Bulletin of the *Transilvania* University of Braşov • Series III: Mathematics, Informatics, Physics,
Vol 5(54) 2012, Special Issue: *Proceedings of the Seventh Congress of Romanian Mathematicians*, 55-66,
published by Transilvania University Press, Braşov and Publishing House of the Romanian Academy

# REMARKS ON NEUMANN BOUNDARY VALUE PROBLEMS WITH VARIABLE EXPONENTS

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**Abstract**

We are interested in elliptic problems with Neumann boundary conditions that are studied in the framework of isotropic and anisotropic spaces with variable exponents. We establish an existence and a uniqueness result concerning a problem with a general $p(\cdot)$ - Laplace type operator. In addition, we present connections to other results, some of them involving the same operator, some of them involving a general $\overrightarrow{p}(\cdot)$ - Laplace type operator.

2000 *Mathematics Subject Classification:* 35J25, 35J60, 35D30, 46E35, 35J20.
*Key words:* variable exponent spaces, Neumann elliptic problem, weak solutions, existence, uniqueness.

## 1 Indroduction

We are working on Lebesgue and Sobolev spaces with variable exponent and we are concerned with the following class of elliptic problems:

$$\begin{cases} -\mathrm{div}(a(x,\nabla u)) + b(x)|u|^{p(x)-2}u = f(x,u) & \text{for} \quad x \in \Omega, \\ a(x,\nabla u)\cdot\nu(x) = g(x,u) & \text{for} \quad x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$ is a bounded domain with smooth boundary. We assume that $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$, $a = a(x,\eta)$, is a Carathéodory function such that it is the continuous derivative with respect to $\eta$ of a function $A : \Omega \times \mathbb{R}^N \to \mathbb{R}$, $A = A(x,\eta)$. More exactly, $a(x,\eta) = \nabla_\eta A(x,\eta)$. The mappings $a$ and $A$ verify the assumptions:
(A1) The equality

$$A(x,0) = 0$$

holds for all $x \in \Omega$;
(A2) There exists a constant $c_0 > 0$ such that

$$|a(x,\eta)| \leq c_0(1 + |\eta|^{p(x)-1}),$$

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for all $x \in \Omega$ and all $\eta \in \mathbb{R}^N$;

(A3) The inequality

$$0 \leq [a(x,\eta_1) - a(x,\eta_2)] \cdot (\eta_1 - \eta_2)$$

holds for all $x \in \Omega$ and all $\eta_1, \eta_2 \in \mathbb{R}^N$, with equality if and only if $\eta_1 = \eta_2$;

(A4) The inequalities

$$|\eta|^{p(x)} \leq a(x,\eta) \cdot \eta \leq p(x)\ A(x,\eta)$$

hold for all $x \in \Omega$ and all $\eta \in \mathbb{R}^N$;

The above set of hypotheses allows us to obtain well known operators by making some suitable choices. Indeed, for $A(x,\eta) = \frac{1}{p(x)}|\eta|^{p(x)}$ we deduce that $a(x,\eta) = |\eta|^{p(x)-2}\eta$ and for $\eta = \nabla u$ we find the $p(\cdot)$-Laplace operator

$$\operatorname{div}(a(x,\nabla u)) = \operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right). \tag{1.2}$$

We get a second example of operator when we choose $A(x,\eta) = \frac{1}{p(x)}[(1+|\eta|^2)^{p(x)/2} - 1]$, thus $a(x,\eta) = (1+|\eta|^2)^{(p(x)-2)/2}\eta$ and for $\eta = \nabla u$ we find the generalized mean curvature operator

$$\operatorname{div}(a(x,\nabla u)) = \operatorname{div}\left((1+|\nabla u|^2)^{(p(x)-2)/2}\nabla u\right).$$

Therefore it is no surprise that general operators described by conditions (A1) - (A4) are considered in other papers too, see for example [2, 12, 13]. In addition, nonstandard operators closely related to these appear in various situations, such is the case of classical Lebesgue-Sobolev spaces (see [11]), or the case of anisotropic Lebesgue-Sobolev spaces with variable exponent (see [1, 3]).

Our goal here is to establish an existence and a uniqueness result for problem (1.1) under some appropriate assumptions on the functions $b$, $f$ and $g$. Since our study will be conducted in the framework of variable exponent spaces, in the next section we introduce some preliminaries.

## 2 Abstract framework

In the present paper, $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) represents a bounded domain with smooth boundary $\partial\Omega$. We set

$$C_+(\overline{\Omega}) = \{h \in C(\overline{\Omega})\ :\ 1 < \min_{x\in\overline{\Omega}} h(x) < \max_{x\in\overline{\Omega}} h(x) < \infty\}$$

and for all $h \in C_+(\overline{\Omega})$ we denote

$$h^+ = \sup_{x\in\Omega} h(x), \qquad h^- = \inf_{x\in\Omega} h(x).$$

Also, we denote

$$h^\star(x) = \begin{cases} Nh(x)/[N-h(x)] & \text{if} \quad h(x) < N, \\ \infty & \text{if} \quad h(x) \geq N, \end{cases}$$

and

$$h^{\partial}(x) = \begin{cases} (N-1)h(x)/[N-h(x)] & \text{if} \quad h(x) < N, \\ \infty & \text{if} \quad h(x) \geq N. \end{cases}$$

Everywhere below we consider $p \in C_+(\overline{\Omega})$. The isotropic Lebesgue space with variable exponent is defined by

$$L^{p(\cdot)}(\Omega) = \{u \,:\, u \text{ is a measurable real–valued function such that } \int_\Omega |u(x)|^{p(x)}\,dx < \infty\}$$

endowed with the Luxemburg norm

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\mu > 0 \;:\; \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \leq 1\right\}.$$

This space is a separable and reflexive Banach space (see [10, Theorem 2.5, Corollary 2.7]) and we recall a significant embedding theorem.

**Theorem 2.1.** *([10, Theorem 2.8]) If $p_1,\ p_2 \in C_+(\overline{\Omega})$ are such that $p_1 \leq p_2$ in $\Omega$, then the embedding $L^{p_2(\cdot)}(\Omega) \hookrightarrow L^{p_1(\cdot)}(\Omega)$ is continuous.*

Moreover, the following Hölder-type inequality

$$\left|\int_\Omega u(x)v(x)\,dx\right| \leq 2\,\|u\|_{L^{p(\cdot)}(\Omega)}\|v\|_{L^{p'(\cdot)}(\Omega)}$$

holds for all $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$ (see [10, Theorem 2.1]), where we denoted by $L^{p'(\cdot)}(\Omega)$ the conjugate space of $L^{p(\cdot)}(\Omega)$, obtained by conjugating the exponent pointwise, that is, $1/p(x) + 1/p'(x) = 1$ (see [10, Corollary 2.7]).

To ease our work, we also introduce the map $\rho_{p(\cdot)} : L^{p(\cdot)}(\Omega) \to \mathbb{R}$,

$$\rho_{p(\cdot)}(u) = \int_\Omega |u(x)|^{p(x)}\,dx,$$

called the $p(\cdot)$-modular of the $L^{p(\cdot)}(\Omega)$ space. We recall next its most important properties (see for example [9, Theorem 1.3, Theorem 1.4]). If $u \in L^{p(\cdot)}(\Omega)$, then:

$$\|u\|_{L^{p(\cdot)}(\Omega)} < 1\ (= 1;\ > 1) \quad \Leftrightarrow \quad \rho_{p(\cdot)}(u) < 1\ (= 1;\ > 1) \tag{2.3}$$

$$\|u\|_{L^{p(\cdot)}(\Omega)} > 1 \quad \Rightarrow \quad \|u\|^{p^-}_{L^{p(\cdot)}(\Omega)} \leq \rho_{p(\cdot)}(u) \leq \|u\|^{p^+}_{L^{p(\cdot)}(\Omega)} \tag{2.4}$$

$$\|u\|_{L^{p(\cdot)}(\Omega)} < 1 \quad \Rightarrow \quad \|u\|^{p^+}_{L^{p(\cdot)}(\Omega)} \leq \rho_{p(\cdot)}(u) \leq \|u\|^{p^-}_{L^{p(\cdot)}(\Omega)}. \tag{2.5}$$

**Remark 2.1.** *If we consider the application $\tilde{\rho}_{p(\cdot)} : L^{p(\cdot)}(\partial\Omega) \to \mathbb{R}$, $\tilde{\rho}_{p(\cdot)}(u) = \int_{\partial\Omega} |u(x)|^{p(x)} dS$, the corresponding properties* (2.3) - (2.5) *remain valid.*

Let us introduce now the definition of the isotropic Sobolev space with variable exponent, $W^{1,p(\cdot)}(\Omega)$. We set

$$W^{1,p(\cdot)}(\Omega) = \left\{ u \in L^{p(\cdot)}(\Omega) : \, |\nabla u| \in L^{p(\cdot)}(\Omega) \right\}$$

endowed with the norm

$$\|u\| = \|u\|_{L^{p(\cdot)}(\Omega)} + \|\nabla u\|_{L^{p(\cdot)}(\Omega)}, \tag{2.6}$$

where by $\|\nabla u\|_{L^{p(\cdot)}(\Omega)}$ we understand $\| \, |\nabla u| \, \|_{L^{p(\cdot)}(\Omega)}$. The space $\left(W^{1,p(\cdot)}(\Omega), \| \cdot \|\right)$ is a separable and reflexive Banach space (see [10, Theorem 1.3]) and the next proposition is very helpful in handling its norm.

**Proposition 2.1.** *([8, Proposition 2.3]) If* $u \in W^{1,p(\cdot)}(\Omega)$ *then*

$$\|u\| > 1 \quad \Rightarrow \quad \|u\|^{p^-} \le \int_\Omega \left[ |\nabla u|^{p(x)} + |u|^{p(x)} \right] dx \le \|u\|^{p^+};$$

$$\|u\| < 1 \quad \Rightarrow \quad \|u\|^{p^+} \le \int_\Omega \left[ |\nabla u|^{p(x)} + |u|^{p(x)} \right] dx \le \|u\|^{p^-}.$$

We also have two embedding results.

**Theorem 2.2.** *([7, Proposition 2.4]) Let* $\Omega \subset \mathbb{R}^N$, $N \ge 2$ *be a bounded domain with smooth boundary. If* $p,\, q \in C(\overline{\Omega})$ *satisfy the condition*

$$1 \le q(x) < p^\star(x), \qquad \forall\, x \in \overline{\Omega},$$

*then there is a compact embedding* $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$.

**Theorem 2.3.** *([6, Corollary 2.4]) Let* $\Omega \subset \mathbb{R}^N$, $N \ge 2$, *be a bounded open set with smooth boundary. Suppose that* $p \in C_+(\overline{\Omega})$ *and* $r \in C(\overline{\Omega})$ *satisfy the condition*

$$1 \le r(x) < p^\partial(x), \qquad \forall\, x \in \partial\Omega.$$

*Then there is a compact boundary trace embedding* $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{r(\cdot)}(\partial\Omega)$.

We refer to [6] for more details regarding the extension of the classical trace to Lebesgue-Sobolev spaces with variable exponent. Everywhere below, when we refer to the trace of $u$ we will write $u$ instead of $u|_{\partial\Omega}$ or $\gamma u$.

## 3 Main Results

In what follows, in addition to hypotheses (A1) - (A4) we assume that function $b : \Omega \to \mathbb{R}$ satisfies:

(B) $b \in L^\infty(\Omega)$ and there exists $b_0 > 0$ such that $b(x) \ge b_0$ for all $x \in \Omega$.

Also, we suppose that $f$, $g$ are Carathéodory functions verifying the conditions:

(F) For $q \in C_+(\overline{\Omega})$ with $q^+ < p^-$, there exist $c_1$, $c_2 > 0$ such that

$$|f(x,t)| \leq c_1 + c_2|t|^{q(x)-1}$$

for all $x \in \Omega$ and all $t \in \mathbb{R}$;
(G) For $r \in C_+(\overline{\Omega})$ with $r^+ < p^-$, there exist $c_3$, $c_4 > 0$ such that

$$|g(x,t)| \leq c_3 + c_4|t|^{r(x)-1}$$

for all $x \in \partial\Omega$ and all $t \in \mathbb{R}$.

Since we deal with the existence and uniqueness of weak solutions for our problem, we introduce the following definition.

**Definition 3.1.** *A function $u \in W^{1,p(\cdot)}(\Omega)$ which verifies*

$$\int_\Omega a(x, \nabla u) \cdot \nabla v \, dx + \int_\Omega b(x)|u|^{p(x)-2}uv \, dx - \int_\Omega f(x,u)v \, dx - \int_{\partial\Omega} g(x,u)v \, dS = 0$$

*for all $v \in W^{1,p(\cdot)}(\Omega)$ is called a weak solution of* (1.1).

We associate to problem (1.1) the energetic functional $I : W^{1,p(\cdot)}(\Omega) \to \mathbb{R}$ defined by

$$I(u) = \int_\Omega A(x, \nabla u) \, dx + \int_\Omega \frac{b(x)}{p(x)}|u|^{p(x)} \, dx - \int_\Omega F(x,u) \, dx - \int_{\partial\Omega} G(x,u) \, dS,$$

where $F : \Omega \times \mathbb{R} \to \mathbb{R}$, $G : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ are given by

$$F(x,s) = \int_0^s f(x,t)dt, \qquad G(x,s) = \int_0^s g(x,t)dt.$$

We also define the functional $\Lambda : W^{1,p(\cdot)}(\Omega) \to \mathbb{R}$ introduced by

$$\Lambda(u) = \int_\Omega A(x, \nabla u) \, dx.$$

**Proposition 3.1.** *([13, Lemma 1]) (i) The functional $\Lambda$ is well-defined on $W^{1,p(\cdot)}(\Omega)$. (ii) The functional $\Lambda$ is of class $C^1(W^{1,p(\cdot)}(\Omega), \mathbb{R})$ and*

$$\langle \Lambda'(u), v \rangle = \int_\Omega a(x, \nabla u) \cdot \nabla v \, dx,$$

*for all $u$, $v \in W^{1,p(\cdot)}(\Omega)$.*

We must mention that the study from [13] is conducted for a $\Lambda_0 : W_0^{1,p(\cdot)}(\Omega) \to \mathbb{R}$, where $W_0^{1,p(\cdot)}(\Omega)$ represents the Sobolev space with zero boundary values defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{W_0^{1,p(\cdot)}(\Omega)} = \|\nabla u\|_{L^{p(\cdot)}(\Omega)}$. However, since the calculus is almost identical, we omit it for brevity. Using Proposition 3.1, a standard calculus shows that $I$ is well defined and $I \in C^1(W^{1,p(\cdot)}(\Omega);\mathbb{R})$ with

$$\langle I'(u), v\rangle = \int_\Omega a(x,\nabla u)\cdot \nabla v\,dx + \int_\Omega b(x)|u|^{p(x)-2}uv\,dx - \int_\Omega f(x,u)v\,dx - \int_{\partial\Omega} g(x,u)v\,dS$$

for all $u$, $v \in W^{1,p(\cdot)}(\Omega)$. Thus any critical point $u \in W^{1,p(\cdot)}(\Omega)$ of $I$ is a weak solution to problem (1.1) and we rely our argumentation on the critical point theory. The main tool to establish our existence result is the following theorem.

**Theorem 3.1.** *([14, 1.2 Theorem]) Suppose $X$ is a reflexive Banach space of norm $\|\cdot\|_X$ and let $I : X \to \mathbb{R}\cup\{\infty\}$ be a function such that:*

*(i) $I$ is coercive on $X$, i.e., $I(u) \to \infty$ as $\|u\|_X \to \infty$.*

*(ii) $I$ is (sequentially) weakly lower semicontinuous on $X$, i.e., for any $u \in X$ and any subsequence $(u_n)_n \subset X$ such that $u_n \rightharpoonup u$ weakly in $X$ there holds*

$$I(u) \le \liminf_{n\to\infty} I(u_n).$$

*Then $I$ is bounded from below on $X$ and attains its infimum in $X$.*

Using the above theorem, we can prove our first main result.

**Theorem 3.2.** *Let $\Omega \subset \mathbb{R}^N$, $N \ge 2$, be a bounded domain with smooth boundary. Assume that $f : \Omega\times\mathbb{R} \to \mathbb{R}$, $g : \partial\Omega\times\mathbb{R} \to \mathbb{R}$ and $a : \Omega\times\mathbb{R}^N \to \mathbb{R}^N$ are Carathéodory functions and that $a = a(x,\eta)$ is the continuous derivative with respect to $\eta$ of a function $A : \Omega\times\mathbb{R}^N \to \mathbb{R}$, $A = A(x,\eta)$. If conditions (A1) - (A4), (B), (F) and (G) are fulfilled, then there exists a weak solution to problem* (1.1).

*Proof.*

We start by showing that $I$ is coercive. By integrating in the relations provided by (F) and (G), we arrive at

$$|F(x,t)| \le c_1|t| + c_2\frac{|t|^{q(x)}}{q(x)} \quad \text{for all } x \in \Omega \text{ and } t \in \mathbb{R},$$

$$|G(x,t)| \le c_3|t| + c_4\frac{|t|^{r(x)}}{r(x)} \quad \text{for all } x \in \partial\Omega \text{ and } t \in \mathbb{R}.$$

Then, due to relations (2.3) – (2.5) and Remark 2.1,

$$\int_\Omega F(x,u)\,dx \le c_1\|u\|_{L^1(\Omega)} + \frac{c_2}{q^+}\left(\|u\|^{q^+}_{L^{q(\cdot)}(\Omega)} + \|u\|^{q^-}_{L^{q(\cdot)}(\Omega)}\right),$$

$$\int_{\partial\Omega} G(x,u)\,dS \le c_3\|u\|_{L^1(\partial\Omega)} + \frac{c_4}{r^+}\left(\|u\|^{r^+}_{L^{r(\cdot)}(\partial\Omega)} + \|u\|^{r^-}_{L^{r(\cdot)}(\partial\Omega)}\right).$$

Taking into account Theorems 2.2 and 2.3, we obtain that, for $u \in W^{1,p(\cdot)}(\Omega)$ with $\|u\| \geq 1$, there exist $k_1$, $k_2$, $k_3$, $k_4 > 0$ such that

$$\int_\Omega F(x,u)\,dx \leq k_1\|u\| + k_2\|u\|^{q^+}, \tag{3.7}$$

$$\int_{\partial\Omega} G(x,u)\,dS \leq k_3\|u\| + k_4\|u\|^{r^+}. \tag{3.8}$$

On the other hand, by (A4) and (B),

$$\int_\Omega A(x,\nabla u)\,dx + \int_\Omega \frac{b(x)}{p(x)}|u|^{p(x)}\,dx \geq \frac{\min\{1,b_0\}}{p^+}\int_\Omega \left[|\nabla u|^{p(x)} + |u|^{p(x)}\right]\,dx.$$

Using Proposition 2.1 in the above inequality, we infer that, for $\|u\| \geq 1$,

$$\int_\Omega A(x,\nabla u)\,dx + \int_\Omega \frac{b(x)}{p(x)}|u|^{p(x)}\,dx \geq \frac{\min\{1,b_0\}}{p^+}\|u\|^{p^-}. \tag{3.9}$$

Putting together (3.7), (3.8) and (3.9), we find out that, for $\|u\| \geq 1$,

$$I(u) \geq \frac{\min\{1,b_0\}}{p^+}\|u\|^{p^-} - k_2\|u\|^{q^+} - k_4\|u\|^{r^+} - (k_1+k_3)\|u\|.$$

By considering the hypotheses on $p$, $q$ and $r$, we get that $I(u) \to \infty$ when $\|u\| \to \infty$, hence $I$ is coercive.

Next, we make the notations

$$\mathcal{F}(u) = \int_\Omega F(x,u)\,dx, \qquad \mathcal{G}(u) = \int_{\partial\Omega} G(x,u)\,dS,$$

and we notice that $\mathcal{F}'$ and $\mathcal{G}'$ are completely continuous, therefore $\mathcal{F}$ and $\mathcal{G}$ are weakly continuous. Following the ideas from [4] and [13] we deduce that $I$ is weakly lower semi-continuous. We are now in position to apply Theorem 3.1. Thus we conclude that problem (1.1) admits at least one weak solution.

□

In order to get the uniqueness of the solution, we introduce new assumptions on $f$ and $g$:

(F0) $f$ is fulfilling the monotonicity condition

$$(f(x,s) - f(x,t))(s-t) < 0,$$

for all $x \in \Omega$ and $s$, $t \in \mathbb{R}$ with $s \neq t$;

(G0) $g$ is fulfilling the monotonicity condition

$$(g(x,s) - g(x,t))(s-t) < 0,$$

for all $x \in \partial\Omega$ and $s$, $t \in \mathbb{R}$ with $s \neq t$.

Now we can formulate our second result.

**Theorem 3.3.** *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain with smooth boundary. Assume that $f : \Omega \times \mathbb{R} \to \mathbb{R}$, $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ and $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ are Carathéodory functions and that $a = a(x, \eta)$ is the continuous derivative with respect to $\eta$ of a function $A : \Omega \times \mathbb{R}^N \to \mathbb{R}$, $A = A(x, \eta)$. If conditions (A1) - (A4), (B), (F), (F0), (G) and (G0) are fulfilled, then problem* (1.1) *admits a unique weak solution.*

*Proof.*

The existence of solutions is guaranteed by Theorem 3.2. Hence we can assume that there exist two weak solutions to problem (1.1), that is, $u_1$ and $u_2$. First, we replace the solution $u$ by $u_1$ in Definition 3.1 and we take $v = u_1 - u_2$. We obtain

$$\int_\Omega a\left(x, \nabla u_1\right) \cdot \nabla(u_1 - u_2)\, dx + \int_\Omega b(x)|u_1|^{p(x)-2}u_1(u_1 - u_2)\, dx$$
$$- \int_\Omega f(x, u_1)(u_1 - u_2)\, dx - \int_{\partial\Omega} g(x, u_1)(u_1 - u_2)\, dS = 0.$$

Then, we replace the solution $u$ by $u_2$ in Definition 3.1 and we take $v = u_2 - u_1$. We get

$$\int_\Omega a\left(x, \nabla u_2\right) \cdot \nabla(u_2 - u_1)\, dx + \int_\Omega b(x)|u_2|^{p(x)-2}u_2(u_2 - u_1)\, dx$$
$$- \int_\Omega f(x, u_2)(u_2 - u_1)\, dx - \int_{\partial\Omega} g(x, u_2)(u_2 - u_1)\, dS = 0.$$

By the two previous relations we deduce that

$$\int_\Omega \left[a\left(x, \nabla u_1\right) - a\left(x, \nabla u_2\right)\right] \cdot (\nabla u_1 - \nabla u_2)\, dx + \int_\Omega b(x) \left[|u_1|^{p(x)-2}u_1 - |u_2|^{p(x)-2}u_2\right](u_1 - u_2)\, dx$$
$$- \int_\Omega \left[f(x, u_1) - f(x, u_2)\right](u_1 - u_2)\, dx - \int_{\partial\Omega} \left[g(x, u_1) - g(x, u_2)\right](u_1 - u_2)\, dS = 0.$$

Using (A3), (F0) and (G0), all the terms in the above equality are positive unless $u_1 = u_2$, therefore we have obtained the uniqueness of the weak solution to problem (1.1). □

# 4 Comments and connections

A first remark is that, by adding new assumptions to (A1) - (A4), (B), (F) and (G), we can establish a multiplicity result. Moreover, this multiplicity result can be achieved by relaxing the initial conditions (F) and (G) by not imposing a particular order between $p$ and $q$, respectively between $p$ and $r$. To be more precise, we refer to the following set of hypotheses:

(A5) The mapping $A$ is even with respect to its second variable, that is,

$$A(x, -\eta) = A(x, \eta)$$

for all $x \in \Omega$ and all $\eta \in \mathbb{R}^N$;

(F1) For $q \in C_+(\overline{\Omega})$ with $q(x) < p^\star(x)$ for all $x \in \overline{\Omega}$, there exists $\tilde{c}_1 > 0$ such that

$$|f(x,t)| \leq \tilde{c}_1 |t|^{q(x)-1}$$

for all $x \in \Omega$ and all $t \in \mathbb{R}$;
(F2) There exists $\alpha_1 > p^+$ such that

$$0 < \alpha_1 F(x,t) \leq t f(x,t)$$

for all $x \in \Omega$ and all $t \in \mathbb{R}$;
(F3) The function $f$ is odd with respect to its second variable, that is,

$$f(x,-t) = -f(x,t)$$

for all $x \in \Omega$ and all $t \in \mathbb{R}$;
(G1) For $r \in C_+(\overline{\Omega})$ with $r(x) < p^\partial(x)$ for all $x \in \partial\Omega$, there exists $\tilde{c}_2 > 0$ such that

$$|g(x,t)| \leq \tilde{c}_2 |t|^{r(x)-1}$$

for all $x \in \partial\Omega$ and all $t \in \mathbb{R}$;
(G2) There exists $\alpha_2 > p^+$ such that

$$0 < \alpha_2 G(x,t) \leq t g(x,t)$$

for all $x \in \partial\Omega$ and all $t \in \mathbb{R}$;
(G3) The function $g$ is odd with respect to its second variable, that is,

$$g(x,-t) = -g(x,t)$$

for all $x \in \partial\Omega$ and all $t \in \mathbb{R}$.

By working under conditions (A1) - (A5), (B), (F1) - (F3) and (G1) - (G3), the existence of a sequence of weak solutions to problem (1.1) is obtained in [2] with the aid of the fountain theorem.

Next, to make some other useful connections, we introduce the vectorial function

$$\overrightarrow{p} : \overline{\Omega} \to \mathbb{R}^N, \quad \overrightarrow{p}(\cdot) = (p_1(\cdot), ..., p_N(\cdot)),$$

where $p_i \in C_+(\overline{\Omega})$ for all $i \in \{1, \ldots, N\}$. Also, we denote

$$p_M(x) = \max\{p_1(x), \ldots, p_N(x)\}$$

and we recall the definition of the anisotropic variable exponent Sobolev space,

$$\begin{aligned} W^{1,\overrightarrow{p}(\cdot)}(\Omega) &= \Big\{ u \in L^{p_M(\cdot)}(\Omega) : \quad \partial_{x_i} u \in L^{p_i(\cdot)}(\Omega) \text{ for all } i \in \{1, \ldots, N\} \Big\} \\ &= \Big\{ u \in L^1_{loc}(\Omega) : \quad u \in L^{p_i(\cdot)}(\Omega),\, \partial_{x_i} u \in L^{p_i(\cdot)}(\Omega) \text{ for all } i \in \{1, \ldots, N\} \Big\}. \end{aligned}$$

The space $W^{1,\overrightarrow{p}(\cdot)}(\Omega)$ endowed with the norm

$$\|u\|_{W^{1,\overrightarrow{p}(\cdot)}(\Omega)} = \|u\|_{L^{p_M(\cdot)}(\Omega)} + \sum_{i=1}^{N} \|\partial_{x_i} u\|_{L^{p_i(\cdot)}(\Omega)}$$

is a reflexive Banach space (see [5, Theorem 2.1, Theorem 2.2]). Notice that for $p_1 = p_2 = \cdots = p_N = p$, we arrive at the isotropic Sobolev space with variable exponent $W^{1,p(\cdot)}(\Omega)$. Moreover, the norm becomes

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{L^{p(\cdot)}(\Omega)} + \sum_{i=1}^{N} \|\partial_{x_i} u\|_{L^{p(\cdot)}(\Omega)},$$

which is a norm equivalent to $\|\cdot\|$ given by (2.6). Thus, the space $W^{1,p(\cdot)}(\Omega)$ can be viewed as a particular case of $W^{1,\overrightarrow{p}(\cdot)}(\Omega)$. However, this is not quite the case for the $p(\cdot)$ - Laplace operator (1.2) and the $\overrightarrow{p}(\cdot)$ - Laplace operator defined as

$$\Delta_{\overrightarrow{p}(x)}(u) = \sum_{i=1}^{N} \partial_{x_i} \left( |\partial_{x_i} u|^{p_i(x)-2} \partial_{x_i} u \right). \tag{4.10}$$

Indeed, when we take $p_1 = p_2 = \cdots = p_N = p$ in (4.10), we do not get exactly the operator introduced by (1.2). For similar reasons, we can not consider problem (1.1) to be a particular case of the problem studied in [3] in the framework of anisotropic variable exponent spaces,

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \partial_{x_i} a_i\left(x, \partial_{x_i} u\right) + b(x)|u|^{p_M(x)-2}u = f(x,u) & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \\ \displaystyle\sum_{i=1}^{N} a_i\left(x, \partial_{x_i} u\right)\nu_i = g(x,u) & \text{on } \partial\Omega, \end{cases} \tag{4.11}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded open set with smooth boundary, $\nu_i$, $i \in \{1, \ldots, N\}$, are the components of the outer normal unit vector, $b$ satisfies (B) and the applications $a_i : \Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions fulfilling the following hypotheses for all $i \in \{1, \ldots, N\}$:

($\tilde{\text{A}}$1) There exists a positive constant $\bar{c}_i$ such that $a_i$ satisfies the growth condition

$$|a_i(x,s)| \leq \bar{c}_i(d_i(x) + |s|^{p_i(x)-1}),$$

for all $x \in \Omega$ and $s \in \mathbb{R}$, where $d_i \in L^{p_i'(\cdot)}(\Omega)$ (with $1/p_i(x) + 1/p_i'(x) = 1$), is a nonnegative function;

($\tilde{\text{A}}$2) If we denote by $A_i : \Omega \times \mathbb{R} \to \mathbb{R}$,

$$A_i(x,s) = \int_0^s a_i(x,t)dt,$$

then the following inequalities hold:

$$|s|^{p_i(x)} \le a_i(x,s)s \le p_i(x)\ A_i(x,s),$$

for all $x \in \Omega$ and $s \in \mathbb{R}$;

($\tilde{\text{A}}3$) $a_i$ is fulfilling

$$(a_i(x,s) - a_i(x,t))(s-t) > 0,$$

for all $x \in \Omega$ and $s,\ t \in \mathbb{R}$ with $s \neq t$.

Also, in addition to (F0) and (G0), the Carathéodory functions $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ are fulfilling conditions very much alike to conditions (F) and (G):

($\tilde{\text{F}}$) There exist a positive constant $k_1$ and $q \in L_+^\infty(\Omega)$ with $q^+ < p_m^-$, such that

$$|f(x,s)| \le k_1 \left(1 + |s|^{q(x)-1}\right),$$

for all $x \in \Omega$ and $s \in \mathbb{R}$, where

$$p_m(x) = \min\{p_1(x), \ldots, p_N(x)\};$$

($\tilde{\text{G}}$) There exist a positive constant $k_2$ and $r \in C(\overline{\Omega})$ with $r^+ < p_m^-$ such that

$$|g(x,s)| \le k_2 \left(1 + |s|^{r(x)-1}\right),$$

for all $x \in \partial\Omega$ and $s \in \mathbb{R}$.

Although problem (1.1) is not a particular case of (4.11), it is clear that the two problems are closely related and in many aspects (4.11) is a generalization of (1.1). Furthermore, as showed in [3], under the previous conditions, an existence and uniqueness result can be provided using the same main argument, that is, Theorem 3.1. Therefore a natural question arises: could we obtain a sequence of weak solutions to (4.11) by applying the same strategy as in [2]? At least for the time being, we can not give a positive answer to this question. A major impediment is that we do not know if the functional $J : W^{1,\vec{p}(\cdot)}(\Omega) \to \mathbb{R}$, $J(u) = \int_\Omega \sum_{i=1}^N A_i\left(x, \partial_{x_i} u\right) dx$ is of type (S+). We remind that $J$ is said to be of type (S+) if any sequence $(u_n)_n \subset W^{1,\vec{p}(\cdot)}(\Omega)$ that is weakly convergent to $u \in W^{1,\vec{p}(\cdot)}(\Omega)$ such that

$$\limsup_{n\to\infty} \langle J'(u_n), u_n - u\rangle \le 0$$

converges strongly to $u$ in $W^{1,\vec{p}(\cdot)}(\Omega)$. The arguments used to show that $J$ is of type (S+) when is defined on $W_0^{1,\vec{p}(\cdot)}(\Omega)$ (see [1, Lemma 2]) fail when we replace this space with $W^{1,\vec{p}(\cdot)}(\Omega)$. For more details we send the reader to [3, Section 4], where is also debated the applicability of the mountain pass theorem and of the Ekeland principle to problem (4.11).

**Acknowledgment.** The author was supported by a grant of the Romanian National Authority for Scientific Research, CNCS - UEFISCDI, project number PN-II-RU-TE-2011-3-0223.

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