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A SURVEY OF TWO - DIMENSIONAL COMPLEX FINSLER SPACES

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Abstract

In this paper, we make a survey of two - dimensional complex Finsler spaces. The tools of this study are the complex Berwald frames $\{l, m, \bar{l}, \bar{m}\}$, $\{\lambda, \mu, \bar{\lambda}, \bar{\mu}\}$ and the Chern-Finsler connection with respect to these frames.

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1 Introduction

The study of two-dimensional real Finsler spaces was initiated by L. Berwald ([9]). His theory is developed based on the choice of an orthonormal frame consisting of the normalized Liouville field and a unit field orthogonal to it. Further substantial contributions on this topic are from M. Matsumoto [12], G.S. Asanov [6], A. Bejancu and H.R. Faran [8], Z. Shen [19], etc.

Based on some ideas from real case, the study of 2 - dimensional complex Finsler spaces is a challenging topic. In a previous paper [17], we constructed the vertical Berwald frame in which the orthogonality is, with respect to the Hermitian structure, defined by the fundamental metric tensor of a 2 - dimensional complex Finsler space, on the holomorphic tangent manifold $T'M$. The main purpose of this paper is to clarify some details about the vertical Berwald frame and then, to investigate 2 - dimensional complex Finsler spaces with respect to some extensions of this frame.

Subsequently, we have made an overview of the paper's content.

In §2, we recall some preliminary properties of the n- dimensional complex Finsler spaces and complete with some others needed.

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In §3, after we review the construction of the Berwald frame of a complex Finsler manifold of dimension two, we prefer to work in a fixed local chart in which a local complex Berwald frame is obtained, which is extended to one on the horizontal part. We also find the expression of the complex Chern-Finsler connection with respect to these local frames. The independence of the obtained results from chosen chart is incessantly studied.

Some characterizations of the complex Finsler manifolds of dimension two comes from the exploration of the $v\bar{v}-$, $h\bar{v}-$ and $v\bar{h}-$ Riemann type tensors, (Theorems 4.1). An immediate interest for the 2 - dimensional complex Berwald spaces is induced by the properties of the $h\bar{v}-$ and $v\bar{h}-$ Riemann type tensors (Propositions 4.1, 4.2. 4.3). Also, the investigations of $h\bar{h}-$ Riemann type tensor lead to some important characterizations of 2 - dimensional complex Finsler spaces which are weakly Kähler, (Propositions 4.4 and 4.5). All these results are in §4.

2 Preliminaries

In the beginning, we will make a short introduction in the complex Finsler geometry and we will set the basic notions and terminology. For more, see [1, 15].

Let M be a $n-$ dimensional complex manifold, $z=(z^k)_{k=\overline{1,n}}$ are the complex coordinates in a local chart.

The complexified of the real tangent bundle T_CM splits into the sum of holomorphic tangent bundle $T'M$ and its conjugate $T''M$. The bundle $T'M$ is itself a complex manifold, and the local coordinates in a local chart will be denoted by $u=(z^k,\eta^k)_{k=\overline{1,n}}$. They are changed into $(z'^k,\eta'^k)_{k=\overline{1,n}}$ by the rules $z'^k=z'^k(z)$ and $\eta'^k=\frac{\partial z'^k}{\partial z^l}\eta^l$.

A *complex Finsler space* is a pair (M,F), where $F:T'M\to\mathbb{R}^+$ is a continuous function satisfying the conditions:

i) $L:=F^2$ is smooth on $\widetilde{T'M}:=T'M\backslash\{0\}$;

ii) $F(z,\eta)\geq 0$, the equality holds if and only if $\eta=0$;

iii) $F(z,\lambda\eta)=|\lambda|F(z,\eta)$ for $\forall\lambda\in\mathbb{C}$;

iv) the Hermitian matrix $\left(g_{i\bar{j}}(z,\eta)\right)$ is positive defined, where $g_{i\bar{j}}:=\frac{\partial^2 L}{\partial\eta^i\partial\bar{\eta}^j}$ is the fundamental metric tensor. Equivalently, it means that the indicatrix is strongly pseudoconvex.

Consequently, from *iii*) we have $\frac{\partial L}{\partial\eta^k}\eta^k=\frac{\partial L}{\partial\bar{\eta}^k}\bar{\eta}^k=L$, $\frac{\partial g_{i\bar{j}}}{\partial\eta^k}\eta^k=\frac{\partial g_{i\bar{j}}}{\partial\bar{\eta}^k}\bar{\eta}^k=0$ and $L=g_{i\bar{j}}\eta^i\bar{\eta}^j$.

Roughly speaking, the geometry of a complex Finsler space consists of the study of the geometric objects of the complex manifold $T'M$ endowed with the Hermitian metric structure defined by $g_{i\bar{j}}$.

Therefore, the first step is to study the sections of the complexified tangent bundle of $T'M$, which is decomposed in the sum $T_C(T'M)=T'(T'M)\oplus T''(T'M)$. Let $VT'M\subset T'(T'M)$ be the vertical bundle, locally spanned by $\{\frac{\partial}{\partial\eta^k}\}$, and $VT''M$ its conjugate.

At this point, the idea of complex nonlinear connection, briefly (*c.n.c.*), is an instrument in 'linearization' of this geometry. A (*c.n.c.*) is a supplementary complex subbundle to $VT'M$ in $T'(T'M)$, i.e. $T'(T'M)=HT'M\oplus VT'M$. The horizontal distribution $H_uT'M$

is locally spanned by $\{\frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N_k^j \frac{\partial}{\partial \eta^j}\}$, where $N_k^j(z,\eta)$ are the coefficients of the $(c.n.c.)$. The pair $\{\delta_k := \frac{\delta}{\delta z^k}, \dot{\partial}_k := \frac{\partial}{\partial \eta^k}\}$ will be called the adapted frame of the $(c.n.c.)$ which obey to the change rules $\delta_k = \frac{\partial z'^j}{\partial z^k}\delta'_j$ and $\dot{\partial}_k = \frac{\partial z'^j}{\partial z^k}\dot{\partial}'_j$. By conjugation, everywhere is obtained an adapted frame $\{\delta_{\bar{k}}, \dot{\partial}_{\bar{k}}\}$ on $T''_u(T'M)$. The dual adapted bases are $\{dz^k, \delta\eta^k\}$ and $\{d\bar{z}^k, \delta\bar{\eta}^k\}$.

Certainly, a main problem in this geometry is to determine a $(c.n.c.)$ related only to the fundamental function of the complex Finsler space (M, F).

The next step is the action of a derivative law D on the sections of $T_C(T'M)$. First, let us consider the *Sasaki* type lift of the metric tensor $g_{i\bar{j}}$,

$$\mathcal{G} = g_{i\bar{j}} dz^i \otimes d\bar{z}^j + g_{i\bar{j}} \delta\eta^i \otimes \delta\bar{\eta}^j. \tag{2.1}$$

A Hermitian connection D, of $(1,0)-$ type, which satisfies in addition $D_{JX}Y = JD_XY$, for all X horizontal vectors and J the natural complex structure of the manifold, is the so called Chern-Finsler connection (cf. [1]), in brief $C - F$. The $C - F$ connection is locally given by the following coefficients (cf. [15]):

$$N_j^k = g^{\overline{m}k} \frac{\partial g_{l\overline{m}}}{\partial z^j} \eta^l = L_{lj}^k \eta^l \; ; \; L_{jk}^i = g^{\bar{l}i} \delta_k g_{j\bar{l}} \; \; ; \; C_{jk}^i = g^{\bar{l}i} \dot{\partial}_k g_{j\bar{l}} \; \; ; \; L_{\bar{j}k}^{\bar{\imath}} = C_{\bar{j}k}^{\bar{\imath}} = 0, \tag{2.2}$$

where here and further on δ_k is the adapted frame of the $C - F$ $(c.n.c.)$ and $D_{\delta_k}\delta_j = L_{jk}^i \delta_i$, $D_{\dot{\partial}_k}\dot{\partial}_j = C_{jk}^i \dot{\partial}_i$, etc. The $C - F$ connection is the main tool in this study.

Denoting by " $_{\mid}$ " , " $|$ " , "$_{\bar{\mid}}$" and "$\bar{|}$", the $h-$, $v-$, $\overline{h}-$, $\overline{v}-$ covariant derivatives with respect to $C - F$ connection, respectively, for any X^i it results

$$\begin{aligned} X^i_{|k} \; &: = \delta_k X^i + X^l L^i_{lk} \; ; \; X^i|_k := \dot{\partial}_k \, X^i + X^l C^i_{lk}; \\ X^i_{|\bar{k}} \; &: = \delta_{\bar{k}} X^i \; ; \; X^i|_{\bar{k}} := \dot{\partial}_{\bar{k}} \, X^i; \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} \eta^i_{|k} &= \eta^i_{|\bar{k}} = \eta^i|_{\bar{k}} = 0 \; ; \; \; \eta^i|_k = \delta^i_k; \\ g_{i\bar{j}|k} &= g_{i\bar{j}|\bar{k}} = g_{i\bar{j}}|_k = g_{i\bar{j}}|_{\bar{k}} = 0; \\ (g_{i\bar{j}}\bar{\eta}^j)_{|k} &= (g_{i\bar{j}}\bar{\eta}^j)_{|\bar{k}} = (g_{i\bar{j}}\bar{\eta}^j)|_k = 0 \; ; \; \; (g_{i\bar{j}}\bar{\eta}^j)|_{\bar{k}} = g_{i\bar{k}}. \end{aligned} \tag{2.4}$$

The nonzero curvatures of the $C - F$ connection are denoted by

$$\begin{aligned} R(\delta_h, \delta_{\bar{k}})\delta_j &= R^i_{j\bar{k}h}\delta_i \; ; \; R(\dot{\partial}_h, \delta_{\bar{k}})\delta_j = \Xi^i_{j\bar{k}h}\delta_i \; ; \; R(\delta_h, \dot{\partial}_{\bar{k}})\delta_j = P^i_{j\bar{k}h}\delta_i \\ R(\delta_h, \delta_{\bar{k}})\dot{\partial}_j &= R^i_{j\bar{k}h}\dot{\partial}_i \; ; \; R(\dot{\partial}_h, \delta_{\bar{k}})\dot{\partial}_j = \Xi^i_{j\bar{k}h}\dot{\partial}_i \; ; \; R(\delta_h, \dot{\partial}_{\bar{k}})\dot{\partial}_j = P^i_{j\bar{k}h}\dot{\partial}_i \\ R(\dot{\partial}_h, \dot{\partial}_{\bar{k}})\delta_j &= S^i_{j\bar{k}h}\delta_i \; ; \; \; R(\dot{\partial}_h, \dot{\partial}_{\bar{k}})\dot{\partial}_j = S^i_{j\bar{k}h}\dot{\partial}_i \; , \end{aligned}$$

where

$$\begin{aligned} R^i_{j\overline{h}k} &= -\delta_{\overline{h}} L^i_{jk} - \delta_{\overline{h}}(N^l_k) C^i_{jl} \; \; ; \; \Xi^i_{j\overline{h}k} = -\delta_{\overline{h}} C^i_{jk} = \Xi^i_{k\overline{h}j}; \\ P^i_{j\overline{h}k} &= -\dot{\partial}_{\overline{h}} L^i_{jk} - \dot{\partial}_{\overline{h}}(N^l_k) C^i_{jl} \; \; ; \; S^i_{j\overline{h}k} = -\dot{\partial}_{\overline{h}} C^i_{jk} = S^i_{k\overline{h}j}. \end{aligned} \tag{2.5}$$

Considering the Riemann tensor

$$\mathbf{R}(W,\overline{Z},X,\overline{Y}) \ : \ = G(R(X,\overline{Y})W,\overline{Z}), \tag{2.6}$$
$$\mathbf{R}(W,\overline{Z},X,\overline{Y}) \ = \ \overline{\mathbf{R}(Z,\overline{W},Y,\overline{X})}$$

for $W, X, \overline{Z}, \overline{Y}$ horizontal or vertical vectors, it results the $h\bar{h}-$, $h\bar{v}-$, $v\bar{h}-$, $v\bar{v}-$ Riemann type tensors: $R_{\bar{j}i\bar{h}k} = g_{l\bar{j}}R^l_{i\bar{h}k}$; $P_{\bar{j}i\bar{h}k} = g_{l\bar{j}}P^l_{i\bar{h}k}$; $\Xi_{\bar{j}i\bar{h}k} = g_{l\bar{j}}\Xi^l_{i\bar{h}k}$; $S_{\bar{j}i\bar{h}k} = g_{l\bar{j}}S^l_{i\bar{h}k}$, which have properties $\overline{R_{i\bar{j}k\bar{h}}} = R_{\bar{j}i\bar{h}k}$; $\overline{\Xi_{i\bar{j}k\bar{h}}} = P_{\bar{j}i\bar{h}k}$; $\overline{P_{i\bar{j}k\bar{h}}} = \Xi_{\bar{j}i\bar{h}k}$; $\overline{S_{i\bar{j}k\bar{h}}} = S_{\bar{j}i\bar{h}k} = S_{\bar{h}i\bar{j}k}$, where $R_{i\bar{j}k\bar{h}} := \overline{R_{\bar{i}j\bar{k}h}}$, etc., (see [15], p. 77).

Further on, everywhere the index 0 means the contraction by η, for example $R^i_{0\bar{h}k} := R^i_{j\bar{h}k}\eta^j$.

Proposition 2.1. *i)* $R^i_{0\bar{h}k} = -\delta_{\bar{h}}N^i_k$; $\ R_{\bar{r}0\bar{h}k} = -g_{i\bar{r}}\delta_{\bar{h}}N^i_k$;

ii) $P^i_{0\bar{h}k} = -g^{\bar{m}i}C_{0\bar{m}\bar{h}|k}$; $\ P_{\bar{r}0\bar{h}k} = -C_{0\bar{r}\bar{h}|k}$; $P^i_{0\bar{0}k} = 0$;

iii) $\Xi^i_{j\bar{h}k} = -C^i_{jk|\bar{h}}$; $\ S^i_{j\bar{h}k} = -C^i_{jk}|_{\bar{h}}$; $\Xi^i_{0\bar{h}k} = \Xi^i_{k\bar{h}0} = S^i_{0\bar{h}k} = S^i_{k\bar{h}0}$;

$\Xi_{\bar{r}j\bar{h}k} = -C_{j\bar{r}k|\bar{h}}$; $\ S_{\bar{r}j\bar{h}k} = -C_{j\bar{r}k}|_{\bar{h}}$, *where we denoted* $C_{j\bar{r}k} := C^i_{jk}g_{i\bar{r}}$ *and* $C_{r\bar{j}\bar{k}}$ *is its conjugate*;

iv) $C_{l\bar{r}\bar{h}|k} = (\dot{\partial}_{\bar{h}}L^i_{lk})g_{i\bar{r}} + (\dot{\partial}_{\bar{h}}N^i_k)C_{i\bar{r}l}$;

v) $C_{l\bar{r}h|k} = (\dot{\partial}_h L^i_{lk})g_{i\bar{r}}$;

vi) $P^i_{j\bar{h}k} - P^i_{0\bar{h}k}|_j - P^i_{0\bar{h}r}C^r_{kj} = 0$.

Proof. i) and iii) results by (2.3), (2.4), (2.5) and $C^i_{0k} = C^i_{k0} = 0$.

For ii) we have

$P_{\bar{r}0\bar{h}k} = g_{i\bar{r}}P^i_{0\bar{h}k} = g_{i\bar{r}}\dot{\partial}_{\bar{h}}N^i_k = -g_{i\bar{r}}\dot{\partial}_{\bar{h}}\left(g^{\bar{m}i}\frac{\partial g_{j\bar{m}}}{\partial z^k}\eta^j\right)$

$= g_{i\bar{r}}g^{\bar{m}l}g^{\bar{s}i}\left(\dot{\partial}_{\bar{h}}g_{l\bar{s}}\right)\frac{\partial g_{j\bar{m}}}{\partial z^k}\eta^j - g_{i\bar{r}}g^{\bar{m}i}\dot{\partial}_{\bar{h}}\left(\frac{\partial g_{j\bar{m}}}{\partial z^k}\right)\eta^j$

$= g^{\bar{m}l}\left(\dot{\partial}_{\bar{h}}g_{l\bar{r}}\right)\frac{\partial g_{j\bar{m}}}{\partial z^k}\eta^j - \frac{\partial}{\partial z^k}\left(\dot{\partial}_{\bar{h}}g_{j\bar{r}}\right)\eta^j = C_{l\bar{r}\bar{h}}N^l_k - \frac{\partial}{\partial z^k}\left(C_{j\bar{r}\bar{h}}\eta^j\right).$

Because $C_{0\overline{rh}} := C_{l\overline{rh}}\eta^l$ it leads to

$C_{0\overline{rh}|k} = (C_{l\overline{rh}}\eta^l)_{|k} = \delta_k(C_{l\overline{rh}}\eta^l) = \frac{\partial}{\partial z^k}\left(C_{l\overline{rh}}\eta^l\right) - N^s_k\dot{\partial}_s\left((\dot{\partial}_{\bar{h}}g_{l\bar{r}})\eta^l\right)$

$= \frac{\partial}{\partial z^k}\left(C_{l\overline{rh}}\eta^l\right) - N^s_k\dot{\partial}_{\bar{h}}\left((\dot{\partial}_s g_{l\bar{r}})\eta^l\right) - N^s_kC_{l\overline{rh}}\delta^l_s = \frac{\partial}{\partial z^k}\left(C_{l\overline{rh}}\eta^l\right) - N^s_kC_{s\overline{rh}}$. From here, result the second relation of ii). The others immediately result by this.

Now, differentiating $N^i_kg_{i\bar{r}} = \frac{\partial g_{j\bar{r}}}{\partial z^k}\eta^j$ with respect to η^l yields $L^i_{lk}g_{i\bar{r}} = \frac{\partial g_{l\bar{r}}}{\partial z^k} - N^i_kC_{i\bar{r}l}$, which differentiated by $\bar{\eta}^h$ leads to iv).

Differentiating $L^i_{lk}g_{i\bar{r}} = \frac{\partial g_{l\bar{r}}}{\partial z^k} - N^i_kC_{i\bar{r}l}$, by $\bar{\eta}^h$ it results v).

It is obvious that $P^i_{0\bar{h}k} = -\dot{\partial}_{\bar{h}}N^i_k$. Hence,

$P^i_{j\bar{h}k} = -\dot{\partial}_{\bar{h}}L^i_{jk} - \dot{\partial}_{\bar{h}}(N^l_k)C^i_{jl} = -\dot{\partial}_{\bar{h}}(\dot{\partial}_jN^i_k) + P^l_{0\bar{h}k}C^i_{jl}$

$= -\dot{\partial}_j(\dot{\partial}_{\bar{h}}N^i_k) + P^l_{0\bar{h}k}C^i_{jl} = \dot{\partial}_jP^i_{0\bar{h}k} + P^l_{0\bar{h}k}C^i_{jl}$

$= P^i_{0\bar{h}k}|_j + P^i_{0\bar{h}r}C^r_{kj}$, i.e. vi). □

Proposition 2.2. *For any* $X \in \Gamma^0(T'M)$ *the following properties hold true:*

i) $X|_{k|j} - X_{|j}|_k = C^i_{jk}X_{|i}$;

ii) $X|_{\bar{k}|j} - X_{|j}|_{\bar{k}} = -P^i_{0\bar{k}j}X|_i$.

Proof. We have
$\left[\delta_j, \dot{\partial}_k\right] X = L^i_{kj}\left(\dot{\partial}_i X\right) = L^i_{kj} X|_i$ and
$\left[\delta_j, \dot{\partial}_{\bar{k}}\right] X = -P^i_{0\bar{k}j}\dot{\partial}_i X = -P^i_{0\bar{k}j} X|_i$.
On the other hand,
$\left[\delta_j, \dot{\partial}_k\right] X = \delta_j\left(\dot{\partial}_k X\right) - \dot{\partial}_k(\delta_j X) = \delta_j\left(X|_k\right) - \dot{\partial}_k(X_{|j})$
$= X|_{k|j} + L^i_{kj} X|_i - X_{|j}|_k - C^i_{jk} X_{|i}$ and
$\left[\delta_j, \dot{\partial}_{\bar{k}}\right] X = \delta_j\left(\dot{\partial}_{\bar{k}} X\right) - \dot{\partial}_{\bar{k}}(\delta_j X) = \delta_j\left(X|_{\bar{k}}\right) - \dot{\partial}_{\bar{k}}(X_{|j})$
$= X|_{\bar{k}|j} - X_{|j}|_{\bar{k}}$.
From the above relations it results i) and ii). □

Let us recall that in [1]'s terminology, the complex Finsler space (M, F) is *strongly Kähler* iff $T^i_{jk} = 0$, *Kähler* iff $T^i_{jk}\eta^j = 0$ and *weakly Kähler* iff $g_{i\bar{l}} T^i_{jk}\eta^j\bar{\eta}^l = 0$, where $T^i_{jk} := L^i_{jk} - L^i_{kj}$. In [10] it is proved that strongly Kähler and Kähler notions actually coincide. We notice that in the particular case of complex Finsler metrics which come from Hermitian metrics on M, so-called *purely Hermitian metrics* in [15], (i.e. $g_{i\bar{j}} = g_{i\bar{j}}(z)$), all those nuances of Kähler coincide. On the other hand, as in Aikou's work [2], a complex Finsler space which is Kähler and $L^i_{jk} = L^i_{jk}(z)$ is named *complex Berwald* space. From Proposition 2.1 iii) and by $\Xi_{i\bar{j}k\bar{h}} = P_{\bar{j}i\bar{h}k}$, a complex Berwald space is a Kähler space with either $\Xi_{i\bar{j}k\bar{h}} = 0$ or $P_{\bar{j}i\bar{h}k} = 0$.

3 The complex Berwald frame

Let (M, F) be a 2 - dimensional complex Finsler space, $(z^k, \eta^k)_{k=\overline{1,2}}$ be complex coordinates on $T'M$ and $VT'M$ be the vertical bundle spanned by $\{\dot{\partial}_k\}$. Further on, the indices $i, j, k, ...$ run over $\{1, 2\}$. Let $g_{i\bar{j}}$ be the fundamental metric tensor of the space and $\mathcal{G}$ the Hermitian metric structure (2.1), defined on $T_C(T'M)$, with respect to the adapted frames of Chern-Finsler (*c.n.c.*).

We set $l := l^i\dot{\partial}_i$ and its dual form is $\omega = l_i\delta\eta^i$, where

$$l^i = \frac{1}{F}\eta^i \text{ and } l_i = \frac{1}{F} g_{i\bar{j}}\bar{\eta}^j = g_{i\bar{j}} l^{\bar{j}}. \tag{3.1}$$

Now, our aim is to construct an orthonormal frame in the vertical bundle $VT'M$, which is 2 - dimensional in any point. Therefore, it is decomposed into $VT'M = \{l\} \oplus \{l\}^\perp$, where $\{l\}^\perp$ is spanned by a complex vector m. Requiring the orthogonality condition $\mathcal{G}(l, \bar{m}) = 0$ and $\mathcal{G}(m, \bar{m}) = 1$, i.e. m is a unit vector and, using $m_i := g_{i\bar{j}} m^{\bar{j}}$, the above two conditions get the linear system $\begin{cases} l_1 m^1 + l_2 m^2 = 0 \\ m_1 m^1 + m_2 m^2 = 1 \end{cases}$.

We try to solve this system following the same technique from [8] for real case. Nevertheless, let us pay more attention to this system. Passing in real coordinates, it contains three real equations with four real unknowns. So that it doesn't admit an unique solution. Formally, solving this system as one linear, it is obtained the 'solutions' $m^1 = \frac{-l_2}{\Delta}$,

$m^2 = \frac{l_1}{\Delta}$, $m_1 = -\Delta l^2$ and $m_2 = \Delta l^1$, where $\Delta = l_1 m_2 - l_2 m_1$, which indeed are not completely determined because Δ depends on m_i. We can say more about these 'solutions'. A straightforward computation proves that $|\Delta| = \sqrt{g}$ and $\Delta' = \mathcal{T}\Delta$ under a change of the local coordinates $(z^k, \eta^k)_{k=\overline{1,2}}$ into $(z'^k, \eta'^k)_{k=\overline{1,2}}$, where $g := \det(g_{i\bar{j}})$ and $\mathcal{T} := \det\left(\frac{\partial z^i}{\partial z'^j}\right)$. Therefore, a natural question is if there exists at least Δ with the above mentioned properties. The answer will come below, when we find two distinct particular solutions for Δ.

Subsequently, our statement will be made for a fixed choice of Δ and then $\{l, m, \bar{l}, \bar{m}\}$ with

$$m = \frac{1}{\Delta}(-l_2\dot{\partial}_1 + l_1\dot{\partial}_2) \tag{3.2}$$

will be called the *complex Berwald frame*. Surely, the dependence of the chosen for Δ will be analyzed everywhere.

But when we work in a fixed local chart, we can choose $\Delta = \sqrt{g}$, i.e. Δ is real, which produces the unique solutions $m^1 = \frac{-l_2}{\sqrt{g}}$, $m^2 = \frac{l_1}{\sqrt{g}}$, $m_1 = -\sqrt{g}l^2$ and $m_2 = \sqrt{g}l^1$. Thus, we have

$$m = \frac{1}{\sqrt{g}}(-l_2\dot{\partial}_1 + l_1\dot{\partial}_2), \tag{3.3}$$

in this fixed chart.

Then $\{l, m, \bar{l}, \bar{m}\}$, with m given by (3.3) will be called the *local complex Berwald frame* of the space.

Note that (3.3) provides only a local frame, because the set of natural local basis in every chart does not have tensorial character. For this reason, considering a change of the local coordinates, we obtain

$$m' = \frac{\mathcal{T}}{|\mathcal{T}|}m \; ; \; m'^i = \frac{\mathcal{T}}{|\mathcal{T}|}\frac{\partial z'^i}{\partial z^k}m^k \; ; \; m'_i = \frac{\overline{\mathcal{T}}}{|\mathcal{T}|}\frac{\partial z^r}{\partial z'^i}m_r,$$

which show that m is not a vector, but it depends on the local change. Therefore, it will say that m from (3.3) is a pseudo-vector.

Although m from (3.3) depends on the local changes of the coordinates, it is very important in our study, in a fixed chart. Certainly, further on we will be very careful with the global validity of our assertions. We will see that together with its horizontal extension it gives rise to some invariants which will characterize two dimensional complex Finsler spaces. A first and useful remark is that the quantities $m_i m^j$, $m^i m^{\bar{j}}$, $m_i m_{\bar{j}}$ and $m_i m$ are independent of the chosen local chart, and hence they have global meaning.

With respect to the local complex Berwald frame, $\dot{\partial}_k$ and $g_{i\bar{j}}$ are decomposed as follows

$$\dot{\partial}_i = l_i l + m_i m \quad \text{and hence} \quad g_{i\bar{j}} = l_i l_{\bar{j}} + m_i m_{\bar{j}}. \tag{3.4}$$

From here we deduce that

$$C^i_{jk} = g^{\bar{m}i}\dot{\partial}_k g_{j\bar{m}} = A l^i m_k m_j + B m^i m_k m_j, \tag{3.5}$$

where we set

$$A := m^j m^k l_h C^h_{kj} \;\; ; \;\; B := m_h m^k m^j C^h_{jk}.$$

The dependence of the vertical terms A and B of the local charts is obvious, $A' = \frac{\mathcal{T}^2}{|\mathcal{T}|^2}A$; $B' = \frac{\mathcal{T}}{|\mathcal{T}|}B$. Thus, A and B are not invariants, but if they are zero in a local chart, then they are zero in any local chart. But, $A\bar{B}^2$ is an invariant to a change of local chart. Moreover, by means of A and B and setting $\Delta = B\sqrt{g}$ with $|B|^2 = 1$ or $\Delta = \sqrt{Ag}$ with $|A|^2 = 1$, we obtain two particular solutions for m from (3.2) which certify the existence of the complex Berwald frames.

Further on, all our work will be with respect to the local complex Berwald frame, where m is given by (3.3).

Therefore, the formulas from Proposition 3.2, in [17], become

$$\begin{aligned}
l(l_i) &= \frac{-1}{2F}l_i \ ; \ \bar{l}(l_i) = \frac{1}{2F}l_i \ ; \ \ l(m_i) = \frac{1}{2F}m_i \ ; \ \bar{l}(m_i) = \frac{-1}{2F}m_i; \\
m(l_i) &= Am_i \ ; \ \ \bar{m}(l_i) = \frac{1}{F}m_i \ ; \ \ m(m_i) = \frac{1}{2}Bm_i - \frac{1}{F}l_i; \ \bar{m}(m_i) = \frac{1}{2}\bar{B}m_i; \\
l(l^i) &= \frac{1}{2F}l^i \ ; \ \ \bar{l}(l^i) = -\frac{1}{2F}l^i \ ; \ \ l(m^i) = -\frac{1}{2F}m^i \ \ ; \ \ \bar{l}(m^i) = \frac{1}{2F}m^i; \\
m(l^i) &= \frac{1}{F}m^i \ ; \ \ \bar{m}(l^i) = 0 \ ; \\
m(m^i) &= -\frac{1}{2}Bm^i - Al^i \ \ ; \ \ \bar{m}(m^i) = -\frac{1}{F}l^i - \frac{1}{2}\bar{B}m^i.
\end{aligned} \tag{3.6}$$

By a direct computation, using the above relations, we obtain formulas for the vertical covariant derivatives of $l, m, \bar{l}$ and $\bar{m}$ with respect to the $C - F$ connection

$$\begin{aligned}
l_i|_j &= \frac{-1}{2F}l_il_j \ ; \ \ l_i|_{\bar{j}} = \frac{1}{2F}l_il_{\bar{j}} + \frac{1}{F}m_im_{\bar{j}}; \\
m_i|_j &= \frac{1}{2F}m_il_j - \frac{1}{F}l_im_j - \frac{B}{2}m_im_j \ ; \ \ m_i|_{\bar{j}} = \frac{-1}{2F}m_il_{\bar{j}} + \frac{\bar{B}}{2}m_im_{\bar{j}}; \\
l^i|_j &= \frac{1}{F}\delta^i_j - \frac{1}{2F}l_jl^i \ ; \ \ l^i|_{\bar{j}} = \frac{-1}{2F}l_{\bar{j}}l^i \ ; \ F|_j = \frac{1}{2}l_j; \\
m^i|_j &= \frac{-1}{2F}l_jm^i + \frac{B}{2}m_jm^i \ ; \ \ m^i|_{\bar{j}} = \frac{1}{2F}l_{\bar{j}}m^i - \frac{1}{F}m_{\bar{j}}l^i - \frac{\bar{B}}{2}m_{\bar{j}}m^i,
\end{aligned} \tag{3.7}$$

and their conjugates.

Moreover, because $\bar{l}(C^h_{kj}) = 0$ and $l(C^h_{kj}) = -\frac{1}{F}C^h_{kj}$ by some computation, it results

$$\begin{aligned}
A|_{\bar{h}} &= \dot{\partial}_{\bar{h}}A = (l_{\bar{h}}\bar{l} + m_{\bar{h}}\bar{m})A = \frac{3A}{2F}l_{\bar{h}} + A|_{\bar{s}}m^{\bar{s}}m_{\bar{h}}; \\
B|_{\bar{h}} &= \dot{\partial}_{\bar{h}}B = (l_{\bar{h}}\bar{l} + m_{\bar{h}}\bar{m})B = \frac{B}{2F}l_{\bar{h}} + B|_{\bar{s}}m^{\bar{s}}m_{\bar{h}}; \\
A|_h &= \dot{\partial}_hA = (l_hl + m_hm)\,A = -\frac{5A}{2F}l_h + A|_sm^sm_h; \\
B|_h &= \dot{\partial}_hB = (l_hl + m_hm)\,B = -\frac{3B}{2F}l_h + B|_sm^sm_h.
\end{aligned} \tag{3.8}$$

Now, via the natural isomorphism between the bundles $VT'M$ and $T'M$, composed with the horizontal lift of $HT'M$, we obtain the following orthonormal local frame on

$H_C T'M$,

$$\{\lambda := l^i \delta_i, \;\; \mu = m^i \delta_i, \;\; \bar{\lambda} := l^{\bar{i}} \delta_{\bar{\imath}}, \;\; \bar{\mu} = m^{\bar{i}} \delta_{\bar{\imath}}\}.$$

Let D be the $C-F$ connection on (M,F). Further on, let us give an explicit expression for $C-F$ connection with respect to horizontal local frame $\{\lambda, \mu, \bar{\lambda}, \bar{\mu}\}$. Moreover, using (3.4) and $L^i_{jk} = g^{\bar{m}i}\delta_k g_{j\bar{m}}$ it results

$$L^i_{jk} \;=\; Jl^i l_j l_k + Ul^i m_j l_k + Vl^i l_j m_k + Xl^i m_j m_k + Om^i l_j l_k + Ym^i m_j l_k + Em^i l_j m_k + Hm^i m_j m_k, \tag{3.9}$$

where we set

$$\begin{aligned} J \;:&= l^j l^k l_i L^i_{jk};\; U := m^j l^k l_i L^i_{jk};\; V := l^j m^k l_i L^i_{jk};\; X := m^j m^k l_i L^i_{jk} \\ O \;:&= l^j l^k m_i L^i_{jk};\; Y := m^j l^k m_i L^i_{jk};\; E := l^j m^k m_i L^i_{jk};\; H := m^j m^k m_i L^i_{jk}. \end{aligned} \tag{3.10}$$

Here the horizontal settled quantities do not have tensorial character, because under the change of charts we have

$$\begin{aligned} J' &= J + \mathcal{T}^r_{ab} l^a l^b l_r \;;\; U' = \frac{\mathcal{T}}{|\mathcal{T}|}(U + \mathcal{T}^r_{ab} m^a l^b l_r) \;; \\ V' &= \frac{\mathcal{T}}{|\mathcal{T}|}(V + \mathcal{T}^r_{ab} l^a m^b l_r) \;;\; X' = \frac{\mathcal{T}^2}{|\mathcal{T}|^2}(X + \mathcal{T}^r_{ab} m^a m^b l_r) \;; \\ O' &= \frac{\overline{\mathcal{T}}}{|\mathcal{T}|}(O + \mathcal{T}^r_{ab} l^a l^b m_r) \;;\; Y' = Y + \mathcal{T}^r_{ab} m^a l^b m_r \;; \\ E' &= E + \mathcal{T}^r_{ab} l^a m^b m_r \;;\; H' = \frac{\mathcal{T}}{|\mathcal{T}|}(H + \mathcal{T}^r_{ab} m^a m^b m_r), \end{aligned} \tag{3.11}$$

where $\mathcal{T}^r_{ab} := \frac{\partial z'^j}{\partial z^a}\frac{\partial z'^k}{\partial z^b}\frac{\partial^2 z^r}{\partial z'^j \partial z'^k}$.

Firstly, the properties of the $C-F$ connection $N^i_k = L^i_{jk}\eta^j$ and $\dot{\partial}_j N^i_k = L^i_{jk}$, (see [15]), permit us to establish some links between the vertical and horizontal terms (3.10) of this connection. Indeed,

$N^i_k = F(Jl^i l_k + Vl^i m_k + Om^i l_k + Em^i m_k)$ and
$L^i_{jk} = (l_j l + m_j m)[F(Jl^i l_k + Vl^i m_k + Om^i l_k + Em^i m_k)]$
$= [\frac{1}{2}J + Fl(J)]l^i l_j l_k + [Fm(J) - V - FAO]l^i m_j l_k + [Fl(V) + \frac{3}{2}V]l^i l_j m_k$
$+[Fm(V) + FAJ + \frac{1}{2}FBV - FAE]l^i m_j m_k + [Fl(O) - \frac{1}{2}O]m^i l_j l_k$
$+[Fm(O) + J - \frac{1}{2}FBO - E]m^i m_j l_k + [Fl(E) + \frac{1}{2}E]m^i l_j m_k$
$+[Fm(E) + V + FAO]m^i m_j m_k$ which together with (3.9) give,

Proposition 3.1. *Let (M,F) be a 2 - dimensional complex Finsler space. Then*
i) $J|_k = \frac{1}{2F}Jl_k + [\frac{1}{F}(U+V) + AO]m_k$;
ii) $V|_k = -\frac{1}{2F}Vl_k + [A(E-J) - \frac{1}{2}BV + \frac{1}{F}X]m_k$;
iii) $O|_k = \frac{3}{2F}Ol_k + [\frac{1}{F}(E+Y-J) + \frac{1}{2}BO]m_k$;
iv) $E|_k = \frac{1}{2F}El_k + [\frac{1}{F}(H-V) - AO]m_k$.

Proof. In the fixed local chart the assertions i)-iv) are true. We must prove their global validity. For example, under the change of a local chart, we have

$V'|'_k + \frac{1}{2F}V'l'_k - [A'(E'-J') - \frac{1}{2}B'V' + \frac{1}{F}X']m'_k$
$= \frac{\mathcal{T}}{|\mathcal{T}|}\frac{\partial z^r}{\partial z'^k}\{V|_r + \frac{1}{2F}Vl_r - [A(E-J) - \frac{1}{2}BV + \frac{1}{F}X]m_r\}$, where $V'|'_k := \dot{\partial}'_k V'$.

Because $V|_r + \frac{1}{2F}Vl_r - [A(E-J) - \frac{1}{2}BV + \frac{1}{F}X]m_r = 0$, by its change rule it results that it is zero in any local chart. Analogous results the geometric character of the others assertions. □

Proposition 3.2. *Let (M, F) be a 2 - dimensional complex Finsler space. Then*
i) It is Kähler if and only if $U = V$ and $Y = E$;
ii) It is weakly Kähler if and only if $U = V$.

Proof. i) By (3.9), $L^i_{jk} - L^i_{kj} = (U-V)l^i m_j l_k + (V-U)l^i l_j m_k + (Y-E)m^i m_j l_k + (E-Y)m^i l_j m_k$. So, $L^i_{jk} - L^i_{kj} = 0$ if and only if $U = V$ and $Y = E$.

To prove ii) we compute $g_{i\bar{l}}T^i_{jk}\eta^j\overline{\eta}^l = F^2(L^i_{jk} - L^i_{kj})l_i l^j = F^2(V-U)m_k$. It results $g_{i\bar{l}}T^i_{jk}\eta^j\overline{\eta}^l = 0$ if and only if $U = V$.

Taking into account the local changes of $U-V$ and $Y-E$, it follows the global validity of these statements. □

Further on, several calculuses imply the following properties.

Proposition 3.3. *With respect to the local Berwald frame, we have:*

$$\begin{aligned}
\lambda(l_i) &= Jl_i + Um_i\ ;\ \bar{\lambda}(l_i) = \bar{\lambda}(l^i) = 0\ ;\ \lambda(l^i) = -Jl^i - Om^i\ ; \qquad (3.12)\\
\lambda(m_i) &= Ol_i - \frac{1}{2}(J-Y)m_i\ ;\ \bar{\lambda}(m_i) = \frac{1}{2}(\bar{J}+\bar{Y})m_i\ ;\\
\lambda(m^i) &= -Ul^i + \frac{1}{2}(J-Y)m^i\ ;\ \bar{\lambda}(m^i) = -\frac{1}{2}(\bar{J}+\bar{Y})m^i\ ;\\
\mu(l_i) &= Vl_i + Xm_i\ ;\ \ \bar{\mu}(l_i) = \bar{\mu}(l^i) = 0\ ;\ \mu(l^i) = -Vl^i - Em^i\ ;\\
\mu(m_i) &= El_i + \frac{1}{2}(H-V)m_i\ ;\ \bar{\mu}(m_i) = \frac{1}{2}(\bar{V}+\bar{H})m_i\ ;\\
\mu(m^i) &= -Xl^i - \frac{1}{2}(H-V)m^i\ ;\ \bar{\mu}(m^i) = -\frac{1}{2}(\bar{V}+\bar{H})m^i\ ;\\
\lambda(g) &= (J+Y)g\ ;\ \mu(g) = (V+H)g\ ;\ \delta_i = l_i\lambda + m_i\mu\ ;\ \lambda(L) = \mu(L) = 0
\end{aligned}$$

and their conjugates.

Then, from (3.12) we deduce that

$$\begin{aligned}
l_{i|j} &= l_{i|\bar{j}} = l^i_{|j} = l^i_{|\bar{j}} = 0; \qquad (3.13)\\
m_{i|j} &= -\frac{1}{2}[(J+Y)l_j + (V+H)m_j]m_i;\ m_{i|\bar{j}} = \frac{1}{2}[(\bar{J}+\bar{Y})l_{\bar{j}} + (\bar{V}+\bar{H})m_{\bar{j}}]m_i;\\
m^i_{|j} &= \frac{1}{2}[(J+Y)l_j + (V+H)m_j]m^i;\ m^i_{|\bar{j}} = -\frac{1}{2}[(\bar{J}+\bar{Y})l_{\bar{j}} + (\bar{V}+\bar{H})m_{\bar{j}}]m^i
\end{aligned}$$

and theirs conjugates.

4 Curvatures of the C-F connection

In this section, we shall compute the curvature coefficients of the $C-F$ connection with respect to the local frames $\{l, m, \bar{l}, \bar{m}\}$ and $\{\lambda, \mu, \bar{\lambda}, \bar{\mu}\}$. By means of these, we characterize the 2 - dimensional complex Finsler spaces.

4.1 The $v\bar{v}-$ Riemann type tensor

Firstly, we study the $v\bar{v}-$ Riemann type tensor $S_{\bar{r}j\bar{h}k}$. Taking into account Proposition 2.1 iii) and the formulas (3.5), (3.7) and (3.8), we have

$S_{\bar{r}j\bar{h}k} = -(Al_{\bar{r}}m_j m_k + Bm_{\bar{r}}m_j m_k)|_{\bar{h}}$

$= [-A|_{\bar{h}} + \frac{3A}{2F}l_{\bar{h}} + (-A\bar{B} + \frac{B}{F})m_{\bar{h}}]l_{\bar{r}}m_j m_k$

$+(-B|_{\bar{h}} + \frac{B}{2F}l_{\bar{h}} - \frac{B\bar{B}}{2}m_{\bar{h}})m_{\bar{r}}m_j m_k$

$= (-A|_{\bar{s}}m^{\bar{s}} - A\bar{B} + \frac{B}{F})m_{\bar{h}}l_{\bar{r}}m_j m_k + (-B|_{\bar{s}}m^{\bar{s}} - \frac{B\bar{B}}{2})m_{\bar{h}}m_{\bar{r}}m_j m_k.$

But, $S_{\bar{r}j\bar{h}k}$ is symmetric in j, k and $\bar{r}, \bar{h}$. Therefore, it results that

$$S_{\bar{r}j\bar{h}k} = \mathbf{I}m_{\bar{h}}m_{\bar{r}}m_j m_k \; ; \; A|_{\bar{s}}m^{\bar{s}} = -A\bar{B} + \frac{B}{F}, \quad \text{where} \quad \mathbf{I} := -B|_{\bar{s}}m^{\bar{s}} - \frac{B\bar{B}}{2}. \tag{4.1}$$

We note that $\mathbf{I}$ is invariable to the changes of the local coordinates thanks to $S_{\bar{r}j\bar{h}k}$ and $m_j m_{\bar{h}} m_k m_{\bar{r}}$ which are tensors. Further on, we point out some properties of the function $\mathbf{I}$, called by us the *vertical curvature invariant*.

4.2 The $v\bar{h}-$ Riemann type tensor

Let $\Xi_{\bar{r}j\bar{h}k}$ be the $v\bar{h}-$ Riemann type tensor. Using the Proposition 2.1 iii) and the formulas (3.5) and (3.13), we have

$$\begin{aligned}\Xi_{\bar{r}j\bar{h}k} = & -[A_{|\bar{h}}l_{\bar{r}} + A(\bar{J}+\bar{Y})l_{\bar{r}}l_{\bar{h}} + A(\bar{V}+\bar{H})l_{\bar{r}}m_{\bar{h}} \\ & +B_{|\bar{h}}m_{\bar{r}} + \frac{B}{2}(\bar{J}+\bar{Y})m_{\bar{r}}l_{\bar{h}} + \frac{B}{2}(\bar{V}+\bar{H})m_{\bar{r}}m_{\bar{h}}]m_j m_k.\end{aligned} \tag{4.2}$$

We wish to investigate the relationship among A, B, $\mathbf{I}$ and to characterize the 2 - dimensional complex Finsler spaces by means of these. For this, contracting the Bianchi identity

$$\Xi_{\bar{r}j\bar{h}k}|_{\bar{s}} - S_{\bar{r}j\bar{s}k|\bar{h}} + \Xi_{\bar{r}j\bar{p}k}\overline{C^{p}_{sh}} = 0, \tag{4.3}$$

(see [15], p. 77), with the tensor $m^{\bar{r}}m^j m^k m^{\bar{s}}$ and taking into account (4.2) and (3.7), we obtain

$\Xi_{\bar{r}j\bar{h}k}|_{\bar{s}}m^{\bar{r}}m^j m^k m^{\bar{s}} = -\{B_{|\bar{h}}|_{\bar{s}}m^{\bar{s}} + \frac{\bar{B}}{2}B_{|\bar{h}}$

$+\frac{1}{2}[-\mathbf{I}(\bar{J}+\bar{Y}) + B(\bar{J}+\bar{Y})|_{\bar{s}}m^{\bar{s}} - \frac{B}{F}(\bar{V}+\bar{H})]l_{\bar{h}}$

$+\frac{1}{2}[B|_{\bar{s}}m^{\bar{s}}(\bar{V}+\bar{H}) + B(\bar{V}+\bar{H})|_{\bar{s}}m^{\bar{s}}]m_{\bar{h}}\};$

$S_{\bar{r}j\bar{s}k|\bar{h}}m^{\bar{r}}m^j m^{\bar{s}}m^k = \mathbf{I}_{|\bar{h}};$

$\Xi_{\bar{r}j\bar{p}k}\overline{C^{p}_{sh}}m^{\bar{r}}m^j m^k m^{\bar{s}} = -[\frac{\bar{A}}{F}B_{|\bar{0}} + \bar{B}B_{|\bar{p}}m^{\bar{p}} + \frac{\bar{A}B}{2}(\bar{J}+\bar{Y}) + \frac{B\bar{B}}{2}(\bar{V}+\bar{H})]m_{\bar{h}}.$

Hence

$$B_{|\bar{h}}|_{\bar{s}}m^{\bar{s}} = -\{\frac{1}{2}[-\mathbf{I}(\bar{J}+\bar{Y})+B(\bar{J}+\bar{Y})|_{\bar{s}}m^{\bar{s}}-\frac{B}{F}(\bar{V}+\bar{H})]l_{\bar{h}} \\ +\frac{1}{2}[(-\mathbf{I}+\frac{B\bar{B}}{2})(\bar{V}+\bar{H})+B(\bar{V}+\bar{H})|_{\bar{s}}m^{\bar{s}} \\ +2\frac{\bar{A}}{F}B_{|\bar{0}}+2\bar{B}B_{|\bar{p}}m^{\bar{p}}+\bar{A}B(\bar{J}+\bar{Y})]m_{\bar{h}}+\mathbf{I}_{|\bar{h}}+\frac{\bar{B}}{2}B_{|\bar{h}}\} \tag{4.4}$$

and its conjugate.

On the other hand, contracting in (4.3) by $m^{\bar{r}}m^{j}m^{k}l^{\bar{s}}$, using
$\Xi_{\bar{r}j\bar{h}k}|_{\bar{s}}m^{\bar{r}}m^{j}m^{k}l^{\bar{s}} = -\frac{1}{F}\{B_{|\bar{h}}|_{\bar{0}}-\frac{1}{2}B_{|\bar{h}}+\frac{B}{2}[-\frac{1}{2}(\bar{J}+\bar{Y})+(J+Y)|_{\bar{0}}]l_{\bar{h}}$
$+\frac{B}{2}[\frac{1}{2}(\bar{V}+\bar{H})+(\bar{V}+\bar{H})|_{\bar{0}}]m_{\bar{h}}\}$ and
$S_{\bar{r}j\bar{s}k|\bar{h}}m^{\bar{r}}m^{j}l^{\bar{s}}m^{k} = \Xi_{\bar{r}j\bar{p}k}\overline{C^{p}_{sh}}m^{\bar{r}}m^{j}m^{k}l^{\bar{s}} = 0$
we have,

$$B_{|\bar{h}}|_{\bar{0}} = \frac{1}{2}B_{|\bar{h}}-\frac{B}{2}[-\frac{\bar{J}+\bar{Y}}{2}+(\bar{J}+\bar{Y})|_{\bar{0}}]l_{\bar{h}} \\ -\frac{B}{2}[\frac{\bar{V}+\bar{H}}{2}+(\bar{V}+\bar{H})|_{\bar{0}}]m_{\bar{h}}, \tag{4.5}$$

and its conjugate.

The conjugates of (4.4), (4.5) and Theorem 4.1 ii) allow us to write

$$\bar{B}_{|k}|_{j} = \frac{1}{2F}\{\bar{B}_{|k}-\bar{B}[-\frac{J+Y}{2}+(J+Y)|_{0}]l_{k} \\ -\bar{B}[\frac{V+H}{2}+(V+H)|_{0}]m_{k}\}l_{j} \\ -\{\frac{1}{2}[-\mathbf{I}(J+Y)+\bar{B}(J+Y)|_{s}m^{s}-\frac{\bar{B}}{F}(V+H)]l_{k} \\ +\frac{1}{2}[(-\mathbf{I}+\frac{B\bar{B}}{2})(V+H)+\bar{B}(V+H)|_{s}m^{s} \\ +2\frac{A}{F}\bar{B}_{|0}+2B\bar{B}_{|s}m^{s}+A\bar{B}(J+Y)]m_{k}+\mathbf{I}_{|k}+\frac{B}{2}\bar{B}_{|k}\}m_{j}. \tag{4.6}$$

It is also worthwhile to note the following identity

$$\bar{B}|_{j|k} = \frac{1}{2F}\bar{B}_{|k}l_{j} \\ -\{\mathbf{I}_{|k}+\frac{\bar{B}}{2}B_{|k}+\frac{B}{2}\bar{B}_{|k}+\frac{1}{2}(-\mathbf{I}-\frac{B\bar{B}}{2})[(J+Y)l_{k}+(V+H)m_{k}]\}m_{j}, \tag{4.7}$$

which is obtained from (3.7), (3.8) and (4.1).

Therefore, (4.6) and (4.7) lead to

$$\begin{aligned}\bar{B}|_{j|k} - \bar{B}_{|k}|_j &= C^i_{jk}\bar{B}_{|i} + \frac{\bar{B}}{2}\{\frac{1}{F}[-\frac{J+Y}{2} + (J+Y)_{|0}]l_k l_j \\ &+\frac{1}{F}[\frac{V+H}{2} + (V+H)_{|0}]m_k l_j \\ &+[\frac{B}{2}(J+Y) + (J+Y)_{|s}m^s - \frac{1}{F}(V+H)]l_k m_j \\ &+[B(V+H) + (V+H)_{|s}m^s + A(J+Y)]m_k m_j - B_{|k}m_j\},\end{aligned} \tag{4.8}$$

because $C^i_{jk}\bar{B}_{|i} = (\frac{A}{F}\bar{B}_{|0} + B\bar{B}_{|s}m^s)m_k m_j$.

4.3 The $h\bar{v}-$ Riemann type tensor

Now let us consider the $h\bar{v}-$ Riemann type tensor $P_{\bar{r}j\bar{h}k}$. By Proposition 2.1.ii) and formulas (3.5) and (3.13), it results that

$$P_{\bar{r}0\bar{h}k} = -F[\bar{A}_{|k} + \bar{A}(J+Y)l_k + \bar{A}(V+H)m_k]m_{\bar{r}}m_{\bar{h}}. \tag{4.9}$$

But, Proposition 2.1 vi) allows us to reconstruct $P_{\bar{r}j\bar{h}k}$. Indeed,

$$P_{\bar{r}j\bar{h}k} = P_{\bar{r}0\bar{h}k}|_j + P_{\bar{r}0\bar{h}s}C^s_{kj} \tag{4.10}$$

and from (4.9), we obtain

$$\begin{aligned}P_{\bar{r}0\bar{h}s}C^s_{kj} &= -F[\frac{A}{F}\bar{A}_{|0} + B\bar{A}_{|s}m^s \\ &+A\bar{A}(J+Y) + B\bar{A}(V+H)]m_{\bar{r}}m_{\bar{h}}m_k m_j\end{aligned} \tag{4.11}$$

and

$$\begin{aligned}P_{\bar{r}0\bar{h}k}|_j &= -\{-\frac{1}{2}\bar{A}_{|k}l_j + FB\bar{A}_{|k}m_j + F\bar{A}_{|k}|_j - \bar{A}(J+Y)l_k l_j \\ &+\bar{A}[FB(J+Y) - (V+H)]l_k m_j + \frac{F\bar{A}B}{2}(V+H)m_k m_j \\ &+F[\bar{A}|_j(J+Y) + \bar{A}(J+Y)|_j]l_k \\ &+F[\bar{A}|_j(V+H) + \bar{A}(V+H)|_j]m_k\}m_{\bar{r}}m_{\bar{h}}.\end{aligned} \tag{4.12}$$

Plugging (4.11) and (4.12) into (4.10), gives

$$\begin{aligned}P_{\bar{r}j\bar{h}k} &= -\{-\frac{1}{2}\bar{A}_{|k}l_j + FB\bar{A}_{|k}m_j + F\bar{A}_{|k}|_j - \bar{A}(J+Y)l_k l_j \\ &+\bar{A}[FB(J+Y) - (V+H)]l_k m_j \\ &+[A\bar{A}_{|0} + FB\bar{A}_{|s}m^s + FA\bar{A}(J+Y) + \frac{3F\bar{A}B}{2}(V+H)]m_k m_j \\ &+F[\bar{A}|_j(J+Y) + \bar{A}(J+Y)|_j]l_k \\ &+F[\bar{A}|_j(V+H) + \bar{A}(V+H)|_j]m_k\}m_{\bar{r}}m_{\bar{h}}.\end{aligned} \tag{4.13}$$

Recall the following property, $P_{\bar{r}j\bar{h}k} = \Xi_{j\bar{r}k\bar{h}} = \overline{\Xi_{\bar{j}r\bar{k}h}}$. Writing it by means of (4.2) and (4.13), we obtain the conditions

$$\begin{aligned}
\overline{A}_{|k}|_0 &= \frac{3}{2}\overline{A}_{|k} \; ; \qquad (4.14)\\
\overline{A}_{|k}|_j m^j &= \frac{1}{F}\overline{B}_{|k} - B\overline{A}_{|k} - [\frac{\bar{B}}{2F}(J+Y) + \bar{A}(J+Y)|_s m^s - \frac{\bar{A}}{F}(V+H)]l_k \\
&\quad -[(\frac{\bar{B}}{2F} + \frac{\bar{A}B}{2})(V+H) + \bar{A}(V+H)|_s m^s + A\bar{A}(J+Y) \\
&\quad +\frac{A}{F}\overline{A}_{|0} + B\overline{A}_{|s}m^s]m_k
\end{aligned}$$

and theirs conjugates.

From both formulas (4.14), it follows that

$$\begin{aligned}
\overline{A}_{|k}|_j &= \frac{3}{2F}\overline{A}_{|k}l_j + \{\frac{1}{F}\overline{B}_{|k} - B\overline{A}_{|k} \qquad (4.15)\\
&\quad -[\frac{\bar{B}}{2F}(J+Y) + \bar{A}(J+Y)|_s m^s - \frac{\bar{A}}{F}(V+H)]l_k \\
&\quad -[(\frac{\bar{B}}{2F} + \frac{\bar{A}B}{2})(V+H) + \bar{A}(V+H)|_s m^s + A\bar{A}(J+Y) \\
&\quad +\frac{A}{F}\overline{A}_{|0} + B\overline{A}_{|s}m^s]m_k\}m_j.
\end{aligned}$$

Moreover, from (3.7), (3.8) and (4.1), we have

$$\begin{aligned}
\overline{A}|_{j|k} &= \frac{3}{2F}\overline{A}_{|k}l_j + \{-B\overline{A}_{|k} - \overline{A}B_{|k} + \frac{1}{F}\overline{B}_{|k} \qquad (4.16)\\
&\quad -(\frac{\bar{B}}{2F} - \frac{\bar{A}B}{2})[(J+Y)l_k + (V+H)m_k]\}m_j.
\end{aligned}$$

By subtracting (4.15) from (4.16), we get

$$\begin{aligned}
\overline{A}|_{j|k} - \overline{A}_{|k}|_j &= C^i_{jk}\overline{A}_{|i} \qquad (4.17)\\
&\quad -\overline{A}\{B_{|k} - [\frac{B}{2}(J+Y) + (J+Y)|_s m^s - \frac{1}{F}(V+H)]l_k \\
&\quad -[B(V+H) + (V+H)|_s m^s + A(J+Y)]m_k\}m_j,
\end{aligned}$$

because $C^i_{jk}\overline{A}_{|i} = (\frac{A}{F}\overline{A}_{|0} + B\overline{A}_{|s}m^s)m_k m_j$.

Theorem 4.1. *Let (M, F) be a 2 - dimensional complex Finsler space.*

i) If $|A| \neq 0$ and $B = 0$, then

$$(J+Y)|_s m^s = \frac{1}{F}(V+H) \; ; \;\; (V+H)|_s m^s = -A(J+Y); \qquad (4.18)$$

ii) If $A\bar{B}^2 \neq 0$ then

$$\begin{aligned}
(J+Y)|_0 &= \frac{J+Y}{2} \; ; \;\; (V+H)|_0 = -\frac{V+H}{2} \; ; \qquad (4.19)\\
B_{|k} &= [\frac{B}{2}(J+Y) + (J+Y)|_s m^s - \frac{1}{F}(V+H)]l_k \\
&\quad +[B(V+H) + (V+H)|_s m^s + A(J+Y)]m_k.
\end{aligned}$$

Proof. Writing the identity i) from Proposition 2.2 for the vertical terms $\overline{A}$ and $\overline{B}$ it involves $\overline{A}|_{k|j} - \overline{A}_{|j}|_k = C^i_{jk}\overline{A}_{|i}$ and $\overline{B}|_{k|j} - \overline{B}_{|j}|_k = C^i_{jk}\overline{B}_{|i}$. But, taking into account (4.17) and (4.8), it follows

$$\begin{aligned}&\overline{A}\{B_{|k} - [\frac{B}{2}(J+Y) + (J+Y)|_s m^s - \frac{1}{F}(V+H)]l_k \\ &-[B(V+H) + (V+H)|_s m^s + A(J+Y)]m_k\}m_j = 0\end{aligned} \tag{4.20}$$

and

$$\begin{aligned}&\bar{B}\{\frac{1}{F}[-\frac{J+Y}{2} + (J+Y)|_0]l_k l_j \\ &+\frac{1}{F}[\frac{V+H}{2} + (V+H)|_0]m_k l_j \\ &+[\frac{B}{2}(J+Y) + (J+Y)|_s m^s - \frac{1}{F}(V+H)]l_k m_j \\ &+[B(V+H) + (V+H)|_s m^s + A(J+Y)]m_k m_j - B_{|k}m_j\} = 0.\end{aligned} \tag{4.21}$$

Hence, we have the cases:

1. If the space is purely Hermitian, the identities (4.20) and (4.21) are identically verified.

2. If the space in non-purely Hermitian, then $|A| \neq 0$ on M, and by (4.20) we obtain

$$\begin{aligned}B_{|k} \;=\; &[\frac{B}{2}(J+Y) + (J+Y)|_s m^s - \frac{1}{F}(V+H)]l_k \\ &+[B(V+H) + (V+H)|_s m^s + A(J+Y)]m_k,\end{aligned} \tag{4.22}$$

which substituted into (4.21) leads to

$$\bar{B}\{[-\frac{J+Y}{2} + (J+Y)|_0]l_k l_j + [\frac{V+H}{2} + (V+H)|_0]m_k l_j\} = 0.$$

But, we have emphasized two kinds of non-purely Hermitian spaces:

a) If $|A| \neq 0$ and $B = 0$ then, together with (4.22), we obtain i).

b) If $A\bar{B}^2 \neq 0$ then $|B| \neq 0$, in any open set, and by (4.22) results $(J+Y)|_0 = \frac{J+Y}{2}$ and $(V+H)|_0 = -\frac{V+H}{2}$, which with (4.21) imply ii).

The independence of the above statement to the changes of local charts results by straightforward computations using (3.11). □

4.4 Two-dimensional complex Berwald spaces

The above considerations offer us the premises for some special characterizations of the 2 - dimensional complex Berwald spaces. Firstly, we write the identity iv) of Proposition 2.1 in terms of the local complex Berwald frame. Some computations give

$$\begin{aligned}\dot{\partial}_{\bar{h}}L^i_{jk} = \{&[\bar{l}(J) + \tfrac{1}{2F}J]l^i l_j l_k + [\bar{l}(U) - \tfrac{1}{2F}U]l^i m_j l_k + [\bar{l}(V) - \tfrac{1}{2F}V]l^i l_j m_k \\ &+[\bar{l}(X) - \tfrac{1}{2F}X]l^i m_j m_k + [\bar{l}(O) + \tfrac{3}{2F}O]m^i l_j l_k + [\bar{l}(Y) + \tfrac{1}{2F}Y]m^i m_j l_k \\ &+[\bar{l}(E) + \tfrac{1}{2F}E]m^i l_j m_k + [\bar{l}(H) - \tfrac{1}{2F}H]m^i m_j m_k\}l_{\bar{h}}\end{aligned}$$

$+\{[\bar{m}(J)-\frac{1}{F}O]l^il_jl_k+[\bar{m}(U)-\frac{1}{F}(Y-J)+\frac{1}{2}\bar{B}U]l^im_jl_k$
$+[\bar{m}(V)-\frac{1}{F}(E-J)+\frac{1}{2}\bar{B}V]l^il_jm_k+[\bar{m}(X)-\frac{1}{F}(H-U-V)+\bar{B}X]l^im_jm_k$
$+[\bar{m}(O)-\frac{1}{2}\bar{B}O]m^il_jl_k+[\bar{m}(Y)+\frac{1}{F}O]m^im_jl_k+[\bar{m}(E)+\frac{1}{F}O]m^il_jm_k$
$+[\bar{m}(H)+\frac{1}{F}(Y+E)+\frac{1}{2}\bar{B}H]m^im_jm_k\}m_{\bar{h}}.$

Using $\Xi_{\bar{r}j\bar{h}k}=-C_{j\bar{r}k|\bar{h}}$ and (4.2) it results

$C_{j\bar{r}k|\bar{h}}=[A_{|\bar{h}}l_{\bar{r}}+A(\bar{J}+\bar{Y})l_{\bar{r}}l_{\bar{h}}+A(\bar{V}+\bar{H})l_{\bar{r}}m_{\bar{h}}+B_{|\bar{h}}m_{\bar{r}}+\frac{B}{2}(\bar{J}+\bar{Y})m_{\bar{r}}l_{\bar{h}}+\frac{B}{2}(\bar{V}+\bar{H})m_{\bar{r}}m_{\bar{h}}]m_jm_k.$

The above outcomes substituted into Proposition 2.1 iv), lead to

Proposition 4.1. *Let (M,F) be a 2 - dimensional complex Finsler space. Then*

i) $J|_{\bar{k}}=-\frac{1}{2F}Jl_{\bar{k}}+\frac{1}{F}Om_{\bar{k}}$; $V|_{\bar{k}}=\frac{1}{2F}Vl_{\bar{k}}+[\frac{1}{F}(E-J)-\frac{1}{2}\bar{B}V]m_{\bar{k}}$;

ii) $\bar{l}(U)-\frac{1}{2F}U=\bar{l}(X)-\frac{1}{2F}X=\bar{l}(O)+\frac{3}{2F}O=\bar{l}(Y)+\frac{1}{2F}Y=\bar{l}(E)+\frac{1}{2F}E$
$=\bar{l}(H)-\frac{1}{2F}H=0$;

iii) $\bar{m}(U)-\frac{1}{F}(Y-J)+\frac{1}{2}\bar{B}U+FA[\bar{m}(O)-\frac{1}{2}\bar{B}O]=0$;

iv) $\bar{m}(V)-\frac{1}{F}(E-J)+\frac{1}{2}\bar{B}V=0$;

v) $\bar{m}(X)-\frac{1}{F}(H-U-V)+\bar{B}X+FA[\bar{m}(E)+\frac{1}{F}O]=0$;

vi) $\frac{1}{F}\bar{A}_{|0}+\bar{A}(J+Y)=\bar{m}(O)-\frac{1}{2}\bar{B}O$;

vii) $\frac{1}{F}\bar{B}_{|0}+\frac{\bar{B}}{2}(J+Y)=\bar{m}(Y)+\frac{1}{F}O+FB[\bar{m}(O)-\frac{1}{2}\bar{B}O]$;

viii) $\bar{A}_{|k}m^k+\bar{A}(V+H)=\bar{m}(E)+\frac{1}{F}O$;

ix) $\bar{B}_{|k}m^k+\frac{\bar{B}}{2}(V+H)=\bar{m}(H)+\frac{1}{F}(Y+E)+\frac{1}{2}\bar{B}H+FB[\bar{m}(E)+\frac{1}{F}O]$.

Next, we rewrite the identity v) from Proposition 2.1, $\dot{\partial}_hL^i_{jk}=C_{j\bar{r}h|k}g^{\bar{r}i}$ with respect to the complex Berwald frame. Taking into account Proposition 3.1, we have

$\dot{\partial}_hL^i_{jk}=\{[l(U)+\frac{1}{2F}U]l^im_jlm_k+[l(X)+\frac{3}{2F}X]l^im_jm_k$
$+[l(Y)-\frac{1}{2F}Y]m^im_jl_k+[l(H)+\frac{1}{2F}H]m^im_jm_k\}l_h$
$+\{[m(U)-A(Y-J)+\frac{1}{2}BU-\frac{1}{F}X]l^im_jl_k+[m(Y)+AO-\frac{1}{F}(H-U)]m^im_jl_k$
$+[m(X)+A(U+V-H)+BX]l^im_jm_k$
$+[m(H)+A(Y+E)+\frac{1}{F}X+\frac{1}{2}BH]m^im_jm_k\}m_h.$

On the other hand, $C_{j\bar{r}h|k}g^{\bar{r}i}=\{[A_{|k}-A(J+Y)l_k-A(V+H)m_k]l^i$
$+[B_{|k}-\frac{B}{2}(J+Y)l_k-\frac{B}{2}(V+H)m_k]m^i\}m_jm_h$. From here we obtain

Proposition 4.2. *Let (M,F) be a 2 - dimensional complex Finsler space. Then*

i) $l(U)+\frac{1}{2F}U=l(X)+\frac{3}{2F}X=l(Y)-\frac{1}{2F}Y=l(H)+\frac{1}{2F}H=0$;

ii) $m(U)-A(Y-J)+\frac{1}{2}BU-\frac{1}{F}X=\frac{1}{F}A_{|0}-A(J+Y)$;

iii) $m(Y)+AO-\frac{1}{F}(H-U)=\frac{1}{F}B_{|0}-\frac{B}{2}(J+Y)$;

iv) $m(X)+A(U+V-H)+BX=A_{|k}m^k-A(V+H)$;

v) $m(H)+A(Y+E)+\frac{1}{F}X+\frac{1}{2}BH=B_{|k}m^k-\frac{B}{2}(V+H)$.

We note that the assertions of Propositions 4.1 and 4.2 are preserved to changes of local charts.

Proposition 4.3. *If* (M,F) *is a* 2 - *dimensional complex Berwald space, then*

$$\begin{aligned}
U|_{\bar{k}} &= \frac{1}{2F}Ul_{\bar{k}}+[\frac{1}{F}(Y-J)-\frac{1}{2}\bar{B}U]m_{\bar{k}};\ Y|_{\bar{k}}=-\frac{1}{2F}Yl_{\bar{k}}-\frac{1}{F}Om_{\bar{k}};\\
O|_{\bar{k}} &= -\frac{3}{2F}Ol_{\bar{k}}+\frac{1}{2}\bar{B}Om_{\bar{k}};\ X|_{\bar{k}}=\frac{1}{2F}Xl_{\bar{k}}+[\frac{1}{F}(H-2V)-\bar{B}X]m_{\bar{k}};\\
H|_{\bar{k}} &= \frac{1}{2F}Hl_{\bar{k}}-(\frac{2}{F}Y+\frac{1}{2}\bar{B}H)m_{\bar{k}}; \qquad (4.23)
\end{aligned}$$

equivalently with

$$\begin{aligned}
A_{|\bar{k}} &= -A(\bar{J}+\bar{Y})l_{\bar{k}}-A(\bar{V}+\bar{H})m_{\bar{k}}; \qquad (4.24)\\
B_{|\bar{k}} &= -\frac{B}{2}(\bar{J}+\bar{Y})l_{\bar{k}}-\frac{B}{2}(\bar{V}+\bar{H})m_{\bar{k}};
\end{aligned}$$

equivalently with

$$\begin{aligned}
U|_{k} &= -\frac{1}{2F}Ul_{k}+[A(Y-J)-\frac{1}{2}BU+\frac{1}{F}X]m_{k}; \qquad (4.25)\\
Y|_{k} &= \frac{1}{2F}Yl_{k}+[\frac{1}{F}(H-U)-AO]m_{k};\\
X|_{k} &= -\frac{3}{2F}Xl_{k}-[2AU-AH+BX]m_{k};\\
H|_{k} &= -\frac{1}{2F}Hl_{k}-[2AY+\frac{1}{2}BH+\frac{1}{F}X]m_{k};
\end{aligned}$$

equivalently with

$$\begin{aligned}
A_{|k} &= A(J+Y)l_{k}+A(V+H)m_{k}; \qquad (4.26)\\
B_{|k} &= \frac{B}{2}(J+Y)l_{k}+\frac{B}{2}(V+H)m_{k}.
\end{aligned}$$

Proof. Under the assumption of Berwald, we have $\dot{\partial}_{\bar{h}}G^i=\dot{\partial}_{\bar{h}}N^i_k=\dot{\partial}_{\bar{h}}L^i_{jk}=0$ which together with Proposition 4.2 induces (4.23). Using Theorem 2.2 and Propositions 4.2 and 4.3 it results the equivalence between (4.23), (4.24), (4.25) and (4.26). By straightforward computations it results their global validity. □

We note that the equivalent sets of relations (4.23), (4.24), (4.25) and (4.26) have a geometric character and are only necessary conditions for complex Berwald space. These become sufficient together with weakly Kähler condition.

4.5 The $h\bar{h}-$ Riemann type tensor

Let us investigate the $h\bar{h}-$ Riemann type tensor $R_{\bar{r}j\bar{h}k}$. By (2.5), (3.5) and (3.12) we can write

$R_{\bar{r}j\bar{h}k}=g_{i\bar{r}}R^i_{j\bar{h}k}$
$=-\left(l_il_{\bar{r}}+m_im_{\bar{r}}\right)\{(l_{\bar{h}}\bar{\lambda}+m_{\bar{h}}\bar{\mu})(L^i_{jk})$
$+[(l_{\bar{h}}\bar{\lambda}+m_{\bar{h}}\bar{\mu})(N^n_k)](Al^im_jm_n+Bm^im_jm_n)\}$

$= -\left(l_i l_{\bar r} + m_i m_{\bar r}\right) l_{\bar h}[\bar\lambda(L^i_{jk}) + F\bar\lambda(L^n_{sk})l^s(Al^i + Bm^i)m_j m_n]$
$-\left(l_i l_{\bar r} + m_i m_{\bar r}\right) m_{\bar h}[\bar\mu(L^i_{jk}) + F\bar\mu(L^n_{sk})l^s(Al^i + Bm^i)m_j m_n]$. It results that

$$\begin{aligned} R_{\bar r j\bar h k} \;=\; & -[\bar\lambda(l_i L^i_{jk}) + FA\bar\lambda(L^n_{sk}l^s)m_j m_n]l_{\bar r}l_{\bar h} \\ & -[\bar\lambda(L^i_{jk})m_i + FB\bar\lambda(L^n_{sk}l^s)m_j m_n]m_{\bar r}l_{\bar h} \\ & -[\bar\mu(l_i L^i_{jk}) + FA\bar\mu(L^n_{sk}l^s)m_j m_n]l_{\bar r}m_{\bar h} \\ & -[\bar\mu(L^i_{jk})m_i + FB\bar\mu(L^n_{skn}l^s)m_j m]m_{\bar r}m_{\bar h}. \end{aligned} \tag{4.27}$$

Further on, our goal is to find the link between the horizontal covariant derivatives of the functions (3.10) and theirs properties. Indeed, from (4.27) it follows that $R_{\bar 0 0\bar h 0} = -LF\bar\lambda(l^j l^k l_i L^i_{jk})l_{\bar h} - LF\bar\mu(l^j l^k l_i L^i_{jk})m_{\bar h} = -LJ_{|\bar 0}l_{\bar h} - LFJ_{|\bar s}m^{\bar s}m_{\bar h}$ and $R_{\bar 0 0\bar 0 k} = -LF\bar\lambda(l^j l_i L^i_{jk})$. The property $\overline{R_{\bar 0 0\bar k 0}} = R_{\bar 0 0\bar 0 k}$ leads to $F\bar\lambda(l^j l_i L^i_{jk}) = \bar J_{|0}l_k + F\bar J_{|s}m^s m_k$, which gives

$$\bar J_{|0} = J_{|\bar 0}\;; \quad \bar J_{|s}m^s = \frac{1}{F}V_{|\bar 0} + \frac{1}{2}V(\bar J + \bar Y). \tag{4.28}$$

Moreover, by (4.27)
$R_{\bar r 0\bar 0 0} = -LF\bar\lambda(l^j l^k l_i L^i_{jk})l_{\bar r} - LF\bar\lambda(l^j l^k m_i L^i_{jk})m_{\bar r} + \frac{1}{2}LFO(\bar J + \bar Y)m_{\bar r}$
$= -LJ_{|\bar 0}l_{\bar r} - LO_{|\bar 0}m_{\bar r} + \frac{1}{2}LFO(\bar J + \bar Y)m_{\bar r}$ and
$R_{\bar 0 r\bar 0 0} = -LF\bar\lambda(l^k l_i L^i_{rk}) - L^2A\bar\lambda(l^k l^s m_n L^n_{sk})m_r + \frac{1}{2}L^2AO(\bar J + \bar Y)m_r$
$= -LF\bar\lambda(l^k l_i L^i_{rk}) - LFAO_{|\bar 0}m_r + \frac{1}{2}L^2AO(\bar J + \bar Y)m_r$.
But, $\overline{R_{\bar r 0\bar 0 0}} = R_{r\bar 0 0\bar 0} = R_{\bar 0 r\bar 0 0}$ leads to
$\bar J_{|0}l_r + [\bar O_{|0} - \frac{1}{2}F\bar O(J + Y)]m_r = F\bar\lambda(l^k l_i L^i_{rk}) + FA[O_{|\bar 0} - \frac{1}{2}FO(\bar J + \bar Y)]m_r$.
The contraction with m^r gives

$$\bar O_{|0} - \frac{1}{2}F\bar O(J + Y) - FAO_{|\bar 0} + \frac{1}{2}LAO(\bar J + \bar Y) = U_{|\bar 0} + \frac{1}{2}FU(\bar J + \bar Y). \tag{4.29}$$

Next, from (4.27) we have
$R_{\bar r 0\bar h 0}m^{\bar r} = -L\bar\lambda(l^j l^k L^i_{jk})m_i l_{\bar h} - L\bar\mu(l^j l^k L^i_{jk})m_i m_{\bar h}$
$= -F[O_{|\bar 0}l_{\bar h} - \frac{1}{2}FO(\bar J + \bar Y)]l_{\bar h} - L[O_{|\bar s}m^{\bar s} - \frac{1}{2}O(\bar V + \bar H)]m_{\bar h}$.
On the other hand
$R_{\bar 0 r\bar 0 h}m^r = -L\bar\lambda(m^r l_i L^i_{rh}) - \frac{1}{2}L(\bar J + \bar Y)(Ul_h + Xm_h) - LFA\bar\lambda(l^s m_n L^n_{sh})$
$+\frac{1}{2}LFA(\bar J + \bar Y)(Ol_h + Em_h)$.
Using $\overline{R_{\bar r 0\bar h 0}m^{\bar r}} = R_{r\bar 0 h\bar 0}m^r = R_{\bar 0 r\bar 0 h}m^r$, we obtain
$[\bar O_{|0} - \frac{1}{2}F\bar O(J + Y)]l_h + F[\bar O_{|s}m^s - \frac{1}{2}\bar O(V + H)]m_h$
$= F\bar\lambda(m^r l_i L^i_{rh}) + LA\bar\lambda(l^s m_n L^n_{sh}) + \frac{1}{2}F(\bar J + \bar Y)(Ul_h + Xm_h)$
$-\frac{1}{2}LA(\bar J + \bar Y)(Ol_h + Em_h)$,
which by transvection with m^h gives

$$\bar O_{|s}m^s - \frac{1}{2}\bar O(V + H) - AE_{|\bar 0} = \frac{1}{F}X_{|\bar 0} + X(\bar J + \bar Y). \tag{4.30}$$

Taking again into account (4.27), it follows
$R_{\bar 0 0\bar h k}m^k = -L\bar\lambda(l^j l_i L^i_{jk})m^k l_{\bar h} - L\bar\mu(l^j l_i L^i_{jk})m^k m_{\bar h}$

$= -F[V_{|\bar{0}} + \frac{1}{2}FV(\bar{J} + \bar{Y})]l_{\bar{h}} - L[V_{|\bar{s}}m^{\bar{s}} + \frac{1}{2}V(\bar{V} + \bar{H})]m_{\bar{h}}$ and
$R_{\bar{0}0\bar{k}h}m^{\bar{k}} = -L\bar{\mu}(l^j l_i L^i_{jh})$.
These relations together with $\overline{R_{\bar{0}0\bar{h}k}m^k m^{\bar{h}}} = R_{0\bar{0}h\bar{k}}m^{\bar{k}}m^h = R_{\bar{0}0\bar{k}h}m^{\bar{k}}m^h$ give

$$\bar{V}_{|s}m^s + \frac{1}{2}\bar{V}(V + H) = V_{|\bar{s}}m^{\bar{s}} + \frac{1}{2}V(\bar{V} + \bar{H}). \tag{4.31}$$

Next, (4.27) involves
$R_{\bar{r}0\bar{h}k}m^{\bar{r}}m^k = -F\bar{\lambda}(l^j m^k m_i L^i_{jk})l_{\bar{h}} - F\bar{\mu}(l^j m^k m_i L^i_{jk})m_{\bar{h}}$
$= -E_{|\bar{0}}l_{\bar{h}} - FE_{|\bar{s}}m^{\bar{s}}m_{\bar{h}}$ and
$R_{\bar{0}r\bar{k}h}m^r m^{\bar{k}} = -F\bar{\mu}(l_i L^i_{rh})m^r - LA\bar{\mu}(l^s L^n_{sh})m_n$.
But, $\overline{R_{\bar{r}0\bar{h}k}m^{\bar{r}}m^k} = R_{r\bar{0}h\bar{k}}m^r m^{\bar{k}} = R_{\bar{0}r\bar{k}h}m^r m^{\bar{k}}$ so that
$-\bar{E}_{|0}l_h - F\bar{E}_{|s}m^s m_h = -F\bar{\mu}(l_i L^i_{rh})m^r - LA\bar{\mu}(l^s L^n_{sh})m_n$.
By transvection with l^h and m^h we obtain

$$\begin{aligned} \frac{1}{F}\bar{E}_{|0} - FAO_{|\bar{s}}m^{\bar{s}} + \frac{1}{2}FAO(\bar{V} + \bar{H}) \;&=\; U_{|\bar{s}}m^{\bar{s}} + \frac{1}{2}U(\bar{V} + \bar{H}); \\ \bar{E}_{|s}m^s - FAE_{|\bar{s}}m^{\bar{s}} \;&=\; X_{|\bar{s}}m^{\bar{s}} + X(\bar{V} + \bar{H}). \end{aligned} \tag{4.32}$$

Using again (4.27), we have
$R_{\bar{r}j\bar{h}k}m^{\bar{r}}m^j m^k = -[\bar{\lambda}(L^i_{jk})m_i m^j m^k + FB\bar{\lambda}(l^s m^k m_n L^n_{sk})]l_{\bar{h}}$
$-[\bar{\mu}(L^i_{jk})m_i m^j m^k + FB\bar{\mu}(l^s m^k m_n L^n_{sk})]m_{\bar{h}}$
$= -(\frac{1}{F}H_{|\bar{0}} + \frac{1}{2}H(\bar{J} + \bar{Y}) + BE_{|\bar{0}})l_{\bar{h}} - (H_{|\bar{s}}m^{\bar{s}} + \frac{1}{2}H(\bar{V} + \bar{H}) + FBE_{|\bar{s}}m^{\bar{s}})m_{\bar{h}}$.
On the other hand,
$R_{\bar{j}r\bar{k}h}m^{\bar{j}}m^r m^{\bar{k}} = -\bar{\mu}(m^r m_i L^i_{rh}) - FB\bar{\mu}(l^s L^n_{sh})m_n$.
But, $\overline{R_{\bar{r}j\bar{h}k}m^{\bar{r}}m^j m^k} = R_{r\bar{j}h\bar{k}}m^{\bar{j}}m^r m^{\bar{k}} = R_{\bar{j}r\bar{k}h}m^{\bar{j}}m^r m^{\bar{k}}$ which leads to
$-(\frac{1}{F}\bar{H}_{|0} + \frac{1}{2}\bar{H}(J + Y) + \bar{B}\bar{E}_{|0})l_h - (\bar{H}_{|s}m^s + \frac{1}{2}\bar{H}(V + H) + F\bar{B}\bar{E}_{|s}m^s)m_h$
$= -\bar{\mu}(m^r m_i L^i_{rh}) - FB\bar{\mu}(l^s L^n_{sh})m_n$.
The transvection with l^h and m^h gives

$$\begin{aligned} \frac{1}{F}\bar{H}_{|0} + \frac{1}{2}\bar{H}(J + Y) + \bar{B}\bar{E}_{|0} \;&=\; Y_{|\bar{s}}m^{\bar{s}} + FBO_{|\bar{s}}m^{\bar{s}} - \frac{1}{2}FBO(\bar{V} + \bar{H}); \\ \bar{H}_{|s}m^s + \frac{1}{2}\bar{H}(V + H) + F\bar{B}\bar{E}_{|s}m^s \;&=\; H_{|\bar{s}}m^{\bar{s}} + \frac{1}{2}H(\bar{V} + \bar{H}) + FBE_{|\bar{s}}m^{\bar{s}}. \end{aligned}$$

Now, $R_{\bar{r}j\bar{h}0}m^{\bar{r}}m^j = -[F\bar{\lambda}(m^j l^k m_i L^i_{jk}) + LB\bar{\lambda}(l^s l^k L^n_{sk})m_n]l_{\bar{h}}$
$-[F\bar{\mu}(m^j l^k m_i L^i_{jk}) + LB\bar{\mu}(l^s l^k L^n_{sk})m_n]m_{\bar{h}}$
$= -[Y_{|\bar{0}} + FBO_{|\bar{0}} - \frac{1}{2}LO(\bar{J} + \bar{Y})]l_{\bar{h}}$
$-F[Y_{|\bar{s}}m^{\bar{s}} + FBO_{|\bar{s}}m^{\bar{s}} - \frac{1}{2}FBO(\bar{V} + \bar{H})]m_{\bar{h}}$
and
$R_{\bar{j}r0h}m^{\bar{j}}m^r = -F\bar{\lambda}(m^r m_i L^i_{rh}) - LB\bar{\lambda}(l^s L^n_{sh})m_n$.
The conjugation $\overline{R_{\bar{r}j\bar{h}0}m^{\bar{r}}m^j l^{\bar{h}}} = R_{r\bar{j}h\bar{0}}m^{\bar{j}}m^r l^h = R_{\bar{j}r\bar{0}h}m^{\bar{j}}m^r l^h$ gives

$$\bar{Y}_{|0} + F\bar{B}\bar{O}_{|0} - \frac{1}{2}L\bar{B}\bar{O}(J + Y) = Y_{|\bar{0}} + FBO_{|\bar{0}} - \frac{1}{2}LBO(\bar{J} + \bar{Y}). \tag{4.33}$$

Proposition 4.4. *Let (M,F) be a 2 - dimensional weakly Kähler complex Finsler space. Then*

Lemma 4.1.

Proposition 4.5. *i)* $\frac{1}{F}\bar{O}_{|0} - \frac{1}{2}\bar{O}(J+Y) - AO_{|\bar{0}} + \frac{1}{2}FAO(\bar{J}+\bar{Y}) = \bar{J}_{|s}m^s$;
ii) $\frac{1}{F}\bar{E}_{|0} - FAO_{|\bar{s}}m^{\bar{s}} + \frac{1}{2}FAO(\bar{V}+\bar{H}) = V_{|\bar{s}}m^{\bar{s}} + \frac{1}{2}V(\bar{V}+\bar{H})$.

Proof. It results by Proposition 3.2, (4.28), (4.29) and (4.32) . By computation using (3.11), we obtain the global validity of these assertions. □

Remark 4.1. *If (M,F) is purely Hermitian ($A=0$) and Kähler, then* $\frac{1}{F}\bar{O}_{|0} - \frac{1}{2}\bar{O}(J+Y) = \bar{J}_{|s}m^s$ *and* $V_{|\bar{s}}m^{\bar{s}} + \frac{1}{2}V(\bar{V}+\bar{H}) = \frac{1}{F}\bar{E}_{|0}$.

Then, using (2.5), $\delta_i = l_i\lambda + m_i\mu$ and (4.27), $R_{\bar{r}j\bar{h}k} = \mathbf{R}(\delta_j, \delta_{\bar{r}}, \delta_k, \delta_{\bar{h}})$ is decomposed into sixteen terms:

$$\begin{aligned} R_{\bar{r}j\bar{h}k} \;=\; & \mathbf{K}l_{\bar{r}}l_j l_{\bar{h}}l_k + \mathbf{W}m_{\bar{r}}m_j m_{\bar{h}}m_k \\ & -[\frac{1}{F}\bar{O}_{|0} - \frac{1}{2}\bar{O}(J+Y)]l_{\bar{r}}m_j l_{\bar{h}}l_k - [\frac{1}{F}O_{|\bar{0}} - \frac{1}{2}O(\bar{J}+\bar{Y})]m_{\bar{r}}l_j l_{\bar{h}}l_k \\ & -\bar{J}_{|s}m^s l_{\bar{r}}l_j l_{\bar{h}}m_k - J_{|\bar{s}}m^{\bar{s}}l_{\bar{r}}l_j m_{\bar{h}}l_k \\ & -[V_{|\bar{s}}m^{\bar{s}} + \frac{1}{2}V(\bar{V}+\bar{H})]l_{\bar{r}}l_j m_{\bar{h}}m_k - \frac{1}{F}\bar{E}_{|0}m_{\bar{r}}l_j l_{\bar{h}}m_k \\ & -\frac{1}{F}E_{|\bar{0}}l_{\bar{r}}m_j m_{\bar{h}}l_k - [\frac{1}{F}Y_{|\bar{0}} + BO_{|\bar{0}} - \frac{1}{2}FBO(\bar{J}+\bar{Y})]m_{\bar{r}}m_j l_{\bar{h}}l_k \\ & -E_{|\bar{s}}m^{\bar{s}}m_{\bar{r}}l_j m_{\bar{h}}m_k - [\frac{1}{F}\bar{H}_{|0} + \frac{1}{2}\bar{H}(J+Y) + \bar{B}\bar{E}_{|0}]m_{\bar{r}}m_j m_{\bar{h}}l_k \\ & -\bar{E}_{|s}m^s l_{\bar{r}}m_j m_{\bar{h}}m_k - [\frac{1}{F}H_{|\bar{0}} + \frac{1}{2}H(\bar{J}+\bar{Y}) + BE_{|\bar{0}}]m_{\bar{r}}m_j l_{\bar{h}}m_k \\ & -[O_{|\bar{s}}m^{\bar{s}} - \frac{1}{2}O(\bar{V}+\bar{H})]m_{\bar{r}}l_j m_{\bar{h}}l_k - [\bar{O}_{|s}m^s - \frac{1}{2}\bar{O}(V+H)]l_{\bar{r}}m_j l_{\bar{h}}m_k, \end{aligned} \tag{4.34}$$

where $\mathbf{K} := -\frac{1}{F}J_{|\bar{0}}$ and $\mathbf{W} := -\,H_{|\bar{s}}m^{\bar{s}} - \frac{1}{2}H(\bar{V}+\bar{H}) - BFE_{|\bar{s}}m^{\bar{s}}$. We call the functions $\mathbf{K}$ and $\mathbf{W}$ the *horizontal curvature invariants.*

Proposition 4.6. *Let (M,F) be a connected 2 - dimensional weakly Kähler complex Finsler space with $R_{\bar{0}k\bar{0}0} = R_{\bar{0}0\bar{0}k}$ and $|A| \neq 0$. Then*
i) $F^2A\mathbf{K} = \Phi_{|0} - F(J+Y)\Phi$;
ii) $E_{|\bar{0}} = -\frac{F\mathbf{K}}{2}\left(1 + A\bar{A}L\right)$;
iii) $V_{|\bar{s}}m^{\bar{s}} + \frac{1}{2}V(\bar{V}+\bar{H}) = -\frac{\mathbf{K}}{2}\left(1 - A\bar{A}L\right)$;
iv) $Y_{|\bar{0}} = -\frac{F\mathbf{K}}{2}\left(1 - A\bar{A}L\right) + F\left|\Phi\right|^2$;
v) $H_{|\bar{0}} + \frac{F}{2}H(\bar{J}+\bar{Y}) = -\frac{ALF\mathbf{K}}{2}\left(\bar{B} - F\bar{A}B\right) + \frac{\bar{A}L}{2}[\Phi_{|k}m^k - (V+H)\Phi] + L\Phi\bar{\Omega}$;
vi) $E_{|\bar{s}}m^{\bar{s}} = -\frac{F\mathbf{K}}{2}\left(\bar{B} - F\bar{A}B\right) - \frac{AF}{2}[\bar{\Phi}_{|\bar{k}}m^{\bar{k}} - (\bar{V}+\bar{H})\bar{\Phi}]$;
vii) $\mathbf{W} = \mathbf{K}\left(1 + A\bar{A}L\right) - F(E_{|\bar{s}}m^{\bar{s}})_{|l}m^l - \frac{3}{2}BFE_{|\bar{s}}m^{\bar{s}} - L\left|\Omega\right|^2$,
where $\Phi := A_{|\bar{0}} + AF(\bar{J}+\bar{Y})$ *and* $\Omega := A_{|\bar{k}}m^{\bar{k}} + A(\bar{V}+\bar{H})$.

Proof. Let us consider the Bianchi identity, (see [15], p. 77),

$$R_{\bar{r}j\bar{h}k}|_l - \Xi_{\bar{r}j\bar{h}l|k} - P_{\bar{r}j\bar{s}k}P^{\bar{s}}_{\bar{0}l\bar{h}} + S_{\bar{r}j\bar{s}l}R^{\bar{s}}_{\bar{0}k\bar{h}} + R_{\bar{r}j\bar{h}n}C^n_{kl} = 0. \tag{4.35}$$

In order to prove the statements i)-vii), we use Theorem 4.8, the covariant derivatives (3.7), (3.13) and the expressions of the $v\bar{v}-$, $h\bar{v}-$, $v\bar{h}-$, $h\bar{h}-$ Riemann type tensors.

Contracting into (4.35) by $\bar{\eta}^r m^j \bar{\eta}^h \eta^k$, using
$R_{\bar{r}j\bar{h}k}|_l \bar{\eta}^r m^j \bar{\eta}^h \eta^k = -R_{\bar{0}j\bar{0}l}m^j = F^2[\bar{O}_{|s}m^s - \frac{1}{2}\bar{O}(V+H)]m_l = -F^3 A\mathbf{K}m_l$;
$P_{\bar{r}j\bar{s}k}\bar{\eta}^r = S_{\bar{r}j\bar{s}l}\bar{\eta}^r = C^n_{kl}\eta^k = 0$ and
$\Xi_{\bar{r}j\bar{h}l|k}\bar{\eta}^r m^j \bar{\eta}^h \eta^k = -F[\Phi_{|0} - F(J+Y)\Phi]m_l$, we obtain i).

The contraction with $\bar{\eta}^r \eta^j \bar{m}^h \eta^k$ of (4.35),
$R_{\bar{r}j\bar{h}k}|_l \bar{\eta}^r \eta^j \bar{m}^h \eta^k = -R_{\bar{0}l\bar{h}0}\bar{m}^h - R_{\bar{0}0\bar{h}l}\bar{m}^h + \frac{1}{L}R_{\bar{0}0\bar{0}0}m_l$
$= F^2[\frac{1}{F}\bar{E}_{|0} + V_{|\bar{s}}m^{\bar{s}} + \frac{1}{2}V(\bar{V}+\bar{H}) + \mathbf{K}]m_l$ and $\Xi_{\bar{r}j\bar{h}k}\eta^j = 0$ lead to

$$\frac{1}{F}\bar{E}_{|0} + V_{|\bar{s}}m^{\bar{s}} + \frac{1}{2}V(\bar{V}+\bar{H}) = -\mathbf{K}.$$

On the other hand, by Lemma 4.2 ii),

$$\frac{1}{F}\bar{E}_{|0} - V_{|\bar{s}}m^{\bar{s}} - \frac{1}{2}V(\bar{V}+\bar{H}) = -LA\bar{A}\mathbf{K}.$$

The last two relations give ii) and iii).

Now, contracting again (4.35) by $\bar{m}^r \eta^j \bar{\eta}^h \eta^k$, we have
$R_{\bar{r}j\bar{h}k}|_l \bar{m}^r \eta^j \bar{\eta}^h \eta^k = F^2(\mathbf{K} + \frac{1}{F}Y_{|\bar{0}} + \frac{1}{F}E_{|\bar{0}})m_l$ and
$P_{\bar{r}j\bar{s}k}P^{\bar{s}}_{\bar{0}l\bar{h}}\bar{m}^r \eta^j \bar{\eta}^h \eta^k = F^2 |\Phi|^2 m_l$.

It results $[\mathbf{K} + \frac{1}{F}Y_{|\bar{0}} + \frac{1}{F}E_{|\bar{0}} - |\Phi|^2]m_l = 0$. Hereby, $Y_{|\bar{0}} = -\mathbf{K}F - E_{|\bar{0}} + F|\Phi|^2$, which together with ii) implies iv).

Next we prove v) and vi). First we contract (4.35) with $\bar{\eta}^r m^j \bar{\eta}^h m^k m^l$ and we obtain $(R_{\bar{0}j\bar{0}k}m^j m^k)|_l m^l - BR_{\bar{0}j\bar{0}k}m^j m^k - \Xi_{\bar{0}j\bar{0}l|k}m^j m^k m^l + R_{\bar{0}j\bar{0}n}C^n_{kl}m^j m^k m^l = 0$. This implies that

$$A|_l m^l L\mathbf{K} = -[\Phi_{|k}m^k + (V+H)\Phi] \tag{4.36}$$

The contraction of (4.35) by $\bar{m}^r \eta^j \bar{\eta}^h m^k m^l$ implies
$(R_{\bar{r}0\bar{0}k}\bar{m}^r m^k)|_l m^l - (R_{\bar{r}l\bar{0}k}\bar{m}^r m^k + P_{\bar{r}0\bar{s}k}P^{\bar{s}}_{\bar{0}l\bar{0}}\bar{m}^r m^k - R_{\bar{r}0\bar{0}n}C^n_{kl}\bar{m}^r m^k)m^l = 0$, which gives $H_{|\bar{0}} + \frac{F}{2}H(\bar{J}+\bar{Y}) = F\bar{E}_{|0}|_l m^l + L\Phi\bar{\Omega}$. Now, this together with ii), (4.36) and (4.1) gives v).

The contraction of (4.35) by $\bar{m}^r \eta^j \bar{m}^h \eta^k m^l$ gives
$(R_{\bar{r}0\bar{h}0}\bar{m}^r \bar{m}^h)|_l m^l + BR_{\bar{r}0\bar{h}0}\bar{m}^r \bar{m}^h - R_{\bar{r}l\bar{h}0}\bar{m}^r \bar{m}^h m^l - R_{\bar{r}0\bar{h}l}\bar{m}^r \bar{m}^h m^l$
$-P_{\bar{r}0\bar{s}0}P^{\bar{s}}_{\bar{0}l\bar{h}}\bar{m}^r \bar{m}^h m^l = 0$, which is equivalent to
$L\mathbf{K}\bar{A}|_l m^l + \frac{1}{F}\bar{H}_{|0} + \frac{1}{2}\bar{H}(J+Y) + \bar{B}\bar{E}_{|0} + E_{|\bar{s}}m^{\bar{s}} + B\bar{A}L\mathbf{K} = F\bar{\Phi}\Omega$.

Using ii), v) and (4.1) it leads to vi).

For vii) we contract (4.35) with $\bar{m}^r \eta^j \bar{m}^h m^k m^l$ and we deduce
$(R_{\bar{r}0\bar{h}k}\bar{m}^r \bar{m}^h m^k)|_l m^l + \frac{B}{2}R_{\bar{r}0\bar{h}k}\bar{m}^r \bar{m}^h m^k$
$+\frac{1}{L}R_{\bar{0}0\bar{h}k}\bar{m}^h m^k - R_{\bar{r}l\bar{h}k}\bar{m}^r \bar{m}^h m^k m^l + \frac{1}{L}R_{\bar{r}0\bar{0}k}\bar{m}^r m^k$

$-P_{\bar{r}0\bar{s}k}P^{\bar{s}}_{\bar{0}l\bar{h}}\bar{m}^r\bar{m}^h m^k m^l + R_{\bar{r}0\bar{h}n}C^n_{kl}\bar{m}^r\bar{m}^h m^k m^l = 0.$

From here we obtain

$-F(E_{|\bar{s}}m^{\bar{s}})|_l m^l - \frac{B}{2}E_{|\bar{s}}m^{\bar{s}} - V_{|\bar{s}}m^{\bar{s}} - \frac{1}{2}V(\bar{V}+\bar{H}) - \mathbf{W} - \frac{1}{F}\bar{E}_{|0} + A\bar{A}L\mathbf{K} - BFE_{|\bar{s}}m^{\bar{s}} = L\,|\Omega|^2$, which leads to vii). The global validity of the above statements results by straightforward computations using (3.11). □

Note that the local Berwald frames are not only a local geometrical machinery, but they also satisfy important properties which contain three main real curvature invariants: **I**, **K** and **W**. The geometry of 2 - dimensional complex finsler spaces can be controlled by means of these curvature invariants, but this makes the subject of a forthcoming paper.

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