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ON THE PROPERTIES OF A CERTAIN CLASS OF ANALYTIC FUNCTIONS

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Abstract

In this paper we investigate some properties of a certain class of analytic functions defined by a differential operator.

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1 Indroduction

Let $\mathcal{H}$ be the class of analytic functions in the unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$.
Denote by $\mathcal{A}$ the class of functions f in $\mathcal{H}$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \mathcal{U}). \tag{1}$$

Let $\mathcal{P}_{\alpha,\beta}$ be the class of functions $p \in \mathcal{H}$ with $p(0) = 1$ such that

$$\Re(e^{i\beta} p(z)) > \alpha \cos \beta \quad (z \in \mathcal{U}) \tag{2}$$

for some real α, β with $0 \leq \alpha < 1$ and $|\beta| < \dfrac{\pi}{2}$.

The function

$$p_{\alpha,\beta}(z) = \frac{1 + [2(1-\alpha)e^{-i\beta} \cos \beta - 1]z}{1 - z} \quad (z \in \mathcal{U}) \tag{3}$$

plays an important role in the class $\mathcal{P}_{\alpha,\beta}$.

Note that for $\alpha = \beta = 0$ the class $\mathcal{P}_{\alpha,\beta}$ reduces to the well-known Carathéodory class of functions which will be denoted by $\mathcal{P}$.

Making use of the properties of functions in the class $\mathcal{P}$ and also of the condition (2), it is easy to obtain the following properties of the functions in the class $\mathcal{P}_{\alpha,\beta}$ (see [7]).

Lemma 1. *Let* $p \in \mathcal{H}$ *with* $p(0) = 1$ *and let* α, β *be real numbers such that* $0 \leq \alpha < 1$ *and* $|\beta| < \dfrac{\pi}{2}$. *Then*

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i. $p \in \mathcal{P}_{\alpha,\beta}$ *if and only if* $\dfrac{e^{i\beta}p(z) - (\alpha\cos\beta + i\sin\beta)}{(1-\alpha)\cos\beta} \in \mathcal{P}$.

ii. $p \in \mathcal{P}_{\alpha,\beta}$ *if and only if* $p \prec p_{\alpha,\beta}$ *in* $\mathcal{U}$*, where* $p_{\alpha,\beta}$ *is defined in (3). The symbol "*$\prec$*" stands for subordonation.*

iii. $p \in \mathcal{P}_{\alpha,\beta}$ *if and only if there exists a function* $w \in \mathcal{H}$ *with* $w(0) = 0$ *and* $|w(z)| < 1$, $z \in \mathcal{U}$ *such that*

$$w(z) = \frac{e^{i\beta}p(z) - e^{i\beta}}{e^{i\beta}p(z) - 2\alpha\cos\beta + e^{-i\beta}} \quad (z \in \mathcal{U}). \tag{4}$$

iv. *If* $p \in \mathcal{P}_{\alpha,\beta}$*, there exists a Borel probability measure* μ *on the unit circle* $T = \{x \in \mathbb{C} : |x| = 1\}$ *such that*

$$p(z) = \int_{|x|=1} \frac{1 + [2(1-\alpha)e^{-i\beta}\cos\beta - 1]xz}{1 - xz} d\mu(x) \quad (z \in \mathcal{U}). \tag{5}$$

v. *If* $p \in \mathcal{P}_{\alpha,\beta}$ *and* $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$*, then*

$$|p_n| \le 2(1-\alpha)\cos\beta \quad (n \ge 1). \tag{6}$$

If $f \in \mathcal{A}$ is given by (1) and $g \in \mathcal{A}$ is given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n$$

then the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z) \quad (z \in \mathcal{U}).$$

Consider $f, g \in \mathcal{H}$. We say that f is subordinate to g, written $f \prec g$, if there exists an analytic function w in $\mathcal{U}$ with $w(0) = 0$ and $|w(z)| < 1$, $z \in \mathcal{U}$ such that $f(z) = g(w(z))$, $z \in \mathcal{U}$. It is known that if $f \prec g$, then $f(0) = g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$. In particular, if g is univalent in $\mathcal{U}$ we have the following equivalence:

$$f(z) \prec g(z) \text{ if and only if } f(0) = g(0) \text{ and } f(\mathcal{U}) \subset g(\mathcal{U}).$$

For a function $f \in \mathcal{A}$ we consider the following differential operator introduced by Răducanu and Orhan in [9]:

$$D^0_{\lambda\mu} f(z) = f(z)$$

$$D^1_{\lambda\mu} f(z) = D_{\lambda\mu} f(z) = \lambda\mu z^2 f''(z) + (\lambda - \mu) z f'(z) + (1 - \lambda + \mu) f(z)$$

$$D_{\lambda\mu}^m f(z) = D_{\lambda\mu}\left(D_{\lambda\mu}^{m-1} f(z)\right) \tag{7}$$

where $0 \le \mu \le \lambda$ and $m \in \mathbb{N} := \{1, 2, \ldots\}$.

If the function f is given by (1) then, from the definition of $D_{\lambda\mu}^m f$, we see that:

$$D_{\lambda\mu}^m f(z) = z + \sum_{n=2}^{\infty} A_n(\lambda, \mu, m) a_n z^n \tag{8}$$

where

$$A_n(\lambda, \mu, m) = [1 + (\lambda\mu n + \lambda - \mu)(n-1)]^m \quad (n \ge 2) \tag{9}$$

If $\lambda = 1$ and $\mu = 0$, we get Sălăgean differential operator [10] and if $\mu = 0$, we obtain the differential operator defined by Al-Oboudi [1].

By using the differential operator $D_{\lambda\mu}^m f$, we define the following class of functions.

Definition 1. *A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{C}_{\lambda\mu}^m(\alpha, \beta)$ if it satisfies the inequality*

$$\Re\left[e^{i\beta}\frac{D_{\lambda\mu}^m f(z)}{z}\right] > \alpha\cos\beta \ , \quad (z \in \mathcal{U}) \tag{10}$$

for $0 \le \alpha < 1$, $\beta \in \mathbb{R}$ with $|\beta| < \dfrac{\pi}{2}$, $0 \le \mu \le \lambda$ and $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Note that $\mathcal{C}_{\lambda\mu}^0(\alpha, o)$ reduces to the class of functions investigated in [14], [3], [4].

In this paper we derive some properties of the class $\mathcal{C}_{\lambda\mu}^m(\alpha, \beta)$. In particular, for this class of functions, we obtain characterization properties, coefficient estimates, distortion theorem and a subordination result.

2 Characterization properties

In this section we obtain three characterization properties for the class $\mathcal{C}_{\lambda\mu}^m(\alpha, \beta)$.

Theorem 1. *Let $f \in \mathcal{A}$. If*

$$\left|\frac{D_{\lambda\mu}^m f(z)}{z} - 1\right| < 1 - \gamma \quad (z \in \mathcal{U}) \tag{11}$$

for $0 \le \gamma < 1$, then $f \in \mathcal{C}_{\lambda\mu}^m(\alpha, \beta)$ provided that

$$|\beta| < \arccos\left(\frac{1-\gamma}{1-\alpha}\right). \tag{12}$$

Proof. From (11) it follows

$$\frac{D_{\lambda\mu}^m f(z)}{z} - 1 = (1-\gamma) w(z),$$

where $|w(z)| < 1$ for $z \in \mathcal{U}$. We have

$$\Re\left[e^{i\beta}\frac{D^m_{\lambda\mu}f(z)}{z}\right] = \Re[e^{i\beta}(1+(1-\gamma)w(z))]$$

$$= \cos\beta + (1-\gamma)\Re[e^{i\beta}w(z)]$$

$$\geq \cos\beta - (1-\gamma)|e^{i\beta}w(z)|$$

$$\geq \cos\beta - (1-\gamma) \geq \alpha\cos\beta,$$

provided that $|\beta| \leq \arccos\left(\frac{1-\gamma}{1-\alpha}\right)$. □

If we set $\gamma = 1-(1-\alpha)\cos\beta$ in Theorem 1, we obtain the next characterization property for the class $\mathcal{C}^m_{\lambda\mu}(\alpha,\beta)$.

Corollary 1. *Let $f \in \mathcal{A}$. If*

$$\left|\frac{D^m_{\lambda\mu}f(z)}{z} - 1\right| < (1-\alpha)\cos\beta \quad (z \in \mathcal{U}) \tag{13}$$

for $0 \leq \alpha < 1$ and $\beta \in \mathbb{R}$, $|\beta| < \frac{\pi}{2}$, then $f \in \mathcal{C}^m_{\lambda\mu}(\alpha,\beta)$.

The following characterization property is given in terms of coefficient inequality.

Theorem 2. *If $f \in \mathcal{A}$, given by (1) satisfies the inequality*

$$\sum_{n=2}^{\infty}\frac{\sec\beta}{1-\alpha}A_n(\lambda,\mu,m)|a_n| \leq 1 \tag{14}$$

then it belongs to the class $\mathcal{C}^m_{\lambda\mu}(\alpha,\beta)$.

Proof. Makind use of Corollary 1, it sufficies to show that the condition (13) is satisfied. From (8) and (9), it follows

$$\left|\frac{D^m_{\lambda\mu}f(z)}{z} - 1\right| = \left|\sum_{n=2}^{\infty}A_n(\lambda,\mu,m)a_n z^{n-1}\right|$$

$$< \sum_{n=2}^{\infty}A_n(\lambda,\mu,m)|a_n| \leq (1-\alpha)\cos\beta.$$

Therefore, $f \in \mathcal{C}^m_{\lambda\mu}(\alpha,\beta)$ and the proof is completed. □

3 Coefficient estimates

The first result on coefficient estimates for the class $\mathcal{C}^m_{\lambda\mu}(\alpha,\beta)$, is the following.

Theorem 3. *If $f \in \mathcal{C}^m_{\lambda\mu}(\alpha,\beta)$ is given by (1), then*

$$|a_n| \le \frac{2(1-\alpha)\cos\beta}{A_n(\lambda,\mu,m)} \quad , \quad n \ge 2. \tag{15}$$

Proof. Since $f \in \mathcal{C}^m_{\lambda\mu}(\alpha,\beta)$, we have

$$\frac{D^m_{\lambda\mu}f(z)}{z} = p(z) \quad (z \in \mathcal{U})$$

where $p(z) \in \mathcal{P}_{\alpha,\beta}$ and $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$.

From (8) and (9), it follows

$$1 + \sum_{n=2}^{\infty} A_n(\lambda,\mu,m) a_n z^{n-1} = 1 + \sum_{n=1}^{\infty} p_n z^n$$

or

$$\sum_{n=2}^{\infty} A_n(\lambda,\mu,m) a_n z^{n-1} = \sum_{n=2}^{\infty} p_{n-1} z^{n-1}.$$

Equating the coefficients of z^{n-1}, we obtain

$$A_n(\lambda,\mu,m) a_n = p_{n-1} \quad , \quad n \ge 2.$$

Making use of Lemma 1 (v.), we get

$$A_n(\lambda,\mu,m)|a_n| = |p_{n-1}| \le 2(1-\alpha)\cos\beta \quad , \quad n \ge 2,$$

and thus

$$|a_n| \le \frac{2(1-\alpha)\cos\beta}{A_n(\lambda,\mu,m)} \quad , \quad n \ge 2.$$

□

In view of Theorem 3 we can derive a distortion result for the class $\mathcal{C}^m_{\lambda\mu}(\alpha,\beta)$.

Theorem 4. *If $f \in \mathcal{C}^m_{\lambda\mu}(\alpha,\beta)$, then for $|z| = r < 1$*

$$|f(z)| \ge r - 2(1-\alpha)\cos\beta r^2 \sum_{n=2}^{\infty} \frac{1}{A_n(\lambda,\mu,m)},$$

$$|f(z)| \le r + 2(1-\alpha)\cos\beta r^2 \sum_{n=2}^{\infty} \frac{1}{A_n(\lambda,\mu,m)}$$

and

$$|f'(z)| \geq 1 - 2(1-\alpha)\cos\beta r \sum_{n=2}^{\infty} \frac{n}{A_n(\lambda,\mu,m)},$$

$$|f'(z)| \leq 1 + 2(1-\alpha)\cos\beta r \sum_{n=2}^{\infty} \frac{n}{A_n(\lambda,\mu,m)}.$$

In order to obtain our next result on coefficient estimates, we need the following lemma.

Lemma 2. *([6]) Let $w(z) = c_1 z + c_2 z^2 + \dots$ be an analytic function with $|w(z)| < 1$ in $\mathcal{U}$. Then, for any complex number ν*

$$|c_2 - \nu c_1^2| \leq \max\{1, |\nu|\}. \tag{16}$$

The equality is attained for $w(z) = z^2$ and $w(z) = z$.

Theorem 5. *Let $\tau \in \mathbb{C}$. If $f \in \mathcal{C}_{\lambda\mu}^m(\alpha,\beta)$ is given by (1), then*

$$|a_3 - \tau a_2^2| \leq$$

$$\leq \frac{2(1-\alpha)\cos\beta}{A_3(\lambda,\mu,m)} \max\left\{1, \frac{|2\tau A_3(\lambda,\mu,m)(1-\alpha)e^{-i\beta}\cos\beta - A_2(\lambda,\mu,m)^2|}{A_2(\lambda,\mu,m)^2}\right\} \tag{17}$$

where $A_2(\lambda,\mu,m) = (2\lambda\mu + \lambda - \mu + 1)^m$ and $A_3(\lambda,\mu,m) = (6\lambda\mu + 2(\lambda-\mu) + 1)^m$. The result is sharp.

Proof. Assume $f \in \mathcal{C}_{\lambda\mu}^m(\alpha,\beta)$. Then $D_{\lambda\mu}^m f(z)/z \in \mathcal{P}_{\alpha,\beta}$. In view of Lemma 1 (ii., iii.), we obtain that there exists an analytic function $w(z) = \sum_{n=1}^{\infty} c_n z^n$, with $|w(z)| < 1$ in $\mathcal{U}$ such that

$$\frac{D_{\lambda\mu}^m f(z)}{z} = \frac{1 + [2(1-\alpha)e^{-i\beta}\cos\beta - 1]w(z)}{1 - w(z)}$$

which is equivalent to

$$(1 - w(z))D_{\lambda\mu}^m f(z) = z + [2(1-\alpha)e^{-i\beta}\cos\beta - 1]zw(z). \tag{18}$$

Equating the coefficients in both sides of (18), we obtain

$$a_2 = \frac{2(1-\alpha)e^{-i\beta}\cos\beta}{A_2(\lambda,\mu,m)} c_1 \tag{19}$$

and

$$a_3 = \frac{2(1-\alpha)e^{-i\beta}\cos\beta}{A_3(\lambda,\mu,m)}(c_2 + c_1^2). \tag{20}$$

From (19) and (20), it follows

$$a_3 - \tau a_2^2 = \frac{2(1-\alpha)e^{-i\beta}\cos\beta}{A_3(\lambda,\mu,m)}[c_2 - \nu c_1^2]$$

where

$$\nu = \frac{2\tau A_3(\lambda,\mu,m)(1-\alpha)e^{-i\beta}\cos\beta - A_2(\lambda,\mu,m)^2}{A_2(\lambda,\mu,m)^2}.$$

Applying Lemma 2, we get

$$|a_3 - \tau a_2^2| \leq \frac{2(1-\alpha)\cos\beta}{A_3(\lambda,\mu,m)}|c_2 - \nu c_1^2|$$

$$\leq \frac{2(1-\alpha)\cos\beta}{A_3(\lambda,\mu,m)} \max\left\{1, \frac{|2\tau A_3(\lambda,\mu,m)(1-\alpha)e^{-i\beta}\cos\beta - A_2(\lambda,\mu,m)^2|}{A_2(\lambda,\mu,m)^2}\right\}.$$

The sharpness of (17) follows from the sharpness of inequality (16). □

4 Subordination result

Denote by $\mathcal{C}(\alpha,\beta,\lambda,\mu,m)$ the class of functions $f \in \mathcal{A}$, given by (1) whose coefficients satisfy the condition (14). In this section we derive a subordination result for the class $\mathcal{C}(\alpha,\beta,\lambda,\mu,m)$.

In order to obtain our main result, we need the following definition and lemma.

Definition 2 (Subordinating factor sequence). *A sequence $\{b_n\}_{n=1}^{\infty}$ of complex numbers is said to be a subordinating factor sequence if, whenever f of the form (1) is analytic, univalent and convex in $\mathcal{U}$, we have the subordination given by*

$$\sum_{n=1}^{\infty} a_n b_n z^n \prec f(z) \quad (z \in \mathcal{U} \ \text{ and } \ a_1 := 1) \tag{21}$$

Lemma 3. *([13]) The sequence $\{b_n\}_{n=1}^{\infty}$ is a subordinating factor sequence if and only if*

$$\Re\left(1 + 2\sum_{n=1}^{\infty} b_n z^n\right) > 0 \quad (z \in \mathcal{U}). \tag{22}$$

Employing the techniques used by Srivastava and Attyia [12], Attyia [2], Frasin [5], Raina and Bansal [8] and Singh [11] we prove the following theorem.

Theorem 6. *Let $f \in \mathcal{C}(\alpha,\beta,\lambda,\mu,m)$ be given by (1) and let $g \in \mathcal{A}$ be a univalent and convex function. Then*

$$\frac{A_2(\lambda,\mu,m)\sec\beta}{2[1-\alpha+A_2(\lambda,\mu,m)\sec\beta]}(f*g)(z) \prec g(z) \ \ (z \in \mathcal{U}). \tag{23}$$

In particular

$$\Re f(z) > -\frac{1-\alpha+A_2(\lambda,\mu,m)\sec\beta}{A_2(\lambda,\mu,m)\sec\beta} \ \ (z \in \mathcal{U}). \tag{24}$$

The constant $\dfrac{A_2(\lambda,\mu,m)\sec\beta}{2[1-\alpha+A_2(\lambda,\mu,m)\sec\beta]}$ *is the best estimate.*

Proof. Let $f \in \mathcal{C}(\alpha, \beta, \lambda, \mu, m)$ and let $g(z) = z + \sum_{n=2}^{\infty} c_n z^n$ be a univalent and convex function. We have

$$\frac{A_2(\lambda, \mu, m) \sec \beta}{2[1 - \alpha + A_2(\lambda, \mu, m) \sec \beta]} (f * g)(z)$$

$$= \frac{A_2(\lambda, \mu, m) \sec \beta}{2[1 - \alpha + A_2(\lambda, \mu, m) \sec \beta]} \left(z + \sum_{n=2}^{\infty} a_n c_n z^n \right). \tag{25}$$

In view of Definition 2, the assertion (23) of the theorem will hold if the sequence

$$\left\{ \frac{A_2(\lambda, \mu, m) \sec \beta}{2[1 - \alpha + A_2(\lambda, \mu, m) \sec \beta]} a_n \right\}_{n=1}^{\infty} \tag{26}$$

is a subordinating factor sequence, with $a_1 = 1$. Making use of Lemma 3, this is equivalent to the following inequality

$$\Re \left\{ 1 + 2 \sum_{n=1}^{\infty} \frac{A_2(\lambda, \mu, m) \sec \beta}{2[1 - \alpha + A_2(\lambda, \mu, m) \sec \beta]} a_n z^n \right\} > 0 \quad (z \in \mathcal{U}). \tag{27}$$

Since $A_n(\lambda, \mu, m)$, $0 \le \mu \le \lambda$, $m \in \mathbb{N}_0$, $n \ge 2$ is an increasing function of n, we have

$$\Re \left\{ 1 + \sum_{n=1}^{\infty} \frac{A_2(\lambda, \mu, m) \sec \beta}{1 - \alpha + A_2(\lambda, \mu, m) \sec \beta} a_n z^n \right\}$$

$$= \Re \left\{ 1 + \frac{A_2(\lambda, \mu, m) \sec \beta}{1 - \alpha + A_2(\lambda, \mu, m) \sec \beta} z \right.$$

$$\left. + \frac{1}{1 - \alpha + A_2(\lambda, \mu, m) \sec \beta} \sum_{n=2}^{\infty} A_n(\lambda, \mu, m) \sec \beta a_n z^n \right\}$$

$$> 1 - \frac{A_2(\lambda, \mu, m) \sec \beta}{1 - \alpha + A_2(\lambda, \mu, m) \sec \beta} r - \frac{1}{1 - \alpha + A_2(\lambda, \mu, m) \sec \beta} \sum_{n=2}^{\infty} A_n(\lambda, \mu, m) \sec \beta |a_n| r^n$$

$$= 1 - \frac{A_2(\lambda, \mu, m) \sec \beta}{1 - \alpha + A_2(\lambda, \mu, m) \sec \beta} r - \frac{1 - \alpha}{1 - \alpha + A_2(\lambda, \mu, m) \sec \beta} r$$

$$= \frac{(1 - r)[1 - \alpha + A_2(\lambda, \mu, m) \sec \beta]}{1 - \alpha + A_2(\lambda, \mu, m) \sec \beta} = 1 - r > 0 \quad (|z| = r < 1).$$

It follows that inequality (27) holds and thus, the assertion (23) of the theorem is proved. The inequality (24) follows from (23) by taking $g(z) = \frac{z}{1 - z}$ $(z \in \mathcal{U})$ which is a univalent and convex function.

To prove the sharpness of the constant $\frac{A_2(\lambda, \mu, m) \sec \beta}{2[1 - \alpha + A_2(\lambda, \mu, m) \sec \beta]}$, we consider the function

$$f_0(z) = z - \frac{1 - \alpha}{A_2(\lambda, \mu, m) \sec \beta} z^2 \quad (0 \le \mu \le \lambda \, , m \in \mathbb{N}_0) \tag{28}$$

which belongs to the class $\mathcal{C}(\alpha,\beta,\lambda,\mu,m)$. From (23), we have

$$\frac{A_2(\lambda,\mu,m)\sec\beta}{2[1-\alpha+A_2(\lambda,\mu,m)\sec\beta]}f_0(z)\prec\frac{z}{1-z}.$$

It is easy to show that for the function $f_0(z)$ defined by (28)

$$\inf_{|z|\le 1}\left\{\Re\left[\frac{A_2(\lambda,\mu,m)\sec\beta}{2[1-\alpha+A_2(\lambda,\mu,m)\sec\beta]}f_0(z)\right]\right\}=-\frac{1}{2}\quad(z\in\mathcal{U})$$

which completes the proof of our theorem. □

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