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ESTIMATES WITH OPTIMAL CONSTANTS FOR THE OPERATOR OF R-TH ORDER

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Abstract

We present general estimates of the degree of approximation by Kirov - Popova operators using Peetre's K-functionals of first order.

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1 Introduction

In [3], Kirov and Popova introduced the operator of r-th order associated with a linear positive operator and proved a Korovkin-type theorem for this operator. To each linear positive operator $L : \mathbf{C}[a,b] \longrightarrow \mathbf{C}[a,b]$ a generalization of r-th order, $r \in \mathbb{N} \cup \{0\}$, is defined by:

$$L_r(f,x) = L\left(T_{r,f,\cdot}(x), x\right) \tag{1}$$

where

$$T_{r,f,y}(x) = \sum_{j=0}^{r} \frac{f^{(j)}(y)}{j!}(x-y)^j \tag{2}$$

is the Taylor polynomial of degree r for function f at y. Using same ideea as in [2] we give a quantitative estimate for the remainder in Taylor's formula using K-functional K_1^{∞} and weighted K-functional $K_{1,\varphi}^{\infty}$ and we obtain estimates for the operator of r-th order. We use the notation e_k for the function $e_k(x) = x^k$, $k \in \mathbb{N} \cup \{0\}$.

For $r \in \mathbb{N}$ and $1 \leq s \leq \infty$, the K-functional $K_r^s(f,t) = K^s\left(f, t^r; \mathbf{C}[a,b], \mathbf{C^r}[a,b]\right)$, $t > 0$ is defined for the Banach space $(\mathbf{C}[a,b], \|\cdot\|)$ and the semi-Banach subspace $\left(\mathbf{C^r}[a,b], |\cdot|_{C^r}\right)$, $|f|_{C^r} = \left\|f^{(r)}\right\|$ by

$$K_r^s(f,t) = \inf_{g \in \mathbf{C^r}[a,b]} \left\{ \|f-g\|^s + t^{rs} \left\|g^{(r)}\right\|^s \right\}^{\frac{1}{s}}, \ 1 \leq s < \infty \tag{3}$$

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and

$$K_r^{\infty}(f,t) = \inf_{g\in \mathbf{C^r}[a,b]} \max\left\{\|f-g\|\,,\, t^r \left\|g^{(r)}\right\|\right\} \tag{4}$$

or shortly

$$K_r^s\,(f,t) = \inf_{g\in \mathbf{C^r}[a,b]} \left\|\left(\|f-g\|\,, t^r \left\|g^{(r)}\right\|\right)\right\|_s,\ 1\le s\le\infty, \tag{5}$$

where $\|\cdot\|_s$, $1\le s<\infty$, is the Minkowski norm in $\mathbb{R}^2$ and $\|\cdot\|_\infty$ is the Chebychev norm in $\mathbb{R}^2$, respectively. These K - functionals are uniform equivalent, more exactly for $f\in\mathbf{C}[a,b]$ and $t>0$ we have

$$K_r^{\infty}\,(f,t)\le K_r^{s_1}\,(f,t)\le K_r^{s_2}\,(f,t)\le K_r^{1}\,(f,t)\le 2K_r^{\infty}\,(f,t)$$

where $1\le s_2\le s_1\le\infty$.

A useful relation between the K-functionals is given by:

Lemma 1. *[7], [1] Let $1\le s<\infty$. Then for $f\in\mathbf{C}[a,b]$ and $t>0$ we have*

$$K_r^s(f,t) = \inf_{u>0}\left(1+\frac{t^{rs}}{u^{rs}}\right)^{\frac{1}{s}} K_r^{\infty}(f,u). \tag{6}$$

In the weighted case, for $r\in\mathbb{N}$ and $\varphi(x)=\sqrt{x(1-x)}$, $x\in[0,1]$ we denote by

$$\mathbf{C}_\varphi[0,1] = \left\{f\in\mathbf{C}(0,1)\,|\,(\exists)\lim_{x\to 0+} f(x)\varphi(x),\ \lim_{x\to 1-} f(x)\varphi(x)\in\mathbb{R}\right\}$$

and

$$\mathbf{W^r_{C_{\varphi^r}}}[0,1] = \left\{f\in\mathbf{C^{r-1}}[0,1]\,|\, f^{(r)}\in\mathbf{C}_{\varphi^r}[0,1]\right\}.$$

We consider the K-functional

$$K^s_{r,\varphi}(f,t) = K^s\left(f,t^r;\mathbf{C}[0,1],\mathbf{W^r_{C_{\varphi^r}}}[0,1]\right),\ t>0,\ 1\le s\le\infty$$

defined for the Banach space $(\mathbf{C}[0,1],\|\cdot\|)$ and the semi-Banach subspace $\left(\mathbf{W^r_{C_{\varphi^r}}}[0,1],|\cdot|_{W^r_{C_{\varphi^r}}}\right)$, $|f|_{W^r_{C_{\varphi^r}}} = \left\|\varphi^r f^{(r)}\right\|$ by

$$K^s_{r,\varphi}\,(f,t) = \inf_{g\in\mathbf{W^r_{C_{\varphi^r}}}[0,1]} \left\|\left(\|f-g\|\,, t^r\left\|\varphi^r g^{(r)}\right\|\right)\right\|_s,\ 1\le s\le\infty. \tag{7}$$

Lemma 2. *Let $1\le s<\infty$ and $r\in\mathbb{N}$. Then for $f\in\mathbf{C}[0,1]$ and $t>0$ we have*

$$K^s_{r,\varphi}(f,t) = \inf_{u>0}\left(1+\frac{t^{rs}}{u^{rs}}\right)^{\frac{1}{s}} K^{\infty}_{r,\varphi}(f,u). \tag{8}$$

Proof. "$\leq$:" Let $g \in \mathbf{W}^{\mathbf{r}}_{\mathbf{C}_\varphi}[0,1]$ and $u > 0$. We have

$$\begin{aligned} K^s_{r,\varphi}(f,t) &\leq \left(\|f-g\|^s + t^{rs}\left\|\varphi^r g^{(r)}\right\|^s\right)^{\frac{1}{s}} \\ &\leq \left(1+\frac{t^{rs}}{u^{rs}}\right)^{\frac{1}{s}} \max\left\{\|f-g\|\,,\, u^r\left\|\varphi^r g^{(r)}\right\|\right\}. \end{aligned}$$

Since g is arbitrary this implies $K^s_{r,\varphi}(f,t) \leq \left(1+\frac{t^{rs}}{u^{rs}}\right)^{\frac{1}{s}} K^\infty_{r,\varphi}(f,u)$. Also since u is arbitrary the above inequality implies $K^s_{r,\varphi}(f,t) \leq \inf\limits_{u>0}\left(1+\frac{t^{rs}}{u^{rs}}\right)^{\frac{1}{s}} K^\infty_{r,\varphi}(f,u)$.

"$\geq$": Let $\varepsilon > 0$. We can choose $g \in \mathbf{W}^{\mathbf{r}}_{\mathbf{C}_\varphi}[0,1]$ such that $\|f-g\| \neq 0$, $\left\|\varphi^r g^{(r)}\right\| \neq 0$ and

$$K^s_{r,\varphi}(f,t) + C\varepsilon \geq \left(\|f-g\|^s + t^{rs}\left\|\varphi^r g^{(r)}\right\|^s\right)^{\frac{1}{s}}$$

where $C = 2 + \frac{t^r r!}{2^r}$. Indeed, by definition of the K-functional there exists $g_1 \in \mathbf{W}^{\mathbf{r}}_{\mathbf{C}_\varphi}[0,1]$ such that

$$K^s_{r,\varphi}(f,t) + \varepsilon \geq \left(\|f-g_1\|^s + t^{rs}\left\|\varphi^r g_1^{(r)}\right\|^s\right)^{\frac{1}{s}}.$$

So, if $\|f-g_1\| \neq 0$ and $\left\|\varphi^r g_1^{(r)}\right\| \neq 0$ we choose $g = g_1$.

If $\|f-g_1\| = 0$ and $\left\|\varphi^r g_1^{(r)}\right\| \neq 0$ we choose $g = g_1 + \varepsilon e_0$. Then $\|f-g\| = \varepsilon \neq 0$, $\left\|\varphi^r g^{(r)}\right\| = \left\|\varphi^r g_1^{(r)}\right\| \neq 0$ and

$$\left(\|f-g\|^s + t^{rs}\left\|\varphi^r g^{(r)}\right\|^s\right)^{\frac{1}{s}} = \left(\varepsilon^s + t^{rs}\left\|\varphi^r g_1^{(r)}\right\|^s\right)^{\frac{1}{s}} \leq K^s_{r,\varphi}(f,t) + 2\varepsilon.$$

If $\|f-g_1\| \neq 0$ and $\left\|\varphi^r g_1^{(r)}\right\| = 0$ we choose $g = g_1 + C_1\varepsilon e_r$, where C_1 is a constant for which $\|f-g\| \neq 0$. We can suppose that $0 < C_1 < 1$. Then $\|f-g\| = \|f-g_1-C_1\varepsilon e_r\| \leq \|f-g_1\| + C_1\varepsilon\|e_r\|$, $\left\|\varphi^r g^{(r)}\right\| = \left\|\varphi^r g_1^{(r)} + \varphi^r r! C_1\varepsilon e_0\right\| = \|\varphi^r\|\, r! C_1\varepsilon \neq 0$ and

$$\begin{aligned} \left(\|f-g\|^s + t^{rs}\left\|\varphi^r g^{(r)}\right\|^s\right)^{\frac{1}{s}} &\leq \|f-g\| + t^r\left\|\varphi^r g^{(r)}\right\| \\ &\leq \|f-g_1\| + C_1\varepsilon\left(\|e_r\| + \|\varphi^r\|\, t^r r!\right) \\ &\leq K^s_{r,\varphi}(f,t) + \varepsilon\left(1 + \|e_r\| + \|\varphi^r\|\, t^r r!\right). \end{aligned}$$

If $\|f-g_1\| = 0$ and $\left\|\varphi^r g_1^{(r)}\right\| = 0$ we choose $g = g_1 + \varepsilon e_r$. Then $\|f-g\| = \varepsilon\|e_r\| \neq 0$, $\left\|\varphi^r g^{(r)}\right\| = \left\|\varphi^r g_1^{(r)} + \varphi^r r! \varepsilon e_0\right\| = \|\varphi^r\|\, r!\varepsilon \neq 0$ and

$$\begin{aligned} \left(\|f-g\|^s + t^{rs}\left\|\varphi^r g^{(r)}\right\|^s\right)^{\frac{1}{s}} &\leq \|f-g\| + t^r\left\|\varphi^r g^{(r)}\right\| \\ &= \varepsilon\left(\|e_r\| + \|\varphi^r\|\, t^r r!\right) \\ &\leq K^s_{r,\varphi}(f,t) + \varepsilon\left(\|e_r\| + \|\varphi^r\|\, t^r r!\right). \end{aligned}$$

On the assumption above we have

$$\inf_{u>0}\left(1+\frac{t^{rs}}{u^{rs}}\right)^{\frac{1}{s}} K^{\infty}_{r,\varphi}(f,u) \le \left(1+t^{rs}\frac{\left\|\varphi^r g^{(r)}\right\|^s}{\|f-g\|^s}\right)^{\frac{1}{s}} K^{\infty}_{r,\varphi}\left(f, \frac{\|f-g\|^{\frac{1}{r}}}{\left\|\varphi^r g^{(r)}\right\|^{\frac{1}{r}}}\right)$$

$$\le \left(1+t^{rs}\frac{\left\|\varphi^r g^{(r)}\right\|^s}{\|f-g\|^s}\right)^{\frac{1}{s}} \cdot \max\left\{\|f-g\|\,,\ \frac{\|f-g\|}{\left\|\varphi^r g^{(r)}\right\|}\left\|\varphi^r g^{(r)}\right\|\right\}$$

$$= \left(\|f-g\|^s + t^{rs}\left\|\varphi^r g^{(r)}\right\|^s\right)^{\frac{1}{s}} \le K^s_{r,\varphi}(f,t) + C\varepsilon\,.$$

Since ε is arbitrary this implies the inequality. □

Remark 1. *For the case of the K-functionals defined for an arbitrary couple of quasi-normed spaces see [1], page 75.*

2 Estimates with K - functionals K_1^s

Lemma 3. *If $f \in \mathbf{C^r}[a,b]$, $r \in \mathbb{N}$, $x, y \in [a,b]$ then for the remainder in Taylor's formula of order r we have the following estimate*

$$|R_{r,f,y}(x)| \le \frac{|y-x|^r}{r!}\left(2+\frac{|y-x|}{t(r+1)}\right) K_1^{\infty}\left(f^{(r)}, t\right),\ (\forall)t>0. \tag{9}$$

Proof. Let $g \in \mathbf{C^{r+1}}[a,b]$. Using the Lagrange form of the remainder we have

$$|R_{r,g,y}(x)| = \left|\frac{g^{(r+1)}(\eta)}{(r+1)!}(x-y)^{r+1}\right| \le \frac{|x-y|^{r+1}}{(r+1)!}\cdot\left\|g^{(r+1)}\right\|$$

and

$$\begin{aligned} |R_{r,f-g,y}(x)| &= \left|(f-g)(x) - \sum_{k=0}^{r}\frac{(f-g)^{(k)}(y)}{k!}(x-y)^k\right| \\ &= \left|R_{r-1,f-g,y}(x) - \frac{(f-g)^{(r)}(y)}{r!}(x-y)^r\right| \\ &= \left|\frac{(f-g)^{(r)}(\xi)}{r!}(x-y)^r - \frac{(f-g)^{(r)}(y)}{r!}(x-y)^r\right| \\ &= \frac{|x-y|^r}{r!}\cdot\left|(f-g)^{(r)}(\xi)-(f-g)^{(r)}(y)\right| \\ &\le 2\frac{|x-y|^r}{r!}\cdot\left\|f^{(r)}-g^{(r)}\right\| \end{aligned}$$

(with η, ξ between y and x). Then

$$\begin{aligned}|R_{r,f,y}(x)| &\leq |R_{r,f-g,y}(x)| + |R_{r,g,y}(x)| \\ &\leq \frac{|x-y|^r}{r!}\left(2\left\|f^{(r)} - g^{(r)}\right\| + \frac{|x-y|}{r+1}\left\|g^{(r+1)}\right\|\right) \\ &\leq \frac{|x-y|^r}{r!}\left(2 + \frac{|x-y|}{t(r+1)}\right)\max\left\{\left\|f^{(r)} - g^{(r)}\right\|, t\left\|g^{(r+1)}\right\|\right\}.\end{aligned}$$

Since g is arbitrary this implies (9). □

Theorem 1. *Let $r \in \mathbb{N}$, $L : \mathbf{C}[a,b] \longrightarrow \mathbf{C}[a,b]$ a positive linear operator and $f \in \mathbf{C^r}[a,b]$. Then $(\forall)x \in [a,b]$, $(\forall)t > 0$ we have*

$$\begin{aligned}&|L_r(f,x) - f(x)| \leq |f(x)| \cdot |L(e_0,x) - 1| + \\ &+\frac{1}{r!}\left(2L\left(|e_1 - xe_0|^r, x\right) + \frac{L\left(|e_1 - xe_0|^{r+1}, x\right)}{t(r+1)}\right) \cdot K_1^\infty\left(f^{(r)}, t\right).\end{aligned} \tag{10}$$

Conversely, if A, B, $C \geq 0$ such that

$$\begin{aligned}&|L_r(f,x) - f(x)| \leq A \cdot |f(x)| \cdot |L(e_0,x) - 1| + \\ &+\left(B \cdot L\left(|e_1 - xe_0|^r, x\right) + C \cdot \frac{L\left(|e_1 - xe_0|^{r+1}, x\right)}{t}\right) \cdot K_1^\infty\left(f^{(r)}, t\right)\end{aligned} \tag{11}$$

holds for any positive linear operator $L : \mathbf{C}[a,b] \longrightarrow \mathbf{C}[a,b]$, any $f \in \mathbf{C^r}[a,b]$, any $x \in [a,b]$ and any $t > 0$ then $A \geq 1$, $B \geq \dfrac{2}{r!}$ and $C \geq \dfrac{1}{(r+1)!}$.

Proof. From Lemma 3 we have

$$\begin{aligned}|L_r(f,x) - f(x)| &= |L(T_{r,f,\cdot}(x), x) - f(x)| \\ &= |L(f(x)e_0 - R_{r,f,\cdot}(x), x) - f(x)| \\ &\leq |f(x)| \cdot |L(e_0,x) - 1| + |L(R_{r,f,\cdot}(x), x)| \\ &\leq |f(x)| \cdot |L(e_0,x) - 1| + \\ &\quad +\frac{1}{r!}\left(2L\left(|e_1 - xe_0|^r, x\right) + \frac{L\left(|e_1 - xe_0|^{r+1}, x\right)}{t(r+1)}\right) \cdot K_1^\infty\left(f^{(r)}, t\right).\end{aligned}$$

which is (10).

We prove now the converse part. If we choose $L(h,x) = 0$ and $f = e_0$ and replace in (11) we obtain $A \geq 1$.

To show that $B \geq \frac{2}{r!}$ we choose $[a,b]=[0,1]$, $L(h,x)=h(0)$, $f(x)=2x^{r+\alpha}$ with $\alpha>0$ and $x=1$. For $g=(r+\alpha)\cdot(r+\alpha-1)\cdots(\alpha+1)\,e_0$ we have

$$\begin{aligned} K_1^{\infty}\left(f^{(r)},t\right) &\leq \max\left\{\left\|f^{(r)}-g\right\|, t\left\|g'\right\|\right\} \\ &= \left\|f^{(r)}-g\right\| = (r+\alpha)\cdot(r+\alpha-1)\cdots(\alpha+1). \end{aligned}$$

From (11) we obtain

$$2 \leq \left(B+\frac{C}{t}\right)(r+\alpha)\cdot(r+\alpha-1)\cdots(\alpha+1),\ (\forall)t>0.$$

Passing to the limit $t\to\infty$ we obtain $B \geq \frac{2}{(r+\alpha)\cdot(r+\alpha-1)\cdots(\alpha+1)}$. Passing to the limit $\alpha\to 0$ we obtain $B\geq\frac{2}{r!}$.

To show that $C \geq \frac{1}{(r+1)!}$ we choose $[a,b]=[0,1]$, $L(h,x)=h(0)$, $f=e_{r+1}$ and $x=1$. We have

$$K_1^{\infty}\left(f^{(r)},t\right) \leq t\left\|f^{(r+1)}\right\| = t(r+1)!.$$

From (11) we obtain $1\leq Bt(r+1)!+(r+1)!C$, $(\forall)t>0$. Passing to the limit $t\to 0$ we obtain $C\geq\frac{1}{(r+1)!}$.

□

Corollary 1. *Under the conditions of the theorem we have*

$$\begin{aligned} &|L_r(f,x)-f(x)| \leq |f(x)|\cdot|L(e_0,x)-1| + \\ &+\frac{1}{r!}\max\left\{2L\left(|e_1-xe_0|^r,x\right), \frac{L\left(|e_1-xe_0|^{r+1},x\right)}{t(r+1)}\right\}\cdot K_1^1\left(f^{(r)},t\right) \end{aligned} \tag{12}$$

and

$$\begin{aligned} &|L_r(f,x)-f(x)| \leq |f(x)|\cdot|L(e_0,x)-1| + \\ &+\frac{1}{r!}\left(2^{s'}L\left(|e_1-xe_0|^r,x\right)^{s'} + \frac{L\left(|e_1-xe_0|^{r+1},x\right)^{s'}}{t^{s'}(r+1)^{s'}}\right)^{\frac{1}{s'}}\cdot K_1^s\left(f^{(r)},t\right) \end{aligned} \tag{13}$$

for $1<s<\infty$, $s'=\frac{s}{s-1}$.

Conversely,

- *if* $(\exists)A,\ B,\ C \geq 0$ *such that*

$$|L(f,x)-f(x)| \leq A \cdot |f(x)|\,|L(e_0,x)-1| + \\ +\max\left\{B \cdot L\left(|e_1 - xe_0|^r, x\right), C\frac{L\left(|e_1-xe_0|^{r+1},x\right)}{t}\right\} \cdot K_1^1(f^{(r)},t)$$

  holds for any positive linear operator $L : \mathbf{C}[a,b] \longrightarrow \mathbf{C}[a,b]$, *any* $f \in \mathbf{C^r}[a,b]$, *any* $x \in [a,b]$ *and any* $t > 0$ *then* $A \geq 1$, $B \geq \dfrac{2}{r!}$ *and* $C \geq \dfrac{1}{(r+1)!}$.

- *if* $(\exists)A,\ B,\ C \geq 0$ *such that*

$$|L(f,x)-f(x)| \leq A \cdot |f(x)|\,|L(e_0,x)-1| + \\ +\left(B \cdot L\left(|e_1 - xe_0|^r, x\right)^{s'} + C\frac{L\left(|e_1-xe_0|^{r+1},x\right)^{s'}}{t^{s'}}\right)^{\frac{1}{s'}} \cdot K_1^s(f^{(r)},t)$$

  holds for any positive linear operator $L : \mathbf{C}[a,b] \longrightarrow \mathbf{C}[a,b]$, *any* $f \in \mathbf{C^r}[a,b]$, *any* $x \in [a,b]$ *and any* $t > 0$ *then* $A \geq 1$, $B \geq \dfrac{2^{s'}}{r!^{s'}}$ *and* $C \geq \dfrac{1}{(r+1)!^{s'}}$.

Proof. For the estimate with K_1^s, $1 \leq s < \infty$ we use estimate (10) and Lemma 1. Let $u > 0$. For $s = 1$ we have:

$$\begin{aligned} |L(f,x)-f(x)| &\leq |f(x)| \cdot |L(e_0,x)-1| + \\ &+\frac{1}{r!}\left(2L\left(|e_1-xe_0|^r,x\right) + \frac{L\left(|e_1-xe_0|^{r+1},x\right)}{u(r+1)}\right) \cdot K_1^\infty(f^{(r)},u) \\ &\leq |f(x)| \cdot |L(e_0,x)-1| + \\ &+\frac{1}{r!}\max\left(2L\left(|e_1-xe_0|^r,x\right), \frac{L\left(|e_1-xe_0|^{r+1},x\right)}{t(r+1)}\right)\left(1+\frac{t}{u}\right)K_1^\infty(f^{(r)},u), \end{aligned}$$

from whence (12).

For $1 < s < \infty$, we denote by $s' = \dfrac{s}{s-1}$ and by Hölder's inequality we have:

$$\begin{aligned} |L(f,x)-f(x)| &\leq |f(x)| \cdot |L(e_0,x)-1| + \\ &+\frac{1}{r!}\left(2L\left(|e_1-xe_0|^r,x\right) + \frac{L\left(|e_1-xe_0|^{r+1},x\right)}{u(r+1)}\right) \cdot K_1^\infty(f^{(r)},u) \\ &\leq |f(x)| \cdot |L(e_0,x)-1| + \\ &+\frac{1}{r!}\left(2^{s'}L\left(|e_1-xe_0|^r,x\right)^{s'} + \frac{L\left(|e_1-xe_0|^{r+1},x\right)^{s'}}{t^{s'}(r+1)^{s'}}\right)^{\frac{1}{s'}}\left(1+\frac{t^s}{u^s}\right)^{\frac{1}{s}}K_1^\infty(f^{(r)},u), \end{aligned}$$

from whence (13).

For the converse part we make the same choices like in Theorem 1. □

3 Estimates with weighted K - functionals $K^s_{1,\varphi}$

Lemma 4. *If $f \in \mathbf{C^r}[0,1]$, $r \in \mathbb{N}$, $x \in (0,1)$ and $y \in [0,1]$ then for the remainder in Taylor's formula of order r we have the following estimate*

$$|R_{r,f,y}(x)| \leq \left(2\frac{|y-x|^r}{r!} + \frac{2^{r+1}}{(2r+1)!!} \cdot \frac{|y-x|^{r+1}}{t\varphi(x)}\right) K^{\infty}_{1,\varphi}\left(f^{(r)}, t\right), \; (\forall) t > 0. \tag{14}$$

Proof. Let $g \in \mathbf{W^{r+1}_{C_\varphi}}[0,1]$ and $x \in (0,1)$. Let us now make use of the fact that function $u \mapsto \dfrac{t-u}{u(1-u)}$, $u \in (0,t)$, $t \in (0,1]$ is decreasing [5]. Let $y \in [0,1]$, $y > x$. Using the integral form of the remainder we have

$$\begin{aligned}
|R_{r,g,y}(x)| &= \left|\frac{1}{r!}\int_y^x g^{(r+1)}(u)(x-u)^r du\right| \leq \frac{1}{r!}\int_x^y \left|\varphi(u) g^{(r+1)}(u)\right| \frac{(u-x)^r}{\varphi(u)} du \\
&\leq \frac{\left\|\varphi g^{(r+1)}\right\|}{r!}\int_x^y (u-x)^r \frac{1}{\varphi(x)}\sqrt{\frac{y-x}{y-u}} du = \frac{\left\|\varphi g^{(r+1)}\right\|\sqrt{y-x}}{r!\varphi(x)}\int_x^y \frac{(u-x)^r}{\sqrt{y-u}} du \\
&= \frac{\left\|\varphi g^{(r+1)}\right\|\sqrt{y-x}}{r!\varphi(x)}(y-x)^{r+\frac{1}{2}}\int_0^\infty \frac{t^r}{(1+t)^{r+\frac{3}{2}}} dt \\
&= \frac{\left\|\varphi g^{(r+1)}\right\|(y-x)^{r+1}}{r!\varphi(x)} B\left(r+1, \frac{1}{2}\right) = \frac{2^{r+1}\left\|\varphi g^{(r+1)}\right\|(y-x)^{r+1}}{(2r+1)!! \cdot \varphi(x)}
\end{aligned}$$

where B is Euler beta function. If $y \in [0,1]$, $y < x$ then $1-y > 1-x$ and

$$\begin{aligned}
|R_{r,g,y}(x)| &= \left|\frac{1}{r!}\int_{1-x}^{1-y} g^{(r+1)}(1-u)(u-1+x)^r du\right| \\
&\leq \frac{1}{r!}\int_{1-x}^{1-y} \left|\varphi(1-u) g^{(r+1)}(1-u)\right| \frac{(u-1+x)^r}{\varphi(1-u)} du \\
&\leq \frac{\left\|\varphi g^{(r+1)}\right\|}{r!}\int_{1-x}^{1-y} (u-1+x)^r \frac{1}{\varphi(1-x)}\sqrt{\frac{x-y}{1-y-u}} du \\
&= \frac{\left\|\varphi g^{(r+1)}\right\|\sqrt{x-y}}{r!\varphi(x)}\int_y^x \frac{(x-u)^r}{\sqrt{u-y}} du = \frac{2^{r+1}\left\|\varphi g^{(r+1)}\right\|(x-y)^{r+1}}{(2r+1)!! \cdot \varphi(x)}.
\end{aligned}$$

So for $y \in [0,1]$ arbitrary we have

$$|R_{r,g,y}(x)| \leq \frac{2^{r+1}}{(2r+1)!!} \cdot \frac{|y-x|^{r+1}}{\varphi(x)} \cdot \left\| \varphi g^{(r+1)} \right\|.$$

We have

$$\begin{aligned} |R_{r,f-g,y}(x)| &= \left| (f-g)(x) - \sum_{k=0}^{r} \frac{(f-g)^{(k)}(y)}{k!} (x-y)^k \right| \\ &= \left| R_{r-1,f-g,y}(x) - \frac{(f-g)^{(r)}(y)}{r!} (x-y)^r \right| \\ &\leq |R_{r-1,f-g,y}(x)| + \frac{\left\|(f-g)^{(r)}\right\|}{r!} |x-y|^r \\ &\leq 2\frac{|y-x|^r}{r!} \cdot \left\| f^{(r)} - g^{(r)} \right\|. \end{aligned}$$

Then

$$\begin{aligned} |R_{r,f,y}(x)| &\leq |R_{r,f-g,y}(x)| + |R_{r,g,y}(x)| \\ &\leq \left(2\frac{|y-x|^r}{r!} \left\| f^{(r)} - g^{(r)} \right\| + \frac{2^{r+1}}{(2r+1)!!} \cdot \frac{|y-x|^{r+1}}{\varphi(x)} \cdot \left\| \varphi g^{(r+1)} \right\| \right) \\ &\leq \left(2\frac{|y-x|^r}{r!} + \frac{2^{r+1}}{(2r+1)!!} \cdot \frac{|y-x|^{r+1}}{t\varphi(x)} \right) \cdot \max\left\{ \left\| f^{(r)} - g^{(r)} \right\|, t \left\| \varphi g^{(r+1)} \right\| \right\}. \end{aligned}$$

Since g is arbitrary this implies (9). □

Theorem 2. *Let $r \in \mathbb{N}$, $L : \mathbf{C}[0,1] \longrightarrow \mathbf{C}[0,1]$ a positive linear operator and $f \in \mathbf{C^r}[0,1]$. Then $(\forall)x \in (0,1)$, $(\forall)t > 0$ we have*

$$\begin{aligned} &|L_r(f,x) - f(x)| \leq |f(x)| \cdot |L(e_0,x) - 1| + \\ &+ \left(\frac{2}{r!} L\left(|e_1 - xe_0|^r, x\right) + \frac{2^{r+1}}{(2r+1)!!} \cdot \frac{L\left(|e_1 - xe_0|^{r+1}, x\right)}{t\varphi(x)} \right) \cdot K_{1,\varphi}^{\infty}\left(f^{(r)}, t\right). \end{aligned} \tag{15}$$

Conversely, if $(\exists)A$, B, $C \geq 0$ such that

$$\begin{aligned} &|L_r(f,x) - f(x)| \leq A \cdot |f(x)| \cdot |L(e_0,x) - 1| + \\ &+ \left(B \cdot L\left(|e_1 - xe_0|^r, x\right) + C \cdot \frac{L\left(|e_1 - xe_0|^{r+1}, x\right)}{t\varphi(x)} \right) \cdot K_{1,\varphi}^{\infty}\left(f^{(r)}, t\right) \end{aligned} \tag{16}$$

holds for any positive linear operator $L : \mathbf{C}[0,1] \longrightarrow \mathbf{C}[0,1]$, any $f \in \mathbf{C^r}[0,1]$, any $x \in (0,1)$ and any $t > 0$ then $A \geq 1$, $B \geq \dfrac{2}{r!}$ and $C \geq \dfrac{2^{r+1}}{(2r+1)!!}$.

Proof. From Lemma 4 we have

$$\begin{aligned}|L_r(f,x)-f(x)| &= |L(T_{r,f,\cdot}(x),x)-f(x)|\\ &= |L(f(x)e_0-R_{r,f,\cdot}(x),x)-f(x)|\\ &\le |f(x)|\cdot|L(e_0,x)-1|+|L(R_{r,f,\cdot}(x),x)|\\ &\le |f(x)|\cdot|L(e_0,x)-1|+\\ &+\left(\frac{2}{r!}L\left(|e_1-xe_0|^r,x\right)+\frac{2^{r+1}}{(2r+1)!!}\cdot\frac{L\left(|e_1-xe_0|^{r+1},x\right)}{t\varphi(x)}\right)\cdot K_{1,\varphi}^{\infty}\left(f^{(r)},t\right).\end{aligned}$$

which is (15).

We prove now the converse part. If we choose $L(h,x)=0$ and $f=e_0$ and replace in (16) we obtain $A\ge 1$.

To show that $B\ge\dfrac{2}{r!}$ we choose $L(h,x)=h(0)$ and $f(x)=2x^{r+\alpha}$ with $\alpha>0$. For $g=(r+\alpha)\cdot(r+\alpha-1)\cdots(\alpha+1)\,e_0$ we have

$$\begin{aligned}K_{1,\varphi}^{\infty}\left(f^{(r)},t\right) &\le \max\left\{\left\|f^{(r)}-g\right\|,t\left\|\varphi g'\right\|\right\}\\ &= \left\|f^{(r)}-g\right\|=(r+\alpha)\cdot(r+\alpha-1)\cdots(\alpha+1)\,.\end{aligned}$$

From (16) we obtain

$$2x^{r+\alpha}\le\left(Bx^r+C\frac{x^{r+1}}{t\varphi(x)}\right)(r+\alpha)\cdot(r+\alpha-1)\cdots(\alpha+1)\,,\ (\forall)t>0.$$

Passing to the limit $t\to\infty$ we obtain $B\ge\dfrac{2x^{\alpha}}{(r+\alpha)\cdot(r+\alpha-1)\cdots(\alpha+1)}$ and passing to the limit $\alpha\to 0$ we obtain $B\ge\dfrac{2}{r!}$.

To show that $C\ge\dfrac{2^{r+1}}{(2r+1)!!}$ we choose $L(h,x)=h(0)$ and $f(x)=x^{r+\frac{1}{2}}$. We have

$$K_{1,\varphi}^{\infty}\left(f^{(r)},t\right)\le t\left\|\varphi f^{(r+1)}\right\|=t\frac{(2r+1)!!}{2^{r+1}}.$$

From (16) we obtain

$$x^{r+\frac{1}{2}}\le Bx^rt\frac{(2r+1)!!}{2^{r+1}}+C\frac{x^{r+1}}{\varphi(x)}\cdot\frac{(2r+1)!!}{2^{r+1}},\ (\forall)t>0.$$

Passing to the limit $t\to 0$ we obtain $C\ge\sqrt{1-x}\cdot\dfrac{2^{r+1}}{(2r+1)!!}$ and passing to the limit $x\to 0$ we obtain $C\ge\dfrac{2^{r+1}}{(2r+1)!!}$. □

Corollary 2. *Under the conditions of the theorem we have*

$$|L_r(f,x)-f(x)| \le |f(x)|\cdot|L(e_0,x)-1|+ \\ +\max\left\{\frac{2}{r!}L\left(|e_1-xe_0|^r,x\right),\frac{2^{r+1}}{(2r+1)!!}\cdot\frac{L\left(|e_1-xe_0|^{r+1},x\right)}{t\varphi(x)}\right\}\cdot K^1_{1,\varphi}\left(f^{(r)},t\right) \tag{17}$$

and

$$|L_r(f,x)-f(x)| \le |f(x)|\cdot|L(e_0,x)-1|+ \\ +\left(\frac{2^{s'}}{r!^{s'}}L\left(|e_1-xe_0|^r,x\right)^{s'}+\frac{2^{s'(r+1)}}{(2r+1)!!^{s'}}\cdot\frac{L\left(|e_1-xe_0|^{r+1},x\right)^{s'}}{t^{s'}\varphi(x)^{s'}}\right)^{\frac{1}{s'}}\cdot K^s_{1,\varphi}\left(f^{(r)},t\right) \tag{18}$$

for $1<s<\infty$, $s'=\dfrac{s}{s-1}$.

Conversely,

- *if* $(\exists)A,\ B,\ C\ge 0$ *such that*

$$|L(f,x)-f(x)| \le A\cdot|f(x)|\,|L(e_0,x)-1|+ \\ +\max\left\{B\cdot L\left(|e_1-xe_0|^r,x\right),C\frac{L\left(|e_1-xe_0|^{r+1},x\right)}{t\varphi(x)}\right\}\cdot K^1_{1,\varphi}(f^{(r)},t)$$

  holds for any positive linear operator $L:\mathbf{C}[0,1]\longrightarrow\mathbf{C}[0,1]$, *any* $f\in\mathbf{C^r}[0,1]$, *any* $x\in(0,1)$ *and any* $t>0$ *then* $A\ge 1$, $B\ge\dfrac{2}{r!}$ *and* $C\ge\dfrac{2^{r+1}}{(2r+1)!!}$.

- *if* $(\exists)A,\ B,\ C\ge 0$ *such that*

$$|L(f,x)-f(x)| \le A\cdot|f(x)|\,|L(e_0,x)-1|+ \\ +\left(B\cdot L\left(|e_1-xe_0|^r,x\right)^{s'}+C\frac{L\left(|e_1-xe_0|^{r+1},x\right)^{s'}}{t^{s'}\varphi(x)^{s'}}\right)^{\frac{1}{s'}}\cdot K^s_{1,\varphi}(f^{(r)},t)$$

  holds for any positive linear operator $L:\mathbf{C}[0,1]\longrightarrow\mathbf{C}[0,1]$, *any* $f\in\mathbf{C^r}[0,1]$, *any* $x\in(0,1)$ *and any* $t>0$ *then* $A\ge 1$, $B\ge\dfrac{2^{s'}}{r!^{s'}}$ *and* $C\ge\dfrac{2^{s'(r+1)}}{(2r+1)!!^{s'}}$.

Proof. For the estimate with $K^s_{1,\varphi}$, $1\le s<\infty$ we use estimate (15) and Lemma 2. For the converse part we make the same choices like in Theorem 2. □

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