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MATHEMATICAL SOLUTION OF SOME INDUSTRIAL PROBLEM IN TECHNICAL ELECTRODYNAMICS

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Abstract

The general differential Maxwell system is solved explicitly in the case of an arbitrary inhomogeneous linear isotropic medium. The analytic mathematical study is done taking into account the initial real industrial phenomena statement of technical electrodynamics

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1 Introduction

The present paper deals with the investigation of the general differential Maxwell system. The latter is the mathematical model of the phenomena in technical and classical electrodynamics that concern the signal transmissions in the various kinds of media, multidimensional circuits and filters with the distributed parameters as well [1].

As far as it is known, the specific case of the aforesaid system was introduced for the first time in [2] in the case of an isotropic homogeneous immobile medium. The medium was given as an arbitrary excited, but such important assumption was not taken into account, as in the framework of the initial problem statement, as in the future research. Moreover, the same paper [2] did not affect the solution of the proposed differential Maxwell system and neglected the outside electromagnetic field tensions.

The next investigating step was done in [3] when the above mentioned differential Maxwell system was studied in the presence of the outside electromagnetic field tensions and excitation of a medium. Additionally, the system's diagonalization problem was raised and fulfilled. The last action meant the reduction of the aforesaid matrix problem to the equivalent totality of the appropriate scalar equations with respect to the only one component of the sought for electromagnetic field vector functions. Then this totality was

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expressed as the unified general scalar wave PDE whose explicit solution [4] based on the preceding conclusions in the simplest cases of media [5].

All mentioned results were independent of the boundary conditions and original matrix structure. Only two restrictions were imposed to the operator matrix elements, - their commutativity in pairs and invertibility.

The proposed paper concerns the study of the general differential Maxwell system when an arbitrary isotropic linear **inhomogeneous** *excited* medium is given. Such problem statement generalizes essentially [2], [3] and spreads all over the vast number of industrial problems in technical and classical electrodynamics whose analytic solution was not done yet.

Therefore, the goal of the suggested article is an explicit solution of the general differential Maxwell system in the case of an arbitrary isotropic **inhomogeneous** linear *excited* medium.

2 The problem statement

Let the general differential Maxwell system be given

$$\begin{cases} \mathbf{rot}\vec{H} = (\sigma \pm \lambda\varepsilon_a)\vec{E} + \varepsilon_a \frac{\partial \vec{E}}{\partial t} + \vec{j^{os}} \\ -\mathbf{rot}\vec{E} = (r \pm \lambda\mu_a)\vec{H} + \mu_a \frac{\partial \vec{H}}{\partial t} + \vec{e}^{os}, \end{cases} \tag{2.1}$$

where: $\vec{E}, \vec{H} = \vec{E}, \vec{H}(x, y, z, t)$ are the sought for vector functions that describe tensions of the electric and magnetic field correspondingly, and their scalar components are $E_k, H_k = E_k, H_k(x, y, z, t) \quad (k = \overline{1, 3})$. The outside current sources and tensions $\vec{j}^{os}, \vec{e}^{os} = \vec{j}^{os}, \vec{e}^{os}(x, y, z, t)$ with scalar components $j_k^{os}, e_k^{os} = j_k^{os}, e_k^{os}(x, y, z, t) \quad (k = \overline{1, 3})$ are known.

Since a medium is accepted as **inhomogeneous** here, the specific conductivity σ, absolute and dielectric permeability μ_a, ε_a depend on the spatial coordinates, just as, $\sigma, \mu_a, \varepsilon_a = \sigma, \mu_a, \varepsilon_a(x, y, z)$. We remind of such assumption as the considerable generalization of the generally accepted classical requirement when $\sigma, \mu_a, \varepsilon_a$ are usual positive physical constants [1], [2]. Further, in (2.1) $\lambda = \mathbf{const} > 0$ is the parameter of the signal that excites the medium, and an additional "symmetrical" number $r > 0$ exists only theoretically at the current stage of study. The change of sign in front of λ means the reaction of the medium to the signal excitation; "+" is an absorption, "-" implies a seizure.

So, the medium is an arbitrary *excited* and **inhomogeneous**.

The scalar functions $E_k, H_k, j_k^{os}, e_k^{os} \quad (k = \overline{1, 3})$ and $\sigma, \mu_a, \varepsilon_a$ are n-times continuously differentiable in some domain of the respective space $\mathbb{R}_4$ and $\mathbb{R}_3$ in terms of either (x, y, z, t) or (x, y, z). The numerical value of n will be introduced a little bit later when it can be confirmed by the results of the first step of solution.

The form of domain is determined when the specific applied industrial and appropriate boundary problem appear.

The same fact holds the specific feature of the aforesaid functions' non smooth behavior too. This characteristic depends also on the concrete industrial problem statement and can be specified only when the corresponding initial and boundary conditions are fixed.

Hence, it should be noted once more, that all suggested here results, as in [3] - [5], are irrespective of the boundary problem statement.

Now we come directly to the explicit solution of the above raised problem by means of the following method that is written below.

The required study is done in two stages. The first step concerns the diagonalization procedure of (2.1) "by blocks" and "by coordinates", represents the operator analogy of Gauss method and bases on the results of [3]. The second step is an explicit solution of the unified wave scalar general PDE which appears as the final result of the diagonalization procedure.

The next section describes the first stage of the proposed solving method.

3 The problem solution in terms of the diagonalization procedure

So, the purpose of the present section is the reduction of the original vector problem, i.e. of (2.1), to the equivalent system of scalar equations where each of them has an only one unknown component of the electromagnetic vector field functions.

Let

$$A = \mathbf{rot}, \quad \partial_0 = \frac{\partial}{\partial t}, \quad \partial_0^* = \partial_0 \pm \lambda, \quad C = \sigma + \varepsilon_a \partial_0^*, \quad D = r + \mu_a \partial_0^* \tag{3.2}$$

be the auxiliary designations of the corresponding differential operators from (2.1). Then this system (2.1) can be rewritten as follows, and the diagonalization "by blocks" [3] begins

$$\left\{ \begin{array}{rl} A\vec{H} - C\vec{E} = \vec{j}^{os} & | \quad D \\ -A\vec{E} - D\vec{H} = \vec{e}^{os} & | \quad (-A) \end{array} \right. \oplus \tag{3.3}$$

$$\Updownarrow$$

$$\left\{ \begin{array}{rl} -(DC + A^2)\vec{E} = D\vec{j}^{os} + A\vec{e}^{os} & | \quad (-A) \\ -A\vec{E} - D\vec{H} = \vec{e}^{os} & | \quad (-(DC + A^2)) \end{array} \right. \oplus \tag{3.4}$$

$$\Updownarrow$$

$$\left\{ \begin{array}{c} -(A^2 + DC)\vec{E} = D\vec{j}^{os} + A\vec{e}^{os} \\ -(A^2 + DC)D\vec{H} = (A^2 + DC)\vec{e}^{os} - AD\vec{j}^{os} - A^2\vec{e}^{os}. \end{array} \right. \tag{3.5}$$

Operator applications (3.3) - (3.5) reflect the diagonalization of (2.1) "by blocks" [3] and reduce this original system to the equivalent unified "vector-scalar" equation with respect to the only one of the electromagnetic field vector functions

$$-(A^2 + DC)\vec{F}_i = \vec{\varphi}_i \quad (i = 1, 2), \tag{3.6}$$

where

$$\vec{F}_1 = \vec{E}, \quad \vec{F}_2 = \vec{H}; \quad \vec{\varphi}_1 = A\vec{e}^{os} + D\vec{j}^{os}, \quad \vec{\varphi}_2 = C\vec{e}^{os} - A\vec{j}^{os}. \tag{3.7}$$

Therefore, the diagonalization "by blocks" is finished and the diagonalization "by coordinates" begins.

Taking into account the following formulae

$$A^2 = (\mathbf{rot})(\mathbf{rot}) = (\mathbf{grad})(\mathbf{div}) - \Delta; \quad \Delta = \sum_{k=1}^{3} \partial_k^2; \\ \partial_1 = \frac{\partial}{\partial x}, \partial_2 = \frac{\partial}{\partial y}, \partial_3 = \frac{\partial}{\partial z}, \tag{3.8}$$

and operator polynomial

$$\tilde{\partial}_0^2 = DC = \mu_a \varepsilon_a (\partial_0^*)^2 + (\sigma \mu_a + r\varepsilon_a)\partial_0^* + r\sigma, \tag{3.9}$$

the operator from the left side of (3.6) can be expressed as follows

$$A^2 + DC = \mathbf{grad}(\partial_1 + \partial_2 + \partial_3) - \Delta + \tilde{\partial}_0^2. \tag{3.10}$$

New symbols in (3.10) are from (3.8), (3.9).

Using (3.10), the former equation (3.6) can be rewritten like that

$$\begin{cases} A_{23}F_{i1} - B_{12}F_{i2} - B_{13}F_{i3} = \varphi_{i1} \\ -B_{12}F_{i1} + A_{13}F_{i2} - B_{23}F_{i3} = \varphi_{i2} \\ -B_{13}F_{i1} - B_{23}F_{i2} + A_{12}F_{i3} = \varphi_{i3}, \end{cases} \quad (i = 1, 2) \tag{3.11}$$

where operators

$$A_{jk} = \partial_j^2 + \partial_k^2 + \tilde{\partial}_0^2 \quad (j \neq k), \quad B_{jk} = \partial_j \partial_k \quad (j \neq k) \quad (j, k = \overline{1,3}) \tag{3.12}$$

and scalar functions

$$F_{ik} = F_{ik}(x, y, z, t), \quad \varphi_{ik} = \varphi_{ik}(x, y, z, t) \quad (k = \overline{1,3}; \quad i = 1, 2) \tag{3.13}$$

are given in terms of (3.7) - (3.9) and

$$\vec{F}_i = \{F_{ik}\}_{k=1}^3 \quad (i = 1, 2); \quad \vec{\varphi}_i = \{\varphi_{ik}\}_{k=1}^3 \quad (i = 1, 2). \tag{3.14}$$

Applying directly the diagonalization process from [3] to (3.11) - (3.14), we obtain the required diagonal matrix that is equivalent to the unified scalar general wave equation regarding the unknown components (3.14) of the function $\vec{F}_i \quad (i = 1, 2)$ from (3.7):

$$\tilde{\partial}_0^2(\tilde{\partial}_0^2 - \Delta)F_{ik} = (\partial_k^2 - \tilde{\partial}_0^2)\varphi_{ik} + \partial_k(\partial_\nu \varphi_{i\nu} + \partial_l \varphi_{il}); \\ \nu \neq l, \quad k \neq \nu, \quad k \neq l \quad (k, \nu, l = \overline{1,3}; \quad i = 1, 2). \tag{3.15}$$

In (3.15), the functions φ_{ik} and partial differential operators are from (3.7), (3.13), (3.14) and (3.8), (3.9) respectively.

Closing this section, it should be noted that the numerical value of n from the previous section 2 can be specified as four. The structure of (3.15) confirms the given number, since n meant the order of the higher continuous derivative of the scalar functions from the original system (2.1).

Thus, the first step of the suggested solution is finished completely, and the last, second above mentioned stage of study remains.

4 Solution of the unified scalar equation

It is clear that (3.15) can be considered from the general mathematical viewpoint as the following PDE

$$\tilde{\partial}_0^2(\tilde{\partial}_0^2 - \Delta)F = f, \tag{4.16}$$

where F is F_{ik} from (3.15) and f is the right part of (3.15).

Using technique of [4], the fourth order equation (4.16) is simplified by the introduction of new unknown function

$$\Phi = \tilde{\partial}_0^2 F. \tag{4.17}$$

Then (4.16), in terms of (4.17), becomes the second-order PDE

$$(\tilde{\partial}_0^2 - \Delta)\Phi = f \tag{4.18}$$

that can be solved effectively by the integral transform method [6].

Namely, after application of the corresponding integral transformation, either by each of spatial variables x, y, z, or using the unified multidimensional transformation by (x, y, z) simultaneously [7], we come to the linear inhomogeneous ODE with constant coefficients in terms of the relevant transforms dependent on the time variable t:

$$(\tilde{\mu}_a\tilde{\varepsilon}_a(\frac{d}{dt} \pm \lambda)^2 + (\tilde{\sigma}\tilde{\mu}_a + r\tilde{\varepsilon}_a)(\frac{d}{dt} \pm \lambda) + (r\tilde{\sigma} - \tilde{\Delta}))\tilde{\Phi} = \tilde{f}. \tag{4.19}$$

"Tilde" means here the transform of the corresponding initial function.

Solving the linear homogeneous ODE that is raised by (4.19), we get the fundamental solution system [8]

$$\begin{array}{l} \{\chi_m = \chi_m(t,p) = e^{\eta_m t}(\cos\xi_m t + i\sin\xi_m t), \quad m = 1,2\}; \\ \quad \eta_m = \mathrm{Re}(\omega_m), \quad \xi_m = \mathrm{Im}(\omega_m); \\ \omega_{1,2} = \frac{1}{2\tilde{\mu}_a\tilde{\varepsilon}_a}\left(-((\tilde{\sigma}\tilde{\mu}_a + r\tilde{\varepsilon}_a) \pm 2\lambda\tilde{\mu}_a\tilde{\varepsilon}_a) \pm \sqrt{(\tilde{\sigma}\tilde{\mu}_a - r\tilde{\varepsilon}_a)^2 + 4\tilde{\mu}_a\tilde{\varepsilon}_a\tilde{\Delta}}\right). \end{array} \tag{4.20}$$

Then the general solution of (4.19) is looking for as

$$\tilde{\Phi} = \sum_{m=1}^{2} C_m\chi_m + \sum_{m=1}^{2} C_m^\star\chi_m, \quad \forall C_m^\star = \mathbf{const} \in \mathbb{R} \quad (m = 1,2), \tag{4.21}$$

where the first sum is the partial solution of (4.19) and the second sum is the general solution of the homogeneous ODE that is generated by (4.19). Further, the unknown functions $C_m = C_m(t,p) \quad (m = 1,2)$ are determined by the following system [8]

$$\begin{array}{c} \left\{ \begin{array}{l} C_1'\chi_1 + C_2'\chi_2 = 0 \\ C_1'\chi_1' + C_2'\chi_2' = \tilde{f}, \end{array} \right. \\ C_m' = \frac{dC_m}{dt}, \quad \chi_m' = \frac{d\chi_m}{dt} \quad (m = 1,2). \end{array} \tag{4.22}$$

Everywhere in (4.20) - (4.22), the numerical values of p describe the set of parameters of the applied integral transformations.

Solving (4.22), we find the sought for functions

$$C_{1,2} = \pm \frac{1}{(\eta_1 - \eta_2) + i(\xi_1 - \xi_2)} \int \frac{\tilde{f}}{e^{\eta_{1,2} t}(\cos \xi_{1,2} t + i \sin \xi_{1,2} t)}\, dt. \tag{4.23}$$

Turning to the original expressions of $\omega_{1,2}$ from (4.20), basing on (4.21), (4.23), the required general solution of (4.19) can be written below

$$\begin{gathered} \tilde{\Phi} = \tfrac{1}{(\eta_1 - \eta_2) + i(\xi_1 - \xi_2)} (e^{\omega_1 t} \textstyle\int e^{-\omega_1 t} \tilde{f} dt - e^{\omega_2 t} \int e^{-\omega_2 t} \tilde{f} dt) + \sum\limits_{m=1}^{2} C_m^{\star} \chi_m, \\ \forall C_m^{\star} = \mathbf{const} \in \mathbb{R}, \quad m = 1, 2. \end{gathered} \tag{4.24}$$

In the right part of (4.24), the first functional item is the partial solution of (4.19), and the second one that is expressed as the sum, represents the general solution of the homogeneous ODE with respect to (4.19).

Since (4.17) is the particular case of (4.18), when $\Delta = 0$, and instead of Φ, f the corresponding functions F, Φ can be considered, the unknown transform $\tilde{F}$ looks like

$$\tilde{F} = \frac{1}{(\eta_1^* - \eta_2^*) + i(\xi_1^* - \xi_2^*)} (e^{\omega_1^* t} \int e^{-\omega_1^* t} \tilde{\Phi} dt - e^{\omega_2^* t} \int e^{-\omega_2^* t} \tilde{\Phi} dt), \tag{4.25}$$

where

$$\omega_{1,2}^* = -\left(\begin{bmatrix} r/\tilde{\mu}_a \\ \tilde{\sigma}/\tilde{\varepsilon}_a \end{bmatrix} \pm \lambda \right), \quad \eta_m^* = \mathrm{Re}(\omega_m^*), \quad \xi_m^* = \mathrm{Im}(\omega_m^*) \quad (m = 1, 2), \tag{4.26}$$

and $\tilde{\Phi}$ is given in (4.24).

Applying to (4.25) the appropriate inverse integral transformation, we find the original required solution of (4.16). This final result closes the given section, means the initial problem's explicit study and the purpose of the present paper's achievement.

5 Concluding remarks

Closing the present article, it should be noted once more that the explicit solution and the given method are independent of the boundary problem statement. Moreover, as far as it is known, the temporal variable t was not considered earlier as mostly important during the application of the integral transform procedure. It means that the integral transformation did not affect the spatial variables but influenced upon t. Such approach complicated the concrete boundary problem's investigation in the case of the spatial dimension whose order was bigger than one.

Thus, there is a tentative hope that the proposed here method will allow to simplify an explicit analytic solution of the applied industrial problems in technical electrodynamics and classical electromagnetic field theory not infringing the original physical phenomenon statement.

The suggested results were announced briefly in [9].

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