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A UNIVALENCE CONDITION

Horiana TUDOR[1]

Abstract

In this paper we obtain sufficient conditions for the analyticity and the univalence of the functions defined by an integral operator.

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1 Introduction

Let A be the class of analytic functions f in the unit disk $U = \{\ z \in \mathbb{C} :\ |z| < 1\}$ with $f(0) = 0,\ f'(0) = 1$. We denote by $U_r = \{\ z \in \mathbb{C} :\ |z| < r\}$ the disk of z-plane, where $r \in (0, 1],\ U_1 = U$ and $I = [0, \infty)$.

In order to prove our main result we need the theory of Löewner chains; we recall the basic result of this theory, from Pommerenke.

A family of functions $\{L(z, t)\}$, $z \in U$, $t \in I$, is a Löewner chain if $L(z, t)$ is analytic and univalent in U for all $t \in I$, and $L(z, t)$ is subordinate to $L(z, s)$ for all $0 \le t \le s$.

Theorem 1. ([3]). *Let $L(z, t) = a_1(t)z + a_2(t)z^2 + \dots,\ \ a_1(t) \neq 0$ be analytic in U_r, for all $t \in I$, locally absolutely continuous in I and locally uniformly with respect to U_r. For almost all $t \in I$, suppose that*

$$z\frac{\partial L(z, t)}{\partial z} = p(z, t)\frac{\partial L(z, t)}{\partial t}, \quad \forall z \in U_r,$$

where $p(z, t)$ is analytic in U and satisfies the condition Re $p(z, t) > 0$, for all $z \in U$, $t \in I$. If $|a_1(t)| \to \infty$ for $t \to \infty$ and $\{L(z, t)/a_1(t)\}$ forms a normal family in U_r, then for each $t \in I$, the function $L(z, t)$ has an analytic and univalent extension to the whole disk U.

We also need the following lemma of Carathéodory:

Lemma 1. ([2]). *Let f be an analytic function in U, $f(0) = 0$ and a positive real number M. If Re $f(z) \le M$ for all $z \in U$, then*

$$|\ f(z)\ | \le \frac{2M \cdot |z|}{1 - |z|}\ , \qquad \text{for all } z \in U.$$

[1]Faculty of Mathematics and Informatics, *Transilvania* University of Braşov, Romania, e-mail: htudor@unitbv.ro

2 Main results

Theorem 2. *Let α, c be complex numbers, $|\alpha - 1| < 1$, $|c| < 1$, $g \in A$ and h an analytic function in U, $h(z) = 1 + c_1 z + \dots$. If the inequality*

$$\left| c|z|^2 + (1 - |z|^2) \left[(\alpha - 1)\frac{zg'(z)}{g(z)} + \frac{zh'(z)}{h(z)} \right] \right| \leq 1 \tag{1}$$

is true for all $z \in U$, then the function

$$H(z) = \left(\alpha \int_0^z g^{\alpha-1}(u)h(u)du \right)^{1/\alpha} \tag{2}$$

is analytic and univalent in U, where the principal branch is intended.

Proof. From the analyticity of function g it follows that function $h_1(z) = \frac{g(z)}{z}$ is analytic in U and since $h_1(0) = 1$ there is a disk U_{r_1} in which $h_1(z) \neq 0$. Therefore we can choose the uniform branch of $(h_1(z))^{\alpha-1}$ equal to 1 at the origin, denoted by h_2. It is easy to see that the function

$$h_3(z,t) = \int_0^{e^{-t}z} u^{\alpha-1}h_2(u)h(u)du$$

can be written as $h_3(z,t) = z^\alpha h_4(z,t)$, where h_4 is also analytic in U_{r_1}. We define

$$L(z,t) = \left[\alpha z^\alpha h_4(z,t) + \frac{\alpha \left(e^{(2-\alpha)t} - e^{-\alpha t} \right) z^\alpha}{1+c} h_2(e^{-t}z)h(e^{-t}z) \right]^{1/\alpha}$$

and it is easy to check that $L(z,t)$ is analytic in a neighborhood U_{r_2}.
Elementary calculation shows that $L(z,t) = a_1(t)z + a_2(t)z^2 + \dots$, where

$$a_1(t) = e^{\frac{2-\alpha}{\alpha}t} \left[\frac{\alpha + (1 + c - \alpha)e^{-2t}}{1+c} \right]^{1/\alpha}$$

Under the assumption of the theorem, we have $a_1(t) \neq 0$ and $\lim_{t\to\infty} |a_1(t)| = \infty$. Since $L(z,t)$ is an analytic function in U_{r_2}, it results that there exist a number $0 < r_3 < r_2$ and a constant $k = k(r_3)$ such that

$$|\ L(z,t)/a_1(t)\ | < k, \qquad z \in U_{r_3},$$

and hence $\{L(z,t)/a_1(t)\}$ forms a normal family in U_{r_3}.

It can be easy see that $\frac{\partial L(z,t)}{\partial t}$ is an analytic function in U_{r_3} and therefore $L(z,t)$ is locally absolutely continuous in I, locally uniform with respect to U_{r_3}. We define

$$p(z,t) = z\frac{\partial L(z,t)}{\partial z} \Big/ \frac{\partial L(z,t)}{\partial t}$$

and we will prove that function $p(z,t)$ has an analytic extension with positive real part in U, for all $t \in I$. Let $w(z,t)$ be the function defined by

$$w(z,t) = \frac{p(z,t) - 1}{p(z,t) + 1}.$$

By simple calculation, we obtain

$$w(z,t) = ce^{-2t} + (1-e^{-2t})\left[\,(\alpha-1)e^{-t}z\frac{g'(e^{-t}z)}{g(e^{-t}z)} + e^{-t}z\frac{h'(e^{-t}z)}{h(e^{-t}z)}\right] \tag{3}$$

We have

$$w(z,0) = c \quad \text{and} \quad w(0,t) = ce^{-2t} + (1-e^{-2t})(\alpha-1)$$

Since $|\alpha-1|<1,\ |c|<1$ we obtain that

$$|\ w(z,0)\ | < 1 \quad \text{and also} \quad |\ w(0,t)\ | < 1 \tag{4}$$

Let t be a fixed positive number, $z\in U,\ z\neq 0$. Since $|e^{-t}z|\leq e^{-t}<1$ for all $z\in\overline{U}=\{z\in\mathbb{C}:|z|\leq 1\}$ we conclude that function $w(z,t)$ is analytic in $\overline{U}$. Using the maximum modulus principle it follows that for each $t>0$, arbitrary fixed, there exists $\theta=\theta(t)\in\mathbb{R}$ such that

$$|w(z,t)| < \max_{|\xi|=1}|w(\xi,t)| = |w(e^{i\theta},t)|, \tag{5}$$

We denote $u=e^{-t}\cdot e^{i\theta}$. Then $|u|=e^{-t}<1$ and from (3) we get

$$w(e^{i\theta},t) = c|u|^2 + (1-|u|^2)\left[\,(\alpha-1)\frac{ug'(u)}{g(u)} + \frac{uh'(u)}{h(u)}\,\right]$$

Since $u\in U$, the inequality (1) implies $|w(e^{i\theta},t)|\leq 1$ and from (4) and (5) we conclude that $|w(z,t)|<1$ for all $z\in U$ and $t\geq 0$.

From Theorem 1 it results that function $L(z,t)$ has an analytic and univalent extension to the whole disk U, for each $t\in I$, in particular $L(z,0)$. But $L(z,0)=H(z)$. Therefore function $H(z)$ defined by (2) is analytic and univalent in U. □

Theorem 3. *Let α be a complex number, $|\alpha-1|<1$, and h an analytic function in U, $h(z)=1+c_1z+\ldots$. If the inequality*

$$Re\frac{zh'(z)}{h(z)} \leq \frac{1-|\alpha-1|}{4} \tag{6}$$

is true for all $z\in U$, then function

$$H(z) = \left(\alpha\int_0^z u^{\alpha-1}h(u)du\right)^{1/\alpha} \tag{7}$$

is analytic and univalent in U, where the principal branch is intended.

Proof. In the particular case $g(z)\equiv z$ and $c=\alpha-1$, the inequality (1) becomes

$$\left|\,(\alpha-1) + (1-|z|^2)\frac{zh'(z)}{h(z)}\,\right| \leq 1 \tag{8}$$

and function $H(z)$ from Theorem 2 is defined by (7).

Under the assumption (6) of the theorem, we can apply Lemma 1 to function $\frac{zh'(z)}{h(z)}$ and we get

$$\left| (\alpha-1)+(1-|z|^2)\frac{zh'(z)}{h(z)} \right| \leq |\alpha-1|+(1-|z|^2)\frac{2|z|}{1-|z|}\cdot\frac{1-|\alpha-1|}{4}$$

$$=|\alpha-1|+\frac{|z|}{2}(\,1+|z|\,)\cdot(\,1-|\alpha-1|\,)\leq|\alpha-1|+(\,1-|\alpha-1|\,)=1$$

Inequality (8) is satisfied and from Theorem 2 it follows that function $H(z)$ defined by (7) is analytic and univalent in U. □

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