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MATHEMATICAL MODEL FOR THE CONTRACTIONAL PHASE OF STARS

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Abstract

An algoritm for constructing tracks for the contractional phase of stars is given. This paper presents an original way of solving the system of equations corresponding to the radiative nucleus by using Taylor's series in close vicinity to the center of stars. We have obtained the form of the Taylor series which are used in the case of undetermination 0/0.

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1 The equations of the interior structure

For the whole radiative nucleus of the star, we consider the following equations to be true: the hydrostatic equilibrum, the mass distribution, the equation of luminosity and the one for temperature, (see, eg. Menzel and others, 1963, [3]; Aller and McLoughlin, 1965, [1]; Cox and Giuli, 1968, [2].

$$\frac{\mathrm{d}P(r)}{\mathrm{d}r} = -G\cdot\frac{M(r)\rho(r)}{r^2} \tag{1}$$

$$\frac{\mathrm{d}M(r)}{\mathrm{d}r} = 4\pi r^2\rho(r) \tag{2}$$

$$\frac{\mathrm{d}L(r)}{\mathrm{d}r} = 4\pi r^2\rho(r)\varepsilon(r) \tag{3}$$

$$\frac{\mathrm{d}T(r)}{\mathrm{d}r} = -\frac{3}{4ac}\cdot\frac{\kappa(r)\rho(r)}{T^3(r)}\cdot\frac{L(r)}{4\pi r^2}, \tag{4}$$

where $P(r), M(r), L(r)$ and $T(r)$ represent the values of pressure, mass, luminosity and temperature in a point placed at the distance r from the center of the star.

We may adopt the perfect gas equation to give pressure as a function of density and temperature:

$$P = \left(\frac{k}{\mu H}\right)\rho T. \tag{5}$$

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During the early contractional phase, the chemical composition of the star does not change with time. In order to get a physical insight into the problem we make some simplifying assumptions; for example we may assume that the star evolves through homologous constraction excepting in the close neighbourhood.

This assumption imposes certain restrictions on the rate of energy generation by gravitational contraction and on the opacity. One of the conditions for the homologous constraction of a gas sphere is satisfied if the opacity obeys Kramers law

$$\kappa = \kappa_0 \rho T^{-3.5} \tag{6}$$

and the ratio of specific heats γ is equal to 5/3..

The formula for the rate of energy generation by gravitational contraction is given by:

$$\varepsilon = \varepsilon_0 T, \tag{7}$$

where

$$\varepsilon_0 = -\frac{3\gamma - 4}{\gamma - 1} \cdot \frac{k}{\mu H} \cdot \frac{1}{R} \cdot \frac{\mathrm{d}R}{\mathrm{d}t}. \tag{8}$$

2 Mathematical solving of the stellar model

Schwarzschild's transformations are applied for system (1)-(4).:

$$\begin{aligned} P(r) &= p \cdot \frac{GM^2}{4\pi R^4} \\ T(r) &= t \cdot \frac{\mu H}{k} \cdot \frac{GM}{R} \\ M(r) &= q \cdot M \\ L(r) &= f \cdot L \\ r &= R \cdot x \end{aligned} \tag{9}$$

The dimensionless variables p, q, f, t, x are introduced. With these variables, the system (1)-(4) becomes:

$$\begin{aligned} &\frac{\mathrm{d}p}{\mathrm{d}x} = -\frac{pq}{tx^2}, \quad \frac{\mathrm{d}q}{\mathrm{d}x} = \frac{px^2}{t} \\ &\frac{\mathrm{d}t}{\mathrm{d}x} = -C \cdot \frac{p^2 f}{t^{8.5} x^2}, \frac{\mathrm{d}f}{\mathrm{d}x} = Dpx^2, \end{aligned} \tag{10}$$

where

$$C = \frac{3}{4ac} \left(\frac{k}{HG} \right)^{7.5} \frac{1}{(4\pi)^3} \cdot \frac{\kappa_0}{\mu^{7.5}} \cdot \frac{L \cdot R^{0.5}}{M^{5.5}} \tag{11}$$

and

$$D = \varepsilon_0 \cdot \frac{\mu H}{k} \cdot \frac{M^2 G}{R \cdot L} = \frac{3\gamma - 4}{\gamma - 1} \cdot \frac{M^2 G}{L} \left(-\frac{1}{R^2} \cdot \frac{\mathrm{d}R}{\mathrm{d}t} \right). \tag{12}$$

The boundary conditions for inward integration from the surface are:

$$q = 1,\ f = 1,\ ,t = 0,\ p = 0,\ \text{at } x = 1. \tag{13}$$

We have adapted zero surface conditions in view of our assumation that radiative equilibrum extends right up to the surface of the star.

We have to effect the outward integrations in terms of the starred variables, as usual, with the help of the equations;

$$\begin{aligned} \frac{\mathrm{d}p^\star}{\mathrm{d}x^\star} &= -\frac{p^\star q^\star}{t^\star (x^\star)^2}, \quad \frac{\mathrm{d}q^\star}{\mathrm{d}x^\star} = \frac{p^\star (x^\star)^2}{t^\star} \\ \frac{\mathrm{d}t^\star}{\mathrm{d}x^\star} &= -\frac{(p^\star)^2 f^\star}{(t^\star)^{8.5}(x^\star)^2}, \quad \frac{\mathrm{d}f^\star}{\mathrm{d}x^\star} = p^\star (x^\star)^2, \end{aligned} \tag{14}$$

where

$$x = x_0 x^\star,\ t = t_0 t^\star,\ f = f_0 f^\star,\ p = p_0 p^\star,\ q = q_0 q^\star \tag{15}$$

and

$$\frac{q_0}{t_0 x_0} = 1,\ \frac{p_0 x_0^3}{t_0 q_0} = 1,\ \frac{C p_0^2 f_0}{t_0^{9.5} x_0} = 1,\ \frac{D p_0 x_0^3}{f_0} = 1. \tag{16}$$

The boundary conditions for solving these equations are:

$$q^\star = 0,\ f^\star = 0,\ t^\star = 1,\ p^\star = p_c^\star \text{ (to be determined)} \tag{17}$$

if we take $t_0 = t_c$. We introduce new parameters by following relations:

$$\begin{aligned} U &= \frac{\mathrm{d}\log M(r)}{\mathrm{d}\log r} = \frac{px^3}{qt} = \frac{p^\star (x^\star)^3}{q^\star t^\star} \\ V &= -\frac{\mathrm{d}\log P(r)}{\mathrm{d}\log r} = \frac{q}{tx} = \frac{q^\star}{t^\star x^\star}. \end{aligned} \tag{18}$$

We integrate system (14) with the limit conditions (17).

System (14) has an undetermination of the type 0/0 about the point $x^\star = 0$. Developing in Taylor series about the point $x^\star = 0$ and replacing in (14), we obtain:

$$\begin{aligned} p(x) &= p_0 - \frac{p_0}{6}\cdot x^2 + \frac{8p_0^3 - 5p_0^5}{360}\cdot x^4 + \dots \\ q(x) &= \frac{p_0}{3}\cdot x^3 - \frac{p_0^2}{30}\cdot x^5 + \dots \\ f(x) &= \frac{p_0}{3}\cdot x^3 - \frac{p_0^2}{30}\cdot x^5 + \dots \\ t(x) &= 1 - \frac{p_0^3}{6}\cdot x^2 + \frac{26p_0^4 - 5p_0^6}{240(3 - 2p_0^3)}\cdot x^4 - \dots \end{aligned} \tag{19}$$

We fit outward and inward solutions at an arbitrary point x_i with the help of $U - V$ curves, (Tatomir, 1986 [5]) and so determine the values of C, D and $p_c^\star$ and the variation of the physical variables throughout the configuration.

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