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THE d-TORSIONS AND CURVATURES ON 2-JET HOLOMORPHIC BUNDLES

Violeta ZALUTCHI[1]

Abstract

We study the curvature and torsion tensors of a $N-$linear connection on the holomorphic jets bundle $J^{(2,0)}M$ of a complex manifold M with respect to a given complex nonlinear connection N. The structures equations for such a connection in the holomorphic jets bundle are written.

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1 Introduction

Let M be a complex manifold, $dim_{\mathbf{C}}M = n$. We briefly revise the decomposition of the complexified tangent bundle $T_{\mathbf{C}}M = T'M \oplus T''M$, where $T'M$ is the holomorphic vector bundle over M, and $T''M$ is the conjugate antiholomorphic vector bundle.

If M is considered as a real manifold, its second order jet manifold $J^2(M)$ is a fiber bundle over M. We have the following decomposition of its complexified space: $J^2_{\mathbf{C}}M = J^{(2,0)}(M) \oplus J^{(1,1)}(M) \oplus J^{(0,2)}(M)$. In the paper [11], the holomorphic structure for the $J^{(2,0)}M$ bundle was studied. Let us remind that on the complex manifold $J^{(2,0)}M$, in a local chart, the coordinates are denoted by $Z = (z^k, \eta^k, \zeta^k)$, $k = \overline{1,n}$, and for the changes in the local bases on M, these will transform in accordance with the following rules:

$$\begin{aligned} z'^i &= z'^i(z) \; ; \; \eta'^i = \frac{\partial z'^i}{\partial z^j}\eta^j; \\ 2\zeta'^i &= \frac{\partial \eta'^i}{\partial z^j}\eta^j + 2\frac{\partial \eta'^i}{\partial \eta^j}\zeta^j, \end{aligned} \tag{1.1}$$

and that $\frac{\partial z'^i}{\partial z^j} = \frac{\partial \eta'^i}{\partial \eta^j} = \frac{\partial \zeta'^i}{\partial \zeta^j}$; $\frac{\partial \eta'^i}{\partial z^j} = \frac{\partial \zeta'^i}{\partial \eta^j}$. The local bases in the holomorphic bundle $T'(J^{(2,0)}M)$ are $\left\{\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \eta^i}, \frac{\partial}{\partial \zeta^j}\right\}$, and in $T''(J^{(2,0)}M)$ are their conjugates.

Two structures, that play a special role in defining the linear and nonlinear connection on $J^{(2,0)}M$ are: the natural complex structure J and the almost second order tangent structure F, which act by the rules: $F(\frac{\partial}{\partial z^j}) = \frac{\partial}{\partial \eta^j}$, $F(\frac{\partial}{\partial \eta^j}) = \frac{\partial}{\partial \zeta^j}$, $F(\frac{\partial}{\partial \eta^j}) = 0$ and their

[1]Faculty of Mathematics and Informatics, *Transilvania* University of Braşov, Romania, e-mail: zalvio@yahoo.com

conjugates everywhere. A complex nonlinear connection, briefly (c.n.c.), is given by a distribution $H(J^{(2,0)}M)$ which is supplementary to $H_1(J^{(2,0)}M)$ in $T'(J^{(2,0)}M)$, where $H_{1,z}(J^{(2,0)}M)$ is spanned by $\left\{\frac{\partial}{\partial\eta^j}, \frac{\partial}{\partial\zeta^j}\right\}$ in a local chart. With $V(J^{(2,0)}M)$ we denote the vertical bundle which is spanned by $\left\{\frac{\partial}{\partial\zeta^j}\right\}$. By conjugation, we obtain the decomposition for $T_C(J^{(2,0)}M)$. A local bases in $H_z(J^{(2,0)}M)$ is called adapted bases to the (c.n.c.) and it is written in this way $\frac{\delta}{\delta z^j} = \frac{\partial}{\partial z^j} - \overset{(1)}{N}{}^i_j \frac{\partial}{\partial\eta^i} - \overset{(2)}{N}{}^i_j \frac{\partial}{\partial\zeta^i}$ and with $F(\frac{\delta}{\delta z^j}) =: \frac{\delta}{\delta\eta^j} = \frac{\partial}{\partial\eta^j} - \overset{(1)}{N}{}^i_j \frac{\partial}{\partial\zeta^i}$ a local adapted bases in $H_{1,z}(J^{(2,0)}M)$ is obtained.

The adapted bases will change as follows: $\frac{\delta}{\delta z^j} = \frac{\partial z'^i}{\partial z^j}\frac{\delta}{\delta z'^i}$ and $\frac{\delta}{\delta\eta^j} = \frac{\partial z'^i}{\partial z^j}\frac{\delta}{\delta z'^i}$. Obviously $\frac{\delta}{\delta\zeta^j} = \frac{\partial z'^i}{\partial z^j}\frac{\delta}{\delta\zeta'^i}$, so these fields are changing as those on the base manifold M. Generally, the tensor measures, which are changed by $\frac{\partial z'^i}{\partial z^j}$ or by their conjugates $\frac{\partial \bar{z}'^i}{\partial \bar{z}^j}$, are called d-tensor fields. The adapted bases on $T''(J^{(2,0)}M)$ is obtained by conjugation.

The dual cobases is $\{dz^i, \delta\eta^i = d\eta^i + \overset{(1)}{M}{}^i_j\, dz^j, \delta\zeta^i = d\zeta^i + \overset{(1)}{M}{}^i_j\, d\eta^j + \overset{(2)}{M}{}^i_j\, dz^j\}$, where

$$\overset{(1)}{M}{}^i_j := \overset{(1)}{N}{}^i_j \,;\; \overset{(2)}{M}{}^i_j := \overset{(2)}{N}{}^i_j + \overset{(1)}{N}{}^i_k \overset{(1)}{N}{}^k_j \,. \tag{1.2}$$

The functions $\overset{(\alpha)}{N}{}^i_j$, where $\alpha = 1, 2$, are the coefficients of the N - (c.n.c.) which determines the decomposition

$$\begin{aligned} T_C\left(J^{(2,0)}M\right) &= H\left(J^{(2,0)}M\right) \oplus H_1\left(J^{(2,0)}M\right) \oplus V\left(J^{(2,0)}M\right) \\ &\oplus \overline{H}\left(J^{(2,0)}M\right) \oplus \overline{H}_1\left(J^{(2,0)}M\right) \oplus \overline{V}(J^{(2,0)}M). \end{aligned}$$

In [11] we introduce the N-complex linear connection notion as a connection on $J^{(2,0)}M$ which preserves the distributions and satisfies a special condition concerning its coefficients. Subsequently we study the torsions and curvatures of an N-(c.l.c.) on $J^{(2,0)}M$, according to the decomposition of $T_C\left(J^{(2,0)}M\right)$. For the sake of simplicity we use further the abbreviations: $\delta_{0k} = \frac{\delta}{\delta z^k}$, $\delta_{1k} = \frac{\delta}{\delta\eta^k}$, $\delta_{2k} = \frac{\partial}{\partial\zeta^k}$ and $\delta_{0\bar{k}} = \frac{\delta}{\delta\bar{z}^k}$, $\delta_{1\bar{k}} = \frac{\delta}{\delta\bar{\eta}^k}$, $\delta_{2\bar{k}} = \frac{\partial}{\partial\bar{\zeta}^k}$.

2 The torsion of a N -(c.l.c.) D

The torsion $\mathbf{T}$ of the N -(c.l.c.) $D\Gamma(N)$ is given by

$$\mathbf{T}(X,Y) = D_X Y - D_Y X - [X,Y], \forall X, Y \in \mathcal{X}\left(J^{(2,0)}M\right).$$

Because a vector field $X \in \mathcal{X}\left(J^{(2,0)}M\right)$, can be written, with respect to the above decomposition, as $X = X^H + X^{H_1} + X^V + X^{\bar{H}} + X^{\bar{H}_1} + X^{\bar{V}}$, we obtain the following vector fields:

$$\mathbf{T}\left(X^{H},Y^{H}\right)\ ;\ \mathbf{T}\left(X^{H},Y^{H_1}\right)\ ;\ \mathbf{T}\left(X^{H},Y^{V}\right)\ ;\ \mathbf{T}\left(X^{H_1},Y^{H_1}\right)\ ;\ \mathbf{T}\left(X^{H_1},Y^{V}\right);$$
$$\mathbf{T}\left(X^{V},Y^{V}\right)\ ;\ \mathbf{T}(X^{H},Y^{\bar{H}})\ ;\ \mathbf{T}(X^{H},Y^{\bar{H}_1})\ ;\ \mathbf{T}(X^{H},Y^{\bar{V}}\ );\ \mathbf{T}(X^{H_1},Y^{\bar{H}_1});$$
$$\mathbf{T}(X^{H_1},Y^{\bar{V}})\ ;\ \mathbf{T}(X^{V},Y^{\bar{V}})\ ;\ \mathbf{T}(X^{H_1},Y^{\bar{H}})\ ;\ \mathbf{T}(X^{V},Y^{\bar{H}})\ ;\ \mathbf{T}(X^{V},Y^{\bar{H}_1}).$$

Then, we have:

Proposition 2.1. *The tensor of torsion* $\boldsymbol{T}$ *of the* N *-(c.l.c.)* D *is given by the following d- tensor fields:*

$$\begin{aligned}
\mathbf{T}\left(X^{H},Y^{H}\right) &= h\mathbf{T}\left(X^{H},Y^{H}\right)+h_1\mathbf{T}\left(X^{H},Y^{H}\right)+v\mathbf{T}\left(X^{H},Y^{H}\right);\\
\mathbf{T}\left(X^{H},Y^{H_1}\right) &= h\mathbf{T}\left(X^{H},Y^{H_1}\right)+h_1\mathbf{T}\left(X^{H},Y^{H_1}\right)+v\mathbf{T}\left(X^{H},Y^{H_1}\right);\\
\mathbf{T}\left(X^{H},Y^{V}\right) &= h\mathbf{T}\left(X^{H},Y^{V}\right)+h_1\mathbf{T}\left(X^{H},Y^{V}\right)+v\mathbf{T}\left(X^{H},Y^{V}\right);\\
\mathbf{T}\left(X^{H_1},Y^{H_1}\right) &= h_1\mathbf{T}\left(X^{H_1},Y^{H_1}\right)+v\mathbf{T}\left(X^{H_1},Y^{H_1}\right);\\
\mathbf{T}\left(X^{H_1},Y^{V}\right) &= h_1\mathbf{T}\left(X^{H_1},Y^{V}\right)+v\mathbf{T}\left(X^{H_1},Y^{V}\right);\\
\mathbf{T}\left(X^{V},Y^{V}\right) &= v\mathbf{T}\left(X^{V},Y^{V}\right);\\
\mathbf{T}(X^{H},Y^{\bar{H}}) &= h\mathbf{T}(X^{H},Y^{\bar{H}})+\bar{h}\mathbf{T}(X^{H},Y^{\bar{H}})+h_1\mathbf{T}(X^{H},Y^{\bar{H}})\\
&\quad+\bar{h}_1\mathbf{T}(X^{H},Y^{\bar{H}})+v\mathbf{T}(X^{H},Y^{\bar{H}})+\bar{v}\mathbf{T}(X^{H},Y^{\bar{H}});\\
\mathbf{T}(X^{H},Y^{\bar{H}_1}) &= h\mathbf{T}(X^{H},Y^{\bar{H}_1})+h_1\mathbf{T}(X^{H},Y^{\bar{H}_1})+\bar{h}_1\mathbf{T}(X^{H},Y^{\bar{H}_1})\\
&\quad+v\mathbf{T}(X^{H},Y^{\bar{H}_1})+\bar{v}\mathbf{T}(X^{H},Y^{\bar{H}_1});\\
\mathbf{T}(X^{H},Y^{\bar{V}}) &= h\mathbf{T}(X^{H},Y^{\bar{V}})+h_1\mathbf{T}(X^{H},Y^{\bar{V}})+v\mathbf{T}(X^{H},Y^{\bar{V}})\\
&\quad+\bar{v}\mathbf{T}(X^{H},Y^{\bar{V}});\\
\mathbf{T}(X^{H_1},Y^{\bar{H}_1}) &= h_1\mathbf{T}(X^{H_1},Y^{\bar{H}_1})+\bar{h}_1\mathbf{T}(X^{H_1},Y^{\bar{H}_1})+v\mathbf{T}(X^{H_1},Y^{\bar{H}_1})\\
&\quad+\bar{v}\mathbf{T}(X^{H_1},Y^{\bar{H}_1});\\
\mathbf{T}(X^{H_1},Y^{\bar{V}}) &= h_1\mathbf{T}(X^{H_1},Y^{\bar{V}})+v\mathbf{T}(X^{H_1},Y^{\bar{V}})+\bar{v}\mathbf{T}(X^{H_1},Y^{\bar{V}});\\
\mathbf{T}(X^{V},Y^{\bar{V}}) &= v\mathbf{T}(X^{V},Y^{\bar{V}})+\bar{v}\mathbf{T}(X^{H_1},Y^{\bar{V}}).
\end{aligned}$$

The local coefficients of the torsion $\boldsymbol{T}$ *are given by*

$$\begin{aligned}
hT\left(\delta_{0h},\delta_{0j}\right) &= T^{i}_{jh}\delta_{0i};\ \ h_1T\left(\delta_{0h},\delta_{0j}\right)=R^{i}_{jh_{(1)}}\delta_{1i};\ \ vT\left(\delta_{0h},\delta_{0j}\right)=R^{i}_{jh_{(2)}}\delta_{2i}\\
hT\left(\delta_{0h},\delta_{1j}\right) &= C^{i}_{jh_{(1)}}\delta_{0i};\ \ h_1T\left(\delta_{0h},\delta_{1j}\right)=P^{i}_{jh_{(11)}}\delta_{1i};\ \ vT\left(\delta_{0h},\delta_{1j}\right)=P^{i}_{jh_{(12)}}\delta_{2i}\\
hT\left(\delta_{0h},\delta_{2j}\right) &= C^{i}_{jh_{(2)}}\delta_{0i};\ \ h_1T\left(\delta_{0h},\delta_{2j}\right)=P^{i}_{jh_{(21)}}\delta_{1i};\ \ vT\left(\delta_{0h},\delta_{2j}\right)=P^{i}_{jh_{(22)}}\delta_{2i}
\end{aligned}$$

$$\begin{aligned}
h_1T\left(\delta_{1h},\delta_{1j}\right) &= Q^{i}_{jh_{(11)}}\delta_{1i};\ \ vT\left(\delta_{1h},\delta_{1j}\right)=Q^{i}_{jh_{(12)}}\delta_{2i};\ \ h_1T\left(\delta_{1h},\delta_{2j}\right)=Q^{i}_{jh_{(21)}}\delta_{1i};\\
vT\left(\delta_{1h},\delta_{2j}\right) &= Q^{i}_{jh_{(22)}}\delta_{2i};\ \ vT\left(\delta_{2h},\delta_{2j}\right)=S^{i}_{jh_{(2)}}\delta_{2i}\\
hT\left(\delta_{0h},\delta_{0\bar{j}}\right) &= T^{i}_{h\bar{j}_{(01)}}\delta_{0i};\ \ \bar{h}T\left(\delta_{0h},\delta_{0\bar{j}}\right)=T^{\bar{i}}_{\bar{j}h_{(11)}}\delta_{0\bar{i}};\ \ h_1T(\delta_{0h},\delta_{0\bar{j}})=R^{i}_{h\bar{j}_{(00)}}\delta_{1i};
\end{aligned}$$

$$
\begin{aligned}
\bar{h}_1T(\delta_{0h},\delta_{0\bar{j}}) &= R^{\bar{\imath}}_{\bar{j}h_{(01)}}\delta_{1\bar{\imath}};\ \ vT\left(\delta_{0h},\delta_{0\bar{j}}\right)=R^{i}_{h\bar{j}_{(02)}}\delta_{2i};\ \ \bar{v}T\left(\delta_{0h},\delta_{0\bar{j}}\right)=R^{\bar{\imath}}_{\bar{j}h_{(12)}}\delta_{2\bar{\imath}};\\
hT\left(\delta_{0h},\delta_{1\bar{j}}\right) &= C^{i}_{h\bar{j}_{(01)}}\delta_{0i};\ \ h_1T\left(\delta_{0h},\delta_{1\bar{j}}\right)=P^{i}_{h\bar{j}_{(01)}}\delta_{1i};\ \ \bar{h}_1T\left(\delta_{0h},\delta_{1\bar{j}}\right)=P^{\bar{\imath}}_{\bar{j}h_{(11)}}\delta_{1\bar{\imath}};\\
vT\left(\delta_{0h},\delta_{1\bar{j}}\right) &= P^{i}_{h\bar{j}_{(12)}}\delta_{2i};\ \ \bar{v}T\left(\delta_{0h},\delta_{1\bar{j}}\right)=R^{\bar{\imath}}_{\bar{j}h_{(01)}}\delta_{2\bar{\imath}};\ \ hT\left(\delta_{0h},\delta_{2\bar{j}}\right)=C^{i}_{h\bar{j}_{(02)}}\delta_{0i};\\
h_1T\left(\delta_{0h},\delta_{2\bar{j}}\right) &= P^{i}_{h\bar{j}_{(20)}}\delta_{1i};\ \ vT\left(\delta_{0h},\delta_{2\bar{j}}\right)=P^{i}_{h\bar{j}_{(21)}}\delta_{2i};\ \ \bar{v}T\left(\delta_{0h},\delta_{2\bar{j}}\right)=T^{\bar{\imath}}_{\bar{j}h_{(11)}}\delta_{2\bar{\imath}};\\
h_1T\left(\delta_{1h},\delta_{1\bar{j}}\right) &= C^{i}_{h\bar{j}_{(01)}}\delta_{1i};\ \ \bar{h}_1T\left(\delta_{1h},\delta_{1\bar{j}}\right)=Q^{\bar{\imath}}_{\bar{j}h_{(11)}}\delta_{1\bar{\imath}};\ \ vT\left(\delta_{1h},\delta_{1\bar{j}}\right)=P^{i}_{h\bar{j}_{(01)}}\delta_{2i}
\end{aligned}
$$

$$
\begin{aligned}
\bar{v}T\left(\delta_{1h},\delta_{1\bar{j}}\right) &= Q^{\bar{\imath}}_{\bar{j}h_{(12)}}\delta_{2\bar{\imath}};\ \ h_1T\left(\delta_{1h},\delta_{2\bar{j}}\right)=C^{i}_{h\bar{j}_{(02)}}\delta_{1i};\ \ vT\left(\delta_{1h},\delta_{2\bar{j}}\right)=P^{i}_{h\bar{j}_{(20)}}\delta_{2i};\\
\bar{v}T\left(\delta_{1h},\delta_{2\bar{j}}\right) &= Q^{\bar{\imath}}_{\bar{j}h_{(11)}}\delta_{2\bar{\imath}};\ \ vT\left(\delta_{2h},\delta_{2\bar{j}}\right)=C^{i}_{h\bar{j}_{(02)}}\delta_{2i};\ \ \bar{v}T\left(\delta_{2h},\delta_{2\bar{j}}\right)=S^{\bar{\imath}}_{\bar{j}h_{(22)}}\delta_{2\bar{\imath}};
\end{aligned}
$$

where

$$
\begin{aligned}
T^{i}_{jh} &= L^{i}_{jh}-L^{i}_{hj};\ \ R^{i}_{jh_{(1)}}=\overset{(1)}{A}{}^{i}_{(jh)};\ \ R^{i}_{jh_{(2)}}=\overset{(2)}{A}{}^{i}_{(jh)}+\overset{(1)}{A}{}^{k}_{(jh)}\overset{(1)}{N}{}^{i}_{k};\\
C^{i}_{jh_{(1)}} &= L^{i}_{jh}-\overset{(1)}{B}{}^{i}_{hj};\ \ P^{i}_{jh_{(11)}}=F^{i}_{jh};\ \ P^{i}_{jh_{(12)}}=\overset{(1)}{A}{}^{i}_{jh}-\overset{(2)}{B}{}^{i}_{hj}-\overset{(1)}{B}{}^{k}_{hj}\overset{(1)}{N}{}^{i}_{k};\\
C^{i}_{jh_{(2)}} &= -C^{i}_{hj};\ \ P^{i}_{jh_{(21)}}=-\overset{(1)}{C}{}^{i}_{hj};\ \ P^{i}_{jh_{(22)}}=L^{i}_{jh}-\overset{(2)}{C}{}^{i}_{hj}-\overset{(1)}{C}{}^{k}_{hj}\overset{(1)}{N}{}^{i}_{k};\\
Q^{i}_{jh_{(11)}} &= F^{i}_{jh}-F^{i}_{hj};\ \ Q^{i}_{jh_{(12)}}=-\overset{(1)}{B}{}^{i}_{(jh)};\ \ Q^{i}_{jh_{(21)}}=-C^{i}_{jh};
\end{aligned}
$$

$$
\begin{aligned}
Q^{i}_{jh_{(22)}} &= F^{i}_{jh}-\overset{(1)}{C}{}^{i}_{jh};\ \ S^{i}_{jh_{(2)}}=C^{i}_{jh}-C^{i}_{hj};\\
T^{i}_{h\bar{j}_{(01)}} &= -L^{i}_{h\bar{j}};\ \ T^{\bar{\imath}}_{\bar{j}h_{(11)}}=L^{\bar{\imath}}_{\bar{j}h};\ \ R^{i}_{h\bar{j}_{(00)}}=-\overset{(1)}{A}{}^{i}_{h\bar{j}};\ \ R^{\bar{\imath}}_{\bar{j}h_{(01)}}=\overset{(1)}{A}{}^{\bar{\imath}}_{\bar{j}h};
\end{aligned}
$$

$$
R^{i}_{h\bar{j}_{(02)}} = -(\overset{(2)}{A}{}^{i}_{h\bar{j}}+\overset{(1)}{A}{}^{m}_{h\bar{j}}\overset{(1)}{N}{}^{i}_{m});\ \ R^{\bar{\imath}}_{\bar{j}h_{(12)}}=\overset{(2)}{A}{}^{\bar{\imath}}_{\bar{j}h}+\overset{(1)}{A}{}^{\bar{m}}_{\bar{j}h}\overset{(1)}{N}{}^{\bar{\imath}}_{\bar{m}};\ \ C^{i}_{h\bar{j}_{(01)}}=-F^{i}_{h\bar{j}};
$$

$$
\begin{aligned}
P^{i}_{h\bar{j}_{(01)}} &= -\overset{(1)}{B}{}^{i}_{h\bar{j}};\ \ P^{\bar{\imath}}_{\bar{j}h_{(11)}}=L^{\bar{\imath}}_{\bar{j}h};\ \ P^{i}_{h\bar{j}_{(12)}}=-(\overset{(2)}{B}{}^{i}_{h\bar{j}}+\overset{(1)}{B}{}^{m}_{h\bar{j}}\overset{(1)}{N}{}^{i}_{m});\\
C^{i}_{h\bar{j}_{(02)}} &= -C^{i}_{h\bar{j}};\ \ P^{i}_{h\bar{j}_{(20)}}=-\overset{(1)}{C}{}^{i}_{h\bar{j}};\ \ Q^{\bar{\imath}}_{\bar{j}h_{(12)}}=\overset{(1)}{B}{}^{\bar{\imath}}_{\bar{j}h};\\
P^{i}_{h\bar{j}_{(21)}} &= -(\overset{(2)}{C}{}^{i}_{h\bar{j}}+\overset{(1)}{C}{}^{m}_{h\bar{j}}\overset{(1)}{N}{}^{i}_{m});\ \ Q^{\bar{\imath}}_{\bar{j}h_{(11)}}=F^{\bar{\imath}}_{\bar{j}h};\ \ S^{\bar{\imath}}_{\bar{j}h_{(22)}}=C^{\bar{\imath}}_{\bar{j}h},
\end{aligned}
$$

because

$$
\begin{aligned}
\mathbf{T}\left(\delta_{0h},\delta_{0j}\right) &= h\mathbf{T}\left(\delta_{0h},\delta_{0j}\right)+h_1\mathbf{T}\left(\delta_{0h},\delta_{0j}\right)+v\mathbf{T}\left(\delta_{0h},\delta_{0j}\right)\\
&= T^{i}_{jh}\delta_{0i}+R^{i}_{jh_{(1)}}\delta_{1i}+R^{i}_{jh_{(2)}}\delta_{2i}
\end{aligned}
$$

and

$$\begin{aligned}\mathbf{T}(\delta_{0h},\delta_{0j}) &= \nabla_{\delta_{0h}}\delta_{0j} - \nabla_{\delta_{0j}}\delta_{0h} - [\delta_{0h},\delta_{0j}] \\ &= L^i_{jh}\delta_{0i} - L^i_{hj}\delta_{0i} + [\overset{(1)}{A^i_{jh}} - \overset{(1)}{A^i_{hj}}]\,\delta_{1i} - [\overset{(2)}{A^i_{(hj)}} + \overset{(1)}{A^k_{(hj)}}\overset{(1)}{N^i_k}]\delta_{2i}.\end{aligned}$$

where

$$\overset{(\beta)}{A^i_{(jk)}} = \overset{(\beta)}{A^i_{jk}} - \overset{(\beta)}{A^i_{kj}} := \delta_{0k}\,\overset{(\beta)}{N^i_j} - \delta_{0j}\,\overset{(\beta)}{N^i_k}\,,\quad \beta = 1,2.$$

Analogue, we compute all the other coefficients of the torsion.

3 The curvature of an N -(c.l.c.) D

The curvature tensor $\mathbf{R}$ of D is given by

$$R(X,Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]}Z,\quad \forall X,Y,Z \in \mathcal{X}\left(J^{(2,0)}M\right)$$

Because $\overline{[X,Y]} = \left[\overline{X},\overline{Y}\right]$, we have $\overline{R(X,Y)Z} = R\left(\overline{X},\overline{Y}\right)\overline{Z}$.
The second tangent structure F satisfies

$$D_X FY = F D_X Y,\ \ \forall X,Y \in \mathcal{X}\left(J^{(2,0)}M\right).$$

Proposition 3.1. *For any N -(c.l.c.) D, the tensor of curvature* $\mathbf{R}$ *satisfies:*
i) $F[R(X,Y)Z] = R(X,Y)(FZ)$
ii) $F^2[R(X,Y)Z] = R(X,Y)\left(F^2Z\right)$, *for any* $X,Y,Z \in \mathcal{X}\left(J^{(2,0)}M\right)$.

Proof. i)

$$\begin{aligned}F[R(X,Y)Z] &= F\left[D_X D_Y Z - D_Y D_X Z - D_{[X,Y]}Z\right] \\ &= D_X(FD_YZ) - D_Y(FD_XZ) - D_{[X,Y]}(FZ) \\ &= D_XD_Y(FZ) - D_YD_X(FZ) - D_{[X,Y]}(FZ) \\ &= R(X,Y)(FZ).\end{aligned}$$

ii) $F^2[R(X,Y)Z] = F^2\left[D_XD_YZ - D_YD_XZ - D_{[X,Y]}Z\right]$
$= D_X\left(F^2D_YZ\right) - D_Y\left(F^2D_XZ\right) - D_{[X,Y]}\left(F^2Z\right)$
$= D_XD_Y\left(F^2Z\right) - D_YD_X\left(F^2Z\right) - D_{[X,Y]}\left(F^2Z\right) = R(X,Y)\left(F^2Z\right)$.

□

Let $Z = Z^H$ be a horizontal vector field and its transformation using the second tangent structure F, i.e., $FZ^H = Z^{H_1}, F^2Z^H = Z^V$.
Therefore, the $\mathbf{R}$ components $R(X,Y)Z^{H_1}$, $R(X,Y)Z^V$ are obtained by:

$$\begin{aligned}R(X,Y)Z^{H_1} &= F\left(R(X,Y)Z^H\right);\ R(X,Y)Z^{\overline{H_1}} = F\left(R(X,Y)Z^{\overline{H}}\right); \\ R(X,Y)Z^V &= F^2\left(R(X,Y)Z^H\right);\ R(X,Y)Z^{\overline{V}} = F^2\left(R(X,Y)Z^{\overline{H}}\right).\end{aligned}$$

We remark:

1) The important component of the tensor of curvature $\mathbf{R}$ is $R(X,Y)Z^H$, for any $X,Y \in \mathcal{X}\left(J^{(2,0)}M\right)$.

2) $R(X,Y)Z^H$ is a horizontal vector field, since, Z^H is a horizontal vector field $\Longrightarrow$ $D_X Z^H$ has the same property $\Longrightarrow R(X,Y)Z^H$ is a horizontal vector field.

Proposition 3.2. *The tensor of curvature* $\mathbf{R}$ *of an* N *-(c.l.c.)* D *has the properties*

$$\begin{aligned}
vR(X,Y)Z^H &= 0;\ h_1R(X,Y)Z^H = 0\,;\ vR(X,Y)Z^{H_1} = 0\,;\\
hR(X,Y)Z^{H_1} &= 0\,; hR(X,Y)Z^V = 0\,;\ h_1R(X,Y)Z^V = 0\,;\\
hR(X,Y)Z^{\overline{H}} &= 0\,;\ \overline{h}R(X,Y)Z^H = 0\,; h_1R(X,Y)Z^{\overline{H}_1} = 0\,;\\
\overline{h}_1R(X,Y)Z^{H_1} &= 0\,; vR(X,Y)Z^{\overline{V}} = 0\,; \overline{v}R(X,Y)Z^V = 0;\\
\overline{v}R(X,Y)Z^H &= 0\,;\ \overline{v}R(X,Y)Z^{H_1} = 0\,;\ \overline{v}R(X,Y)Z^{\overline{H}} = 0\,;\\
\overline{v}R(X,Y)Z^{\overline{H}_1} &= 0\,; \overline{h}R(X,Y)Z^{\overline{V}} = 0\,;\ \overline{h}R(X,Y)Z^{\overline{H}_1} = 0\,;\\
\overline{h}_1R(X,Y)Z^{\overline{H}} &= 0\,;\ \overline{h}_1R(X,Y)Z^{\overline{V}} = 0\,; \overline{h}R(X,Y)Z^{H_1} = 0\,;\\
\overline{h}R(X,Y)Z^V &= 0\,;\ \overline{h}_1R(X,Y)Z^H = 0\,;\ \overline{h}_1R(X,Y)Z^V = 0.
\end{aligned}$$

Therefore, we can write:

$R(X,Y)Z = R(X,Y)Z^H + R(X,Y)Z^{H_1} + R(X,Y)Z^V$
$+R(X,Y)Z^{\overline{H}} + R(X,Y)Z^{\overline{H}_1} + R(X,Y)Z^{\overline{V}}$, $\forall X,Y,Z \in \mathcal{X}\left(T_C J^{(2,0)}M\right)$.

The transformation of the tensor of curvature $\mathbf{R}$, using the second tangent structure F, leads to the following rules:

$$\begin{aligned}
F\left\{R\left(X^H,Y^H\right)Z^H\right\} &= R\left(X^H,Y^H\right)Z^{H_1};\\
F\left\{R\left(X^{H_1},Y^H\right)Z^H\right\} &= R\left(X^{H_1},Y^H\right)Z^{H_1}\\
F\left\{R\left(X^V,Y^H\right)Z^H\right\} &= R\left(X^V,Y^H\right)Z^{H_1};\\
F\left\{R\left(X^{H_1},Y^V\right)Z^H\right\} &= R\left(X^{H_1},Y^V\right)Z^{H_1}\\
F\left\{R\left(X^{\overline{H}},Y^{\overline{H}}\right)Z^{\overline{H}}\right\} &= R\left(X^{\overline{H}},Y^{\overline{H}}\right)Z^{\overline{H}_1};\\
F\left\{R\left(X^{\overline{H}_1},Y^{\overline{H}}\right)Z^{\overline{H}}\right\} &= R\left(X^{\overline{H}_1},Y^{\overline{H}}\right)Z^{\overline{H}_1}\\
F\left\{R\left(X^{\overline{V}},Y^{\overline{H}}\right)Z^{\overline{H}}\right\} &= R\left(X^{\overline{V}},Y^{\overline{H}}\right)Z^{\overline{H}_1};\\
F\left\{R\left(X^{\overline{H}_1},Y^{\overline{V}}\right)Z^{\overline{H}}\right\} &= R\left(X^{\overline{H}_1},Y^{\overline{V}}\right)Z^{\overline{H}_1}
\end{aligned}$$

and using transformation F^2, we obtain:

$$\begin{aligned}
F^2\left\{R\left(X^H,Y^H\right)Z^H\right\} &= R\left(X^H,Y^H\right)Z^V;\\
F^2\left\{R\left(X^{H_1},Y^H\right)Z^H\right\} &= R\left(X^{H_1},Y^H\right)Z^V\\
F^2\left\{R\left(X^V,Y^H\right)Z^H\right\} &= R\left(X^V,Y^H\right)Z^V;\\
F^2\left\{R\left(X^{H_1},Y^V\right)Z^H\right\} &= R\left(X^{H_1},Y^V\right)Z^V\\
F^2\left\{R\left(X^{\overline{H}},Y^{\overline{H}}\right)Z^{\overline{H}}\right\} &= R\left(X^{\overline{H}},Y^{\overline{H}}\right)Z^{\overline{V}};\\
F^2\left\{R\left(X^{\overline{H}_1},Y^{\overline{H}}\right)Z^{\overline{H}}\right\} &= R\left(X^{\overline{H}_1},Y^{\overline{H}}\right)Z^{\overline{V}}\\
F^2\left\{R\left(X^{\overline{V}},Y^{\overline{H}}\right)Z^{\overline{H}}\right\} &= R\left(X^{\overline{V}},Y^{\overline{H}}\right)Z^{\overline{V}};\\
F^2\left\{R\left(X^{\overline{H}_1},Y^{\overline{V}}\right)Z^{\overline{H}}\right\} &= R\left(X^{\overline{H}_1},Y^{\overline{V}}\right)Z^{\overline{V}}.
\end{aligned}$$

The d -tensors from the above formulas are called the *curvatures tensors* of an N -(c.l.c.) D.

Proposition 3.3. *The nonzero components of the N -(c.l.c) D are*

$$\begin{aligned}
R(\delta_{0m},\delta_{0j})\delta_{\alpha h} &= R^i_{hjm}\delta_{\alpha i}\,;R(\delta_{0m},\delta_{0j})\delta_{\alpha\bar{h}} = R^{\bar{i}}_{\bar{h}jm}\delta_{\alpha\bar{i}}\,;\\
R(\delta_{0m},\delta_{0\bar{j}})\delta_{\alpha h} &= R^i_{h\bar{j}m}\delta_{\alpha i};R(\delta_{\beta m},\delta_{0j})\delta_{\alpha h} = \overset{(\beta)}{P}{}^i_{hjm}\ \delta_{\alpha i}\,;\\
R(\delta_{\beta m},\delta_{0j})\delta_{\alpha\bar{h}} &= \overset{(\beta)}{P}{}^{\bar{i}}_{\bar{h}jm}\ \delta_{\alpha\bar{i}}\,;R(\delta_{\beta m},\delta_{0\bar{j}})\delta_{\alpha h} = \overset{(\beta)}{P}{}^i_{h\bar{j}m}\ \delta_{\alpha i};\\
R(\delta_{\beta m},\delta_{0\bar{j}})\delta_{\alpha\bar{h}} &= \overset{(\beta)}{Q}{}^{\bar{i}}_{\bar{h}\bar{j}m}\ \delta_{\alpha\bar{i}}\,;\ R(\delta_{1m},\delta_{1j})\delta_{\alpha h} = S^i_{hjm}\delta_{\alpha i}\,;\\
R(\delta_{1m},\delta_{1j})\delta_{\alpha\bar{h}} &= S^{\bar{i}}_{\bar{h}jm}\delta_{\alpha\bar{i}}\,;\ R(\delta_{1m},\delta_{1\bar{j}})\delta_{\alpha h} = S^i_{h\bar{j}m}\delta_{\alpha i}\,;\\
R(\delta_{2m},\delta_{2j})\delta_{\alpha h} &= O^i_{hjm}\delta_{\alpha i}\,;\ R(\delta_{2m},\delta_{2j})\delta_{\alpha\bar{h}} = O^{\bar{i}}_{\bar{h}jm}\delta_{\alpha\bar{i}};\\
R(\delta_{2m},\delta_{2\bar{j}})\delta_{\alpha h} &= O^i_{h\bar{j}m}\delta_{\alpha i}\,;\ R(\delta_{2m},\delta_{1j})\delta_{\alpha h} = G^i_{hjm}\delta_{\alpha i}\,;\\
R(\delta_{2m},\delta_{1j})\delta_{\alpha\bar{h}} &= G^{\bar{i}}_{\bar{h}jm}\delta_{\alpha\bar{i}}\,;\ R(\delta_{2m},\delta_{1\bar{j}})\delta_{\alpha h} = G^i_{h\bar{j}m}\delta_{\alpha i}\,;\\
R(\delta_{2m},\delta_{1\bar{j}})\delta_{\alpha\bar{h}} &= N^{\bar{i}}_{\bar{h}\bar{j}m}\delta_{\alpha\bar{i}},
\end{aligned}$$

and the corresponding conjugates, where $\alpha = 0, 1, 2$ and $\beta = 1, 2$.

Computing $R(X,Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]}Z$, we obtain the components of the curvature:

$$R^i_{hjm} = \delta_{0m}L^i_{hj} - \delta_{0j}L^i_{hm} + L^k_{hj}L^i_{km} - L^k_{hm}L^i_{kj} + \overset{(1)}{A}{}^k_{(jm)}\ F^i_{hk} + C^i_{hk}R^k_{jm(2)};$$

$$R^{\bar{i}}_{\bar{h}jm} = \delta_{0m}L^{\bar{i}}_{\bar{h}j} - \delta_{0j}L^{\bar{i}}_{\bar{h}m} + L^{\bar{k}}_{\bar{h}j}L^{\bar{i}}_{\bar{k}m} - L^{\bar{k}}_{\bar{h}m}L^{\bar{i}}_{\bar{k}j} + \overset{(1)}{A}{}^k_{(jm)}\ F^{\bar{i}}_{\bar{h}k} + C^{\bar{i}}_{\bar{h}k}R^k_{jm(2)};$$

$$R^i_{h\bar{j}m} = \delta_{0m}L^i_{h\bar{j}} - \delta_{0\bar{j}}L^i_{hm} + L^k_{h\bar{j}}L^i_{km} - L^k_{hm}L^i_{k\bar{j}} - \overset{(1)}{A}{}^k_{m\bar{j}}\ F^i_{hk} + \overset{(1)}{A}{}^{\bar{k}}_{\bar{j}m}\ F^i_{h\bar{k}}$$

$+C^i_{hk}R^k_{m\bar{j}(02)} + C^i_{h\bar{k}}R^{\bar{k}}_{\bar{j}m(12)};$

$\overset{(1)}{P}{}^i_{hjm} = \delta_{1m}L^i_{hj} - \delta_{0j}F^i_{hm} + L^k_{hj}F^i_{km} - F^k_{hm}L^i_{kj} + \overset{(1)}{B}{}^k_{jm}\, F^i_{hk} - C^i_{hk}P^k_{jm(12)};$

$\overset{(1)}{P}{}^{\bar{i}}_{\bar{h}jm} = \delta_{1m}L^{\bar{i}}_{\bar{h}j} - \delta_{0j}F^{\bar{i}}_{\bar{h}m} + L^{\bar{k}}_{\bar{h}j}F^{\bar{i}}_{\bar{k}m} - F^{\bar{k}}_{\bar{h}m}L^{\bar{i}}_{\bar{k}j} + \overset{(1)}{B}{}^k_{jm}\, F^{\bar{i}}_{\bar{h}k} - C^{\bar{i}}_{\bar{h}k}P^k_{jm(12)};$

$\overset{(1)}{P}{}^i_{h\bar{j}m} = \delta_{1m}L^i_{h\bar{j}} - \delta_{0\bar{j}}F^i_{hm} + L^k_{h\bar{j}}F^i_{km} - F^k_{hm}L^i_{k\bar{j}} - \overset{(1)}{A}{}^k_{m\bar{j}}\, C^i_{hk} + \overset{(1)}{B}{}^{\bar{k}}_{\bar{j}m}\, F^i_{h\bar{k}}$
$-C^i_{h\bar{k}}P^{\bar{k}}_{\bar{j}m(12)};$

$\overset{(2)}{P}{}^i_{hjm} = \delta_{2m}L^i_{hj} - \delta_{0j}C^i_{hm} + L^k_{hj}C^i_{km} - C^k_{hm}L^i_{kj} + \overset{(1)}{C}{}^k_{jm}\, F^i_{hk}$
$+C^i_{hk}(L^k_{mj} - P^k_{mj(22)});$

$\overset{(2)}{P}{}^{\bar{i}}_{\bar{h}jm} = \delta_{2m}L^{\bar{i}}_{\bar{h}j} - \delta_{0j}C^{\bar{i}}_{\bar{h}m} + L^{\bar{k}}_{\bar{h}j}C^{\bar{i}}_{\bar{k}m} - C^{\bar{k}}_{\bar{h}m}L^{\bar{i}}_{\bar{k}j} + \overset{(1)}{C}{}^k_{jm}\, F^{\bar{i}}_{\bar{h}k}$
$+C^{\bar{i}}_{\bar{h}k}(L^k_{mj} - P^k_{mj(22)});$

$\overset{(2)}{P}{}^i_{h\bar{j}m} = \delta_{2m}L^i_{h\bar{j}} - \delta_{0\bar{j}}C^i_{hm} + L^k_{h\bar{j}}C^i_{km} - C^k_{hm}L^i_{k\bar{j}} + \overset{(1)}{C}{}^{\bar{k}}_{\bar{j}m}\, F^i_{h\bar{k}} - C^i_{h\bar{k}}P^{\bar{k}}_{\bar{j}m(21)};$

$\overset{(1)}{Q}{}^{\bar{i}}_{\bar{h}\bar{j}m} = \delta_{1m}L^{\bar{i}}_{\bar{h}\bar{j}} - \delta_{0\bar{j}}F^{\bar{i}}_{\bar{h}m} + L^{\bar{k}}_{\bar{h}\bar{j}}F^{\bar{i}}_{\bar{k}m} - F^{\bar{k}}_{\bar{h}m}L^{\bar{i}}_{\bar{k}\bar{j}} + \overset{(1)}{B}{}^{\bar{k}}_{\bar{j}m}\, F^{\bar{i}}_{\bar{h}\bar{k}}$
$-C^{\bar{i}}_{\bar{h}\bar{k}}(\overset{(1)}{A}{}^k_{m\bar{j}} + P^{\bar{k}}_{\bar{j}m(12)});$

$\overset{(2)}{Q}{}^{\bar{i}}_{\bar{h}\bar{j}m} = \delta_{2m}L^{\bar{i}}_{\bar{h}\bar{j}} - \delta_{0\bar{j}}C^{\bar{i}}_{\bar{h}m} + L^{\bar{k}}_{\bar{h}\bar{j}}C^{\bar{i}}_{\bar{k}m} - C^{\bar{k}}_{\bar{h}m}L^{\bar{i}}_{\bar{k}\bar{j}} + \overset{(1)}{C}{}^{\bar{k}}_{\bar{j}m}\, F^{\bar{i}}_{\bar{h}\bar{k}} - C^{\bar{i}}_{\bar{h}\bar{k}}P^{\bar{k}}_{\bar{j}m(21)};$

$S^i_{hjm} = \delta_{1m}F^i_{hj} - \delta_{1j}F^i_{hm} + F^k_{hj}F^i_{km} - F^k_{hm}F^i_{kj} + \overset{(1)}{B}{}^k_{(jm)}\, C^i_{hk};$

$S^{\bar{i}}_{\bar{h}jm} = \delta_{1m}F^{\bar{i}}_{\bar{h}j} - \delta_{1j}F^{\bar{i}}_{\bar{h}m} + F^{\bar{k}}_{\bar{h}j}F^{\bar{i}}_{\bar{k}m} - F^{\bar{k}}_{\bar{h}m}F^{\bar{i}}_{\bar{k}j} + \overset{(1)}{B}{}^k_{(jm)}\, C^{\bar{i}}_{\bar{h}k};$

$S^i_{h\bar{j}m} = \delta_{1m}F^i_{h\bar{j}} - \delta_{1\bar{j}}F^i_{hm} + F^k_{h\bar{j}}F^i_{km} - F^k_{hm}F^i_{k\bar{j}} - \overset{(1)}{B}{}^k_{m\bar{j}}\, C^i_{hk} + \overset{(1)}{B}{}^{\bar{k}}_{\bar{j}m}\, C^i_{h\bar{k}};$
$O^i_{hjm} = \delta_{2m}C^i_{hj} - \delta_{2j}C^i_{hm} + C^k_{hj}C^i_{km} - C^k_{hm}C^i_{kj};$
$O^{\bar{i}}_{\bar{h}jm} = \delta_{2m}C^{\bar{i}}_{\bar{h}j} - \delta_{2j}C^{\bar{i}}_{\bar{h}m} + C^{\bar{k}}_{\bar{h}j}C^{\bar{i}}_{\bar{k}m} - C^{\bar{k}}_{\bar{h}m}C^{\bar{i}}_{\bar{k}j};$
$O^i_{h\bar{j}m} = \delta_{2m}C^i_{h\bar{j}} - \delta_{2\bar{j}}C^i_{hm} + C^k_{h\bar{j}}C^i_{km} - C^k_{hm}C^i_{k\bar{j}};$

$G^i_{hjm} = \delta_{2m}F^i_{hj} - \delta_{1j}C^i_{hm} + F^k_{hj}C^i_{km} - C^k_{hm}F^i_{kj} + \overset{(1)}{C}{}^k_{jm}\, C^i_{hk};$

$G^{\bar{i}}_{\bar{h}jm} = \delta_{2m}F^{\bar{i}}_{\bar{h}j} - \delta_{1j}C^{\bar{i}}_{\bar{h}m} + F^{\bar{k}}_{\bar{h}j}C^{\bar{i}}_{\bar{k}m} - C^{\bar{k}}_{\bar{h}m}F^{\bar{i}}_{\bar{k}j} + \overset{(1)}{C}{}^k_{jm}\, C^{\bar{i}}_{\bar{h}k};$

$G^i_{h\bar{j}m} = \delta_{2m}F^i_{h\bar{j}} - \delta_{1\bar{j}}C^i_{hm} + F^k_{h\bar{j}}C^i_{km} - C^k_{hm}F^i_{k\bar{j}} + \overset{(1)}{C}{}^{\bar{k}}_{\bar{j}m}\, C^i_{h\bar{k}};$

$N^{\bar{i}}_{\bar{h}\bar{j}m} = \delta_{2m}F^{\bar{i}}_{\bar{h}\bar{j}} - \delta_{1\bar{j}}C^{\bar{i}}_{\bar{h}m} + F^{\bar{k}}_{\bar{h}\bar{j}}C^{\bar{i}}_{\bar{k}m} - C^{\bar{k}}_{\bar{h}m}F^{\bar{i}}_{\bar{k}\bar{j}} + \overset{(1)}{C}{}^{\bar{k}}_{\bar{j}m}\, C^{\bar{i}}_{\bar{h}\bar{k}},$
where $\alpha = 0, 1, 2$.

4 The structure equations

We shall study the structure equations of the N -linear connection D.

Let $\left\{ \frac{\delta}{\delta z^i}, \frac{\delta}{\delta \eta^i}, \frac{\partial}{\partial \zeta^i} \right\}$, $i = 1, ...n$, be the adapted bases of the N-(c.n.c.). An N-(c.l.c.) on $J^{(2,0)}M$ is $D\Gamma(N) = \left(L^i_{jk}, F^i_{jk}, C^i_{jk}, L^{\bar{i}}_{\bar{j}k}, F^{\bar{i}}_{\bar{j}k}, C^{\bar{i}}_{\bar{j}k} \right)$ where

$$D_{\delta_{0k}}\delta_{\alpha j} = L^i_{jk}\delta_{\alpha i}; D_{\delta_{1k}}\delta_{\alpha j} = F^i_{jk}\delta_{\alpha i}; D_{\delta_{2k}}\delta_{\alpha j} = C^i_{jk}\delta_{\alpha i} \tag{4.1}$$
$$D_{\delta_{0k}}\delta_{\alpha \bar{j}} = L^{\bar{i}}_{\bar{j}k}\delta_{\alpha \bar{i}}; D_{\delta_{1k}}\delta_{\alpha \bar{j}} = F^{\bar{i}}_{\bar{j}k}\delta_{\alpha \bar{i}}; D_{\delta_{2k}}\delta_{\alpha \bar{j}} = C^{\bar{i}}_{\bar{j}k}\delta_{\alpha \bar{i}}$$

and $\alpha = 0, 1, 2$.

The connection form of an N-(c.l.c.) has the following simplified expression:

$$\omega^i_j = L^i_{jk}dz^k + F^i_{jk}\delta\eta^k + C^i_{jk}\delta\zeta^k + L^i_{j\bar{k}}d\bar{z}^k + F^i_{j\bar{k}}\delta\bar{\eta}^k + C^i_{j\bar{k}}\delta\bar{\zeta}^k \tag{4.2}$$

and by conjugation $\omega^{\bar{i}}_{\bar{j}} = \overline{\omega^i_j}$ is obtained.

Let D be a N-(c.n.c.) and ω^i_j the connection 1-forms of D.

Proposition 4.1. *If $D'' = d''$ then N-(c.l.c.) D is of $(1, 0)$-type and $L^{\bar{i}}_{\bar{j}k} = F^{\bar{i}}_{\bar{j}k} = C^{\bar{i}}_{\bar{j}k} = 0$, and their conjugates.*

Proof. $D" = d" \Leftrightarrow \omega^{\bar{i}}_{\bar{j}}\delta_{0\bar{i}} + \omega^{\bar{i}}_{\bar{j}}\delta_{1\bar{i}} + \omega^{\bar{i}}_{\bar{j}}\delta_{2\bar{i}} = 0 \Leftrightarrow \omega^{\bar{i}}_{\bar{j}} = 0 \Leftrightarrow \overline{\omega^i_j} = 0 \Leftrightarrow L^{\bar{i}}_{\bar{j}k} = F^{\bar{i}}_{\bar{j}k} = C^{\bar{i}}_{\bar{j}k} = L^i_{j\bar{k}} = F^i_{j\bar{k}} = C^i_{j\bar{k}} = 0$ □

Proposition 4.2. *D is of $(1, 0)$-type if and only if the natural complex operator performs $D_{JX}Y = JD_XY$.*

A vector field $X \in \mathcal{X}\left(T_C J^{(2,0)}M\right)$ is decomposed in the form

$$\begin{aligned} X &= X^H + X^{H_1} + X^V + X^{\overline{H}} + X^{\overline{H}_1} + X^{\overline{V}} \\ &= X^{0i}\delta_{0i} + X^{1i}\delta_{1i} + X^{2i}\delta_{2i} + X^{0\bar{i}}\delta_{0\bar{i}} + X^{1\bar{i}}\delta_{1\bar{i}} + X^{2\bar{i}}\delta_{2\bar{i}} \end{aligned} \tag{4.3}$$

We get for DX

$$\begin{aligned} DX = &\left\{dX^{0i} + X^{0i}\omega^i_m\right\}\delta_{0i} + \left\{dX^{1i} + X^{1i}\omega^i_m\right\}\delta_{1i} \\ &+ \left\{dX^{2i} + X^{2i}\omega^i_m\right\}\delta_{2i} + \left\{dX^{0\bar{i}} + X^{0\bar{i}}\omega^i_m\right\}\delta_{0\bar{i}} \\ &+ \left\{dX^{1\bar{i}} + X^{1\bar{i}}\omega^i_m\right\}\delta_{1\bar{i}} + \left\{dX^{2\bar{i}} + X^{2\bar{i}}\omega^i_m\right\}\delta_{2\bar{i}}. \end{aligned}$$

In order to obtain the structure equations of D, first we prove:

Lemma 4.1. *The exterior differentials of the 1-forms dz^i, $\delta\eta^i$, $\delta\zeta^i$ are expressed as follows:*

$$\begin{aligned} d(dz^i) &= 0; \\ d(\delta\eta^i) &= \frac{1}{2}\overset{(1)}{A}{}^i_{(jm)}\, dz^m \wedge dz^j + \overset{(1)}{B}{}^i_{jm}\, \delta\eta^m \wedge dz^j + \overset{(1)}{C}{}^i_{jm}\, \delta\zeta^m \wedge dz^j \\ &+ \overset{(1)}{A}{}^i_{j\overline{m}}\, d\bar{z}^m \wedge dz^j + \overset{(1)}{B}{}^i_{j\overline{m}}\, \delta\bar{\eta}^m \wedge dz^j + \overset{(1)}{C}{}^i_{j\overline{m}}\, \delta\bar{\zeta}^m \wedge dz^j; \end{aligned} \tag{4.4}$$

$$
\begin{aligned}
d(\delta\zeta^i) &= \frac{1}{2}R^i_{mj(2)}dz^m\wedge dz^j + P^i_{jm(12)}dz^m\wedge\delta\eta^j - P^i_{j\bar{m}(12)}\delta\overline{\eta}^m\wedge dz^j\\
&+ (L^i_{mj}-P^i_{mj(22)})\delta\zeta^m\wedge dz^j - R^i_{j\overline{m}(02)}d\overline{z}^m\wedge dz^j + \overset{(1)}{C}{}^i_{j\overline{m}}\,\delta\overline{\zeta}^m\wedge\delta\eta^j\\
&- P^i_{j\bar{m}(21)}\delta\overline{\zeta}^m\wedge dz^j + \overset{(1)}{C}{}^i_{jm}\,\delta\zeta^m\wedge\delta\eta^j + \overset{(1)}{A}{}^i_{j\overline{m}}\,d\overline{z}^m\wedge\delta\eta^j\\
&+ \frac{1}{2}\overset{(1)}{B}{}^i_{(jm)}\,\delta\eta^m\wedge\delta\eta^j + \overset{(1)}{B}{}^i_{j\overline{m}}\,\delta\overline{\eta}^m\wedge\delta\eta^j
\end{aligned}
$$

where the coefficients of $\delta\zeta^{(\alpha)j}\wedge\delta\zeta^{(\alpha)m}$ *are skew-symmetric* ($\alpha,\beta=0,1,2,\zeta^0=z,\zeta^1=\eta,\zeta^2=\zeta$).

Theorem 4.1. *The structure equations of an N-linear connection D on the total space are given by:*

$$
\begin{aligned}
d(dz^i)-dz^k\wedge\omega^i_k &= -\overset{(0)}{\Omega}{}^i;\\
d(\delta\eta^i)-\delta\eta^k\wedge\omega^i_k &= -\overset{(1)}{\Omega}{}^i;\\
d(\delta\zeta^i)-\delta\zeta^k\wedge\omega^i_k &= -\overset{(2)}{\Omega}{}^i;\\
d\omega^i_j-\omega^k_j\wedge\omega^i_k &= -\Omega^i_j
\end{aligned}
\tag{4.5}
$$

where $\overset{(\alpha)}{\Omega}{}^i$, $\alpha=1,2,3$ *are the 2-forms of torsion*

$$
\begin{aligned}
\overset{(0)}{\Omega}{}^i &= \frac{1}{2}T^i_{jk}dz^j\wedge dz^k + P^i_{jk(11)}dz^j\wedge\delta\eta^k - C^i_{kj(2)}dz^j\wedge\delta\zeta^k - T^i_{j\bar{k}(01)}dz^j\wedge d\overline{z}^k\\
&- C^i_{j\bar{k}(01)}dz^j\wedge\delta\overline{\eta}^k - C^i_{j\bar{k}(02)}dz^j\wedge\delta\overline{\zeta}^k;
\end{aligned}
\tag{4.6}
$$

$$
\begin{aligned}
\overset{(1)}{\Omega}{}^i &= \frac{1}{2}R^i_{jk(1)}dz^j\wedge dz^k - C^i_{jk(1)}dz^j\wedge\delta\eta^k - P^i_{kj(21)}dz^j\wedge\delta\zeta^k - R^i_{j\bar{k}(00)}dz^j\wedge d\overline{z}^k\\
&- P^i_{j\bar{k}(01)}dz^j\wedge\delta\overline{\eta}^k - P^i_{j\bar{k}(20)}dz^j\wedge\delta\overline{\zeta}^k + \frac{1}{2}Q^i_{jk(11)}\delta\eta^j\wedge\delta\eta^k - Q^i_{kj(21)}\delta\zeta^j\wedge\delta\eta^k\\
&+ T^i_{k\bar{j}(01)}d\overline{z}^j\wedge\delta\eta^k + C^i_{k\bar{j}(01)}\delta\overline{\eta}^j\wedge\delta\eta^k;
\end{aligned}
$$

$$
\begin{aligned}
\overset{(2)}{\Omega}{}^i &= \frac{1}{2}R^i_{jm}dz^m\wedge dz^j + P^i_{mj(12)}\delta\eta^m\wedge dz^j + P^i_{mj(22)}\delta\zeta^m\wedge dz^j + R^i_{j\overline{m}(02)}d\overline{z}^m\wedge dz^j\\
&+ P^i_{j\overline{m}(12)}\delta\overline{\eta}^m\wedge dz^j + P^i_{j\overline{m}(21)}\delta\overline{\zeta}^m\wedge dz^j + Q^i_{mj(22)}\delta\zeta^m\wedge\delta\eta^j - C^i_{m\bar{j}(01)}\delta\zeta^m\wedge\delta\overline{\eta}^j\\
&+ R^i_{j\overline{m}(00)}d\overline{z}^m\wedge\delta\eta^j + P^i_{j\overline{m}(01)}\delta\overline{\eta}^m\wedge\delta\eta^j + P^i_{j\overline{m}(20)}\delta\overline{\zeta}^m\wedge\delta\eta^j - T^i_{m\bar{j}(01)}\delta\zeta^m\wedge d\overline{z}^j\\
&- \frac{1}{2}Q^i_{mj(12)}\delta\eta^m\wedge\delta\eta^j + \frac{1}{2}S^i_{mj(2)}\delta\zeta^m\wedge\delta\zeta^j - C^i_{m\bar{j}(02)}\delta\zeta^m\wedge\delta\overline{\zeta}^j.
\end{aligned}
$$

and where Ω^i_m *are the 2-form of curvature*

$$\begin{aligned}
\Omega^i_j &= \frac{1}{2}R^i_{jmr}\ dz^m\wedge dz^r+\overset{(1)}{P^i_{j\bar{m}\bar{r}}}\ d\overline{z}^m\wedge\delta\overline{\eta}^r+\overset{(2)}{P^i_{j\bar{m}\bar{r}}}\ d\overline{z}^m\wedge\delta\overline{\zeta}^r+O^i_{j\bar{m}r}\ \delta\overline{\zeta}^m\wedge\delta\zeta^r\\
&+ \frac{1}{2}S^i_{j\bar{m}\bar{r}}\ \delta\overline{\eta}^m\wedge\delta\overline{\eta}^r+\overset{(1)}{P^i_{jmr}}\ dz^m\wedge\delta\eta^r+\overset{(2)}{P^i_{jmr}}\ dz^m\wedge\delta\zeta^r+G^i_{jmr}\ \delta\eta^m\wedge\delta\zeta^r\\
&+ \frac{1}{2}S^i_{jmr}\ \delta\eta^m\wedge\delta\eta^r+R^i_{j\bar{m}r}\ d\overline{z}^m\wedge dz^r+\overset{(1)}{Q^i_{jm\bar{r}}}\ dz^m\wedge\delta\overline{\eta}^r+\overset{(2)}{Q^i_{jm\bar{r}}}\ dz^m\wedge\delta\overline{\zeta}^r\\
&+ \frac{1}{2}O^i_{jmr}\ \delta\zeta^m\wedge\delta\zeta^r+\overset{(1)}{P^i_{j\bar{m}r}}\ d\overline{z}^m\wedge\delta\eta^r+\overset{(2)}{P^i_{j\bar{m}r}}\ d\overline{z}^m\wedge\delta\zeta^r+N^i_{jm\bar{r}}\ \delta\eta^m\wedge\delta\bar{\zeta}^r\\
&+ \frac{1}{2}R^i_{j\bar{m}\bar{r}}\ d\overline{z}^m\wedge d\overline{z}^r+S^i_{j\bar{m}r}\ \delta\overline{\eta}^m\wedge\delta\eta^r+G^i_{j\bar{m}r}\ \delta\overline{\eta}^m\wedge\delta\zeta^r+G^i_{j\bar{m}\bar{r}}\ \delta\overline{\eta}^m\wedge\delta\overline{\zeta}^r\\
&+ \frac{1}{2}O^i_{j\bar{m}\bar{r}}\delta\overline{\zeta}^m\wedge\delta\overline{\zeta}^r.
\end{aligned}$$

Proof.

$$\begin{aligned}
d\omega^i_j &= \frac{1}{2}(\delta_{0m}L^i_{jr}-\delta_{0r}L^i_{jm}+F^i_{jl}\overset{(1)}{A^l_{(rm)}}+C^i_{jl}R^l_{mr(2)})dz^m\wedge dz^r\\
&+ (\delta_{1m}L^i_{jr}-\delta_{0r}F^i_{jm}+F^i_{jl}\overset{(1)}{B^l_{rm}}-C^i_{jl}P^l_{mr(12)})\delta\eta^m\wedge dz^r\\
&+ [\delta_{2m}L^i_{jr}-\delta_{0r}C^i_{jm}+F^i_{jl}\overset{(1)}{C^l_{rm}}+C^i_{jl}(L^l_{mr}-P^l_{mr(22)})]\delta\zeta^m\wedge dz^r\\
&+ \frac{1}{2}(\delta_{1m}F^i_{jr}-\delta_{1r}F^i_{jm}-C^i_{jl}Q^l_{rm(12)})\delta\eta^m\wedge\delta\eta^r
\end{aligned}$$

$$\begin{aligned}
&+ \frac{1}{2}(\delta_{2m}C^i_{jr}-\delta_{2r}C^i_{jm})\delta\zeta^m\wedge\delta\zeta^r+(\delta_{2m}F^i_{jr}-\delta_{1r}C^i_{jm}+C^i_{jl}\overset{(1)}{C^l_{rm}})\delta\zeta^m\wedge\delta\eta^r\\
&+ (\delta_{0\bar{m}}L^i_{jr}-\delta_{0r}L^i_{j\overline{m}}+F^i_{jl}\overset{(1)}{A^l_{r\bar{m}}}-F^i_{j\bar{l}}\overset{(1)}{A^{\bar{l}}_{\bar{m}r}}-C^i_{j\bar{l}}R^{\bar{l}}_{\bar{m}r(12)}-C^i_{jl}R^l_{r\bar{m}(02)})d\overline{z}^m\wedge dz^r\\
&+ (\delta_{0m}F^i_{j\bar{r}}-\delta_{1\bar{r}}L^i_{jm}-F^i_{jl}\overset{(1)}{B^l_{m\bar{r}}}+C^i_{j\bar{l}}\overset{(1)}{A^{\bar{l}}_{\bar{r}m}}+C^i_{jl}P^l_{m\bar{r}(12)})dz^m\wedge\delta\overline{\eta}^r\\
&+ (\delta_{0m}C^i_{j\bar{r}}-\delta_{2\bar{r}}L^i_{jm}-F^i_{jl}\overset{(1)}{C^l_{m\bar{r}}}+C^i_{ml}P^l_{j\bar{r}(21)})dz^m\wedge\delta\overline{\zeta}^r
\end{aligned}$$

$$\begin{aligned}
&+ (\delta_{1m}L^i_{j\bar{r}}-\delta_{0\bar{r}}F^i_{jm}+F^i_{j\bar{l}}\overset{(1)}{B^{\bar{l}}_{\bar{r}m}}-C^i_{j\bar{l}}P^{\bar{l}}_{\bar{r}m(12)}-C^i_{jl}\overset{(1)}{A^l_{m\bar{r}}})\delta\eta^m\wedge d\overline{z}^r\\
&+ (\delta_{2m}L^i_{j\bar{r}}-\delta_{0\bar{r}}C^i_{jm}+F^i_{j\bar{l}}\overset{(1)}{C^{\bar{l}}_{\bar{r}m}}-C^i_{j\bar{l}}P^{\bar{l}}_{\bar{r}m(21)})\delta\zeta^m\wedge\delta\overline{z}^r\\
&+ \frac{1}{2}(\delta_{0\overline{m}}L^i_{j\bar{r}}-\delta_{0\bar{r}}L^i_{j\overline{m}}+F^i_{j\bar{l}}\overset{(1)}{A^{\bar{l}}_{(\bar{r}\bar{m})}}+C^i_{j\bar{l}}R^{\bar{l}}_{\bar{m}\bar{r}(2)})d\overline{z}^m\wedge d\overline{z}^r
\end{aligned}$$

$$
\begin{aligned}
&+ && (\delta_{1\overline{m}}L^i_{j\overline{r}} - \delta_{0\overline{r}}F^i_{j\overline{m}} + F^i_{j\bar{l}}\stackrel{(1)}{B}{}^{\bar{l}}_{\bar{r}\bar{m}} - C^i_{j\bar{l}}P^{\bar{l}}_{\bar{m}\bar{r}(12)})\delta\overline{\eta}^m \wedge d\overline{z}^r \\
&+ && [\delta_{2\overline{m}}L^i_{j\overline{r}} - \delta_{0\overline{r}}C^i_{j\overline{m}} + F^i_{j\bar{l}}\stackrel{(1)}{C}{}^{\bar{l}}_{\bar{r}\bar{m}} + C^i_{j\bar{l}}(L^{\bar{l}}_{\bar{m}\bar{r}} - P^{\bar{l}}_{\bar{m}\bar{r}(22)})]\delta\overline{\zeta}^m \wedge d\overline{z}^r \\
&+ && (\delta_{1\bar{m}}F^i_{jr} - \delta_{1r}F^i_{j\bar{m}} + C^i_{jl}\stackrel{(1)}{B}{}^{l}_{r\bar{m}} - C^i_{j\bar{l}}\stackrel{(1)}{B}{}^{\bar{l}}_{\bar{m}r})\delta\overline{\eta}^m \wedge \delta\eta^r \\
&+ && (\delta_{1\overline{m}}C^i_{jr} - \delta_{2r}F^i_{j\bar{m}} - C^i_{j\bar{l}}\stackrel{(1)}{C}{}^{\bar{l}}_{\bar{m}r})\delta\overline{\eta}^m \wedge \delta\zeta^r \\
&+ && \frac{1}{2}(\delta_{1\overline{m}}F^i_{j\overline{r}} - \delta_{1\bar{r}}F^i_{j\overline{m}} + C^i_{j\bar{l}}\stackrel{(1)}{B}{}^{\bar{l}}_{(\bar{r}\bar{m})})\delta\overline{\eta}^m \wedge \delta\overline{\eta}^r
\end{aligned}
$$

$$
\begin{aligned}
&+ && (\delta_{2\overline{m}}F^i_{j\overline{r}} - \delta_{1\bar{r}}C^i_{j\overline{m}} + C^i_{j\bar{l}}\stackrel{(1)}{C}{}^{\bar{l}}_{\bar{r}\bar{m}})\delta\overline{\zeta}^m \wedge \delta\overline{\eta}^r + \frac{1}{2}(\delta_{2\overline{m}}C^i_{j\bar{r}} - \delta_{2\bar{r}}C^i_{j\bar{m}})\delta\overline{\zeta}^m \wedge \delta\overline{\zeta}^r \\
&+ && (\delta_{2\bar{m}}F^i_{jr} - \delta_{1r}C^i_{j\overline{m}} + C^i_{jl}\stackrel{(1)}{C}{}^{l}_{r\bar{m}})\delta\overline{\zeta}^m \wedge \delta\eta^r + (\delta_{2\overline{m}}C^i_{jr} - \delta_{2r}C^i_{j\overline{m}})\delta\overline{\zeta}^m \wedge \delta\zeta^r
\end{aligned}
$$

and

$$
\begin{aligned}
\omega^k_j \wedge \omega^i_k &= && \frac{1}{2}(L^k_{jm}L^i_{kr} - L^k_{jr}L^i_{km})dz^m \wedge dz^r + \frac{1}{2}(F^k_{jm}F^i_{kr} - F^k_{jr}F^i_{km})\delta\eta^m \wedge \delta\eta^r \\
&+ && \frac{1}{2}(C^k_{jm}C^i_{kr} - C^k_{jr}C^i_{km})\delta\zeta^m \wedge \delta\zeta^r + \frac{1}{2}(L^k_{j\overline{m}}L^i_{k\bar{r}} - L^k_{j\bar{r}}L^i_{k\overline{m}})d\overline{z}^m \wedge d\overline{z}^r \\
&+ && \frac{1}{2}(F^k_{j\overline{m}}F^i_{k\bar{r}} - F^k_{j\bar{r}}F^i_{k\overline{m}})\delta\overline{\eta}^m \wedge \delta\overline{\eta}^r + \frac{1}{2}(C^k_{j\overline{m}}C^i_{k\bar{r}} - C^k_{j\bar{r}}C^i_{k\overline{m}})\delta\overline{\zeta}^m \wedge \delta\overline{\zeta}^r \\
&+ && (L^k_{jm}F^i_{kr} - F^k_{jr}L^i_{km})dz^m \wedge \delta\eta^r + (C^k_{jm}L^i_{kr} - L^k_{jr}C^i_{km})\delta\zeta^m \wedge dz^r \\
&+ && (L^k_{j\overline{m}}L^i_{kr} - L^k_{jr}L^i_{k\overline{m}})d\overline{z}^m \wedge dz^r + (L^k_{jm}F^i_{k\bar{r}} - F^k_{j\bar{r}}L^i_{km})dz^m \wedge \delta\overline{\eta}^r \\
&+ && (L^k_{jm}C^i_{k\bar{r}} - C^k_{j\bar{r}}L^i_{km})dz^m \wedge \delta\overline{\zeta}^r + (C^k_{jm}F^i_{kr} - F^k_{jr}C^i_{km})\delta\zeta^m \wedge \delta\eta^r \\
&+ && (F^k_{jm}L^i_{k\bar{r}} - F^i_{km}L^k_{j\bar{r}})\delta\eta^m \wedge d\overline{z}^r + (F^k_{j\overline{m}}F^i_{kr} - F^k_{jr}F^i_{k\overline{m}})\delta\overline{\eta}^m \wedge \delta\eta^r \\
&+ && (C^k_{j\overline{m}}F^i_{kr} - F^k_{jr}C^i_{k\overline{m}})\delta\overline{\zeta}^m \wedge \delta\eta^r + (C^k_{jm}L^i_{k\bar{r}} - L^k_{m\bar{r}}C^i_{km})\delta\zeta^m \wedge d\overline{z}^r \\
&+ && (F^k_{j\overline{m}}C^i_{kr} - C^k_{jr}F^i_{k\overline{m}})\delta\overline{\eta}^m \wedge \delta\zeta^r + (C^k_{j\overline{m}}C^i_{kr} - C^k_{jr}C^i_{k\overline{m}})\delta\overline{\zeta}^m \wedge \delta\zeta^r \\
&+ && (F^k_{j\overline{m}}L^i_{k\bar{r}} - L^k_{j\bar{r}}F^i_{k\overline{m}})\delta\overline{\eta}^m \wedge d\overline{z}^r + (C^k_{j\overline{m}}L^i_{k\bar{r}} - L^k_{j\bar{r}}C^i_{k\overline{m}})\delta\overline{\zeta}^m \wedge d\overline{z}^r \\
&+ && (F^i_{k\bar{r}}C^k_{j\overline{m}} - C^i_{k\overline{m}}F^k_{j\bar{r}})\delta\overline{\zeta}^m \wedge \delta\overline{\eta}^r
\end{aligned}
$$

□

References

[1] M. Abate, G.Patrizio, *Finsler Metrics - A Global Approach*, Lecture Notes in Math., **1591**, Springer-Verlag, 1994.

[2] T. Aikou, *Finsler geometry on complex vector bundles*, Riemannian Finsler Geometry, MSRI Pub. **50**, 2004.

[3] I. Bucataru, *Characterization of the nonlinear connection in the higher order geometry*, BJGA Vol. **2**(1997), nr. 2, 13-22.

[4] N. Branzei, *Geodesics and Jacobi fields in second order geometry*, Ed. Univ. Transilvania Brasov, 2007.

[5] K. Chandler, P-M. Wong, *Finsler geometry of holomorphic jet bundles*, Riemann-Finsler geometry, MSRI Publ., **50** (2004), 107-196.

[6] M. Green, P. Griffiths, *Two applications of algebraic geometry to entire holomorphic mappings*, The Chern Symp. Berkeley, 1979, Springer, New York, 1980.

[7] R. Miron, *The geometry of higher order Lagrange spaces. Applications to Mechanics and Physics*, Kluwer Acad. Publ., FTPH, **82**, 1997.

[8] G. Munteanu, *Complex spaces in Finsler, Lagrange and Hamilton geometries*, Kluwer Acad. Publ., FTPH **141**, 2004.

[9] D.J. Saunders, *The geometry of Jet Bundles*, London Math. Soc., Lecture Note Series, **142**, Cambridge U.P. 1989.

[10] W. Stoll, P.-M. Wong, *On Holomorphic jet bunldes*, preprint, arxiv: math/0003226v1/2000.

[11] V. Zalutchi, *The geometry of (2,0)-jet Bundles*, D.G.D.S, Vol **12**, 2010, 311-320.

[12] P.-Mann Wong, *A survey of complex Finsler geometry*, Advanced Studied in Pure Math., Vol. **48**, Math. Soc. of Japan, (2007), 375-433.