk}$ *are arbitrary d-tensor fields,* $\omega = \omega_i dx^i + \dot{\omega}_i \delta y^i + \ddot{\omega}^i \delta p_i$ *is an arbitrary 1-form and* Ω *is the operator of Obata's type given by (1.9).*

2. If $X^i{}_j = X_{ij} = X^i{}_{jk} = Y^i{}_{jk} = Z_i^{jk} = 0$, in Theorem 2.1 we have:

Theorem 2.3. *Let* $\overset{0}{D}$ *be a given* $\overset{0}{N}$*-linear connection with local coefficients* $\overset{0}{D}\,\Gamma(\overset{0}{N}) = \left(\overset{0}{H^i}{}_{jk}, \overset{0}{C^i}{}_{jk}, \overset{0}{C_i^{jk}}\right)$. *Then the following N-linear conection K with local coefficients* $K\Gamma(N) = \left(H^i{}_{jk}, C^i{}_{jk}, C_i^{jk}\right)$ *given by (2.10) is general conformal metrical with respect to* $\hat{g}$*:*

$$\begin{cases} H^i_{jk} = \overset{0}{H^i}{}_{jk} + \frac{1}{2} g^{im}(g_{\overset{0}{mj|k}} - K_{mjk}), \\ C^i_{jk} = \overset{0}{C^i}{}_{jk} + \frac{1}{2} g^{im}(g_{mj}\overset{0}{|}_k - Q_{mjk}), \\ C_i^{jk} = \overset{0}{C_i^{jk}} + \frac{1}{2} g^{jm}(g_{mi}\overset{0}{|}^k - \dot{Q}_{mi}{}^k), \end{cases} \tag{2.10}$$

where $\overset{0}{|}_k$, $\overset{0}{|}_k$ *and* $\overset{0}{|}^k$ *denote the* $h-$, w_1- *and* w_2- *covariant derivatives with respect to* $\overset{0}{D}$, *and* $K_{ijk}, Q_{ijk}, \dot{Q}_{mi}{}^k$ *are arbitrary d-tensor fields of the types (0,3), (0,3) and (2,1) respectively, with the properties (2.2).*

3. If we take a general conformal metrical N-linear connection with respect to $\hat{g}$ as $\overset{0}{D}$, in Theorem 2.1. we have:

Theorem 2.4. *Let* $\overset{0}{D}$ *be on* $T^{*^2}M$ *a fixed general conformal metrical N-linear connection with respect to* $\hat{g}$ *with the local coefficients* $\overset{0}{D}\,\Gamma(\overset{0}{N}) = \left(\overset{0}{H^i}{}_{jk}, \overset{0}{C^i}{}_{jk}, \overset{0}{C_i^{jk}}\right)$. *The set of all general conformal metrical N-linear connections, with respect to* $\hat{g}$, *with local coefficients* $D\Gamma(N) = \left(H^i{}_{jk}, C^i{}_{jk}, C_i^{jk}\right)$ *is given by:*

$$\begin{cases} N^i_j = \overset{0}{N^i}{}_j - X^i_j, \\ N_{ij} = \overset{0}{N}_{ij} - X_{ij}, \\ H^i_{jk} = \overset{0}{H^i}{}_{jk} + \left(\overset{0}{C^i}{}_{jl} + \frac{1}{2} g^{im} Q_{mjl}\right) X^l_k - \left(\overset{0}{C_j^{il}} + \frac{1}{2} g^{im} \dot{Q}_{mj}{}^l\right) X_{kl} + \Omega^{ir}_{sj} X^s_{rk}, \\ C^i_{jk} = \overset{0}{C^i}{}_{jk} + \Omega^{ir}_{sj} Y^s{}_{rk}, \\ C_i^{jk} = \overset{0}{C_i^{jk}} + \Omega^{jr}_{si} Z_r{}^{sk}, \quad (i,j,k = 1,2,...,n), \end{cases} \tag{2.11}$$

with

$$X^k{}_{\overset{0}{i|j}} = 0,\ X_{\overset{0}{ik|j}} = 0,\ (i, j, k = 1, 2, ..., n), \tag{2.12}$$

where $\overset{0}{|}_k$, $\overset{0}{|}_k$ *and* $\overset{0}{|}^k$ *denote the* $h-$, w_1- *and* w_2- *covariant derivatives with respect to* $\overset{0}{D}$, $K_{ijk}, Q_{ijk}, \dot{Q}_{mi}{}^k$ *are arbitrary d-tensor fields of the types (0,3), (0,3) and (2,1) respectively, with the properties (2.2).*

4. If $K_{ijk} = Q_{ijk} = \dot{Q}_{mi}{}^k = 0$ and $X^i_j = X_{ij} = 0$ in Theorem 2.1 we obtain the set of all metrical N-linear connection in the case when the nonlinear connection is fixed, result given in ([4]).

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