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DYNAMIC EQUATIONS ON TIME SCALES SEEN AS GENERALIZED DIFFERENTIAL EQUATIONS

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Abstract

In the present paper we prove, in the most natural framework, that dynamic equations on time scales can be treated as generalized differential equations. More precisely, we use the Henstock-Kurzweil vector integral and impose only a uniform integrability condition. Our result generalizes the main result of [24], where the embeddability of dynamic equations on time scales into generalized differential equations was proved under some assumptions of Lipschitz continuity-type (and consequently involving the Lebesgue integral).

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Key words: dynamic equation on time scales, generalized differential equation, Henstock-Kurzweil integral.

1 Introduction

Since the czech mathematician J. Kurzweil introduced the theory of generalized differential equations in [16], generalized differential equations were considered in many works, such as [22], [25], [9] or [18]. There are also several monographs treating this subject, e.g. [21] or [26].

Two decades after the publication of [16], the necessity of considering such a theory was once again motivated in [1]: in general, the space of ordinary differential equations is not complete but, by embedding ordinary equations in the space of generalized differential equations, we get a complete and compact space, where techniques of topological dynamics can be applied.

More recently, this theory was shown to be connected to that of impulsive differential equations (see [9], [11]), to the theory of retarded functional differential equations (as in [10]) or to that of discrete systems (e.g. [22]).

What's more, in [24] the author shows that even dynamic equations on time scale domains can be seen as generalized differential equations. The analysis on time scale domains, introduced in 1988 in the PhD Thesis of S. Hilger (see [15]), allows a unified treatment of

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continuous and discrete problems, but not exclusively (we refer to [3], [4] and the references therein).
While the Kurzweil integration is the most appropriate to the theory of generalized differential equations, the main result of [24] works with the Lebesgue-Stieltjes integral as the involved functions are supposed to satisfy some continuity conditions of Lipschitz-type.
In the present paper, we prove that dynamic equations on time scales can be treated as generalized differential equations under more natural hypotheses: of integrability in Henstock-Kurzweil sense. It is worthwhile to remind that the Henstock-Kurzweil integral (see [13], [7], [23] for functions defined on a real interval or [8] on time scales) gives us the possibility to study more general problems, taking into consideration the fact that classical theories of integration do not cover the case of highly oscillatory functions.

2 Preliminaries

We recall some basic elements from the theory of generalized Kurzweil integration. We call a gauge a positive function δ. A partition of the real interval $[a,b]$ is a finite family $([\alpha_{i-1},\alpha_i],\tau_i)_{i=1}^n$ of non-overlapping intervals covering $[a,b]$ with tags $\tau_i \in [\alpha_{i-1},\alpha_i]$; a partition is said to be δ-fine if for each $i \in \{1,...,n\}$, $[\alpha_{i-1},\alpha_i] \subset [\tau_i - \delta(\tau_i), \tau_i + \delta(\tau_i)]$.

Definition 1. *A function $F : [a,b] \times [a,b] \to \mathbb{R}^n$ is said to be generalized Kurzweil integrable if there exists a vector $\int_a^b DF(\tau,t) \in \mathbb{R}^n$ such that for every $\varepsilon > 0$ there exists a gauge $\delta_\varepsilon : [a,b] \to \mathbb{R}_+$ with the property that for every δ_ε-fine partition of $[a,b]$:*

$$\left\| \sum_{i=1}^n (F(\tau_i,\alpha_i) - F(\tau_i,\alpha_{i-1})) - \int_a^b DF(\tau,t) \right\| < \varepsilon.$$

The vector $\int_a^b DF(\tau,t)$ is called the generalized Kurzweil integral of F.

A particular case is the Henstock-Kurzweil-Stieltjes (shortly, HK-Stieltjes) integral (HKS) $\int_a^b f(s)dg(s)$, that can be obtained for the function $F(\tau,t) = f(\tau)g(t)$, where $f : [a,b] \to \mathbb{R}^n$ and $g : [a,b] \to \mathbb{R}$. Moreover, when $g(t) = t$ the preceding definition describes the Henstock-Kurzweil (HK) integral.
For more on the generalized Kurzweil integral and its importance in the theory of differential equations, we refer to [21].

We now remind of several features from the time scale theory; for a survey on this subject, see [3] or [4] and the references therein.
A time scale $\mathbb{T}$ is a nonempty closed set of real numbers with the subspace topology inherited from the topology of $\mathbb{R}$ (such as, $\mathbb{T} = \mathbb{R}, \mathbb{T} = \mathbb{N}$ or $\mathbb{T} = q^{\mathbb{Z}} = \{q^t : t \in \mathbb{Z}\}$, where $q > 1$). For two points a, b in $\mathbb{T}$ we denote by $[a,b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \le t \le b\}$ the time scales interval.

Definition 2. *The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ are defined by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$, respectively $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$. Also, $\inf \emptyset = \sup \mathbb{T}$ (i.e. $\sigma(M) = M$ if $\mathbb{T}$ has a maximum M) and $\sup \emptyset = \inf \mathbb{T}$ (i.e. $\rho(m) = m$ if $\mathbb{T}$ has a minimum m).*

A point $t \in \mathbb{T}$ is called right-dense, right-scattered, left-dense, left-scattered, dense, respectively isolated if $\sigma(t) = t$, $\sigma(t) > t$, $\rho(t) = t$, $\rho(t) < t$, $\rho(t) = t = \sigma(t)$ and $\rho(t) < t < \sigma(t)$, respectively. Also, we will use the function $\mu(t) = \sigma(t) - t$ that is called the graininess function.

Definition 3. *Let $f : \mathbb{T} \to \mathbb{R}^n$ and $t \in \mathbb{T}$. f is called Δ-differentiable at the point t if there exists an element of $\mathbb{R}^n$ (called Δ-derivative $f^\Delta(t)$) with the property that for any $\varepsilon > 0$ there exists a neighborhood of t on which*

$$\left\| f(\sigma(t)) - f(s) - f^\Delta(t)[\sigma(t) - s] \right\| \leq \varepsilon |\sigma(t) - s|.$$

Several simple properties of Δ-derivatives were proved in [4] (Theorem 1.3):
i) f is continuous at the points where it is Δ-differentiable;
ii) if f is continuous at the right-scattered point t, then f is Δ-differentiable at t and

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)};$$

iii) if t is right-dense, then f is Δ-differentiable at t if and only if the limit

$$\lim_{s \to t, s > t} \frac{f(s) - f(t)}{s - t}$$

exists and is finite. In this case, its value equals to $f^\Delta(t)$.

Remark that the time scale calculus gives the possibility to unify (and generalize) the treatment of differential and difference equations since, in particular,

(i) $f^\Delta = f'$ is the usual derivative if $\mathbb{T} = \mathbb{R}$,

(ii) $f^\Delta = \Delta f$ is the usual forward difference operator if $\mathbb{T} = \mathbb{Z}$.

The space $C([a, b]_\mathbb{T}, \mathbb{R}^n)$ of continuous functions is endowed with the usual (Banach space) norm $\|f\|_C = \sup_{t \in [a,b]_\mathbb{T}} \|f(t)\|$.
The symbol μ_Δ stands for the Lebesgue measure on $\mathbb{T}$ (for its definition and properties we refer the reader to [6]). For properties of Riemann delta-integral we refer to [14] and for Lebesgue integral on time scales to [2], [3], [4] or [14].
In order to recall the Henstock-Kurzweil-Δ-integral, let $\delta = (\delta_L, \delta_R)$ be a Δ-gauge, that is a pair of positive functions such that $\delta_L(t) > 0$ on $(a, b]$, $\delta_R(t) > 0$ and $\delta_R(t) \geq \sigma(t) - t$ on $[a, b)$. A partition $\mathcal{D} = \{[\alpha_{i-1}, \alpha_i]_\mathbb{T}; \tau_i\ , i = 1, 2, \ldots n\}$ of $[a, b]_\mathbb{T}$ is δ-fine whenever:

$$\tau_i \in [\alpha_{i-1}, \alpha_i] \subset [\tau_i - \delta_L(\tau_i), \tau_i + \delta_R(\tau_i)], \forall 1 \leq i \leq n$$

(such a partition exists for arbitrary positive pair of functions, see Lemma 1.9 in [17]).

Definition 4. ([8], see also [7], [23] for the particular case $\mathbb{T} = \mathbb{R}$)
i) A function $f : [a, b]_\mathbb{T} \to \mathbb{R}^n$ is Henstock-Kurzweil-Δ-integrable on $[a, b]_\mathbb{T}$ if there exists

an element $(HK)\int_a^b f(s)\Delta s \in \mathbb{R}^n$ *satisfying the following property: given* $\varepsilon > 0$*, there exists a* Δ*-gauge* δ_ε *on* $[a,b]_{\mathbb{T}}$ *such that*

$$\left\| \sum_{i=1}^{n} f(\tau_i)\mu_\Delta([\alpha_{i-1},\alpha_i]_{\mathbb{T}}) - (HK)\int_a^b f(s)\Delta s \right\| < \varepsilon$$

for every δ_ε*-fine division* $\mathcal{D} = \{[\alpha_{i-1},\alpha_i]_{\mathbb{T}}, \tau_i\}$ *of* $[a,b]_{\mathbb{T}}$*. We call it the Henstock-Kurzweil-*Δ*-integral of* f *on* $[a,b]_{\mathbb{T}}$*.*

On the other hand, a family of Henstock-Kurzweil-Δ-integrable functions is said to be uniformly HK-Δ-integrable if the Δ-gauge δ_ε can be chosen to be the same for all elements of the family.
The space of HK-Δ-integrable $\mathbb{R}^n$-valued functions will be denoted by $\mathcal{HK}([a,b]_{\mathbb{T}},\mathbb{R}^n)$ and we provide it with the Alexiewicz norm:

$$\|f\|_A = \sup_{t\in[a,b]_{\mathbb{T}}} \left\| (\text{HK})\int_a^t f(s)\Delta s \right\|.$$

When interested in differential equations, of a great importance are the properties of the primitives which allow to transfer the differential problem into an integral one. In this direction, it was proved (in [7], see also [23]) that in the particular case where $\mathbb{T} = \mathbb{R}$ the primitive in Henstock-Kurzweil sense (HK) $\int_0^{\cdot} f(s)ds$ is continuous and a.e. differentiable. In order to present a similar result on time scales (as it was done in [20]), we refer to [6], where the integrability of a function on time scales is shown to be equivalent to the integrability of its extension (defined below) to a real interval. Thus, if the time scale $\mathbb{T}$ is contained in a real interval $[a,b]$, then a function $f : \mathbb{T} \to \mathbb{R}^n$ is integrable if and only if the function $f^* : \mathbb{T}^* = [a,b] \to \mathbb{R}^n$ given by $f^*(t) = f(t^*)$, where $t^* = \inf\{s \in \mathbb{T},\ s \geq t\}$, is integrable. In fact,

$$f^*(t) = \begin{cases} f(t), \text{ if } t \in \mathbb{T}; \\ f(t_i), \text{ if } t \in (t_i, \sigma(t_i)) \text{ for } t_i \in R_{\mathbb{T}}. \end{cases}$$

(here the set $R_{\mathbb{T}}$ is the set of all right-scattered points that is, by Lemma 3.1 in [6], at most countable). By the same method as in Proposition 2.19 in [12], the next result can be proved:

Proposition 1. *Let* $g : [a,b]_{\mathbb{T}} \to \mathbb{R}^n$ *be HK-*Δ*-integrable. Then its primitive*

$$G(t) = (\text{HK})\int_a^t g(s)\Delta s$$

is Δ*-a.e. differentiable and* $G^\Delta = g$*,* Δ*-a.e.*

The following convergence result on time scales will be used in the sequel:

Theorem 1. *(Theorem 1.10 in [20]) Let* $(g_n)_{n\in\mathbb{N}} \subset \mathcal{HK}([a,b]_{\mathbb{T}},\mathbb{R}^n)$ *be a pointwisely bounded sequence such that:*
i) $g_n(t) \to g(t)$ *for* $t \in [a,b]_{\mathbb{T}} \setminus E$*, where* $E \subset [a,b]_{\mathbb{T}}$ *a* Δ*-null measure set;*
ii) $(g_n)_n$ *is uniformly HK-*Δ*-integrable.*
Then $g \in \mathcal{HK}([a,b]_{\mathbb{T}},\mathbb{R}^n)$ *and* $\|g_n - g\|_A \to 0$*.*

The primitives of Henstock-Kurzweil integrable functions are characterized by the notion of ACG^* function, that we recall bellow:

Definition 5. ([13]) *i) A function $F : [a, b]_{\mathbb{T}} \to \mathbb{R}$ is absolutely continuous in the restricted sense (shortly, AC^*) on $E \subset [a, b]_{\mathbb{T}}$ if, for any $\varepsilon > 0$, there exists $\eta_\varepsilon > 0$ such that, whenever $\{[c_i, d_i]_{\mathbb{T}}, 1 \le i \le N\}$ is a finite collection of non-overlapping intervals that have endpoints in E and satisfy $\sum_{i=1}^{N} \mu_\Delta([c_i, d_i]_{\mathbb{T}}) < \eta_\varepsilon$, one has $\sum_{i=1}^{N} osc(F, [c_i, d_i]_{\mathbb{T}}) < \varepsilon$;*
ii) F is said to be generalized absolutely continuous in the restricted sense (shortly, ACG^*) if it is continuous and the whole interval can be written as a countable union of sets on each of which F is AC^*.

It is well known that

Proposition 2. *A function $F : [a, b]_{\mathbb{T}} \to \mathbb{R}^n$ is ACG^* if and only if it is Δ-differentiable almost everywhere, F^Δ is HK-Δ-integrable and*

$$\text{(HK)} \int_a^t F^\Delta(s)\Delta s = F(t) - F(a),\ \forall t \in [a, b]_{\mathbb{T}}.$$

Related to this, a result proved in [5] (Proposition 3.2) asserts that

Proposition 3. *If $g : [a, b] \to \mathbb{R}$ is ACG^* and $f : [a, b] \to \mathbb{R}^n$ is HK-Stieltjes integrable with respect to g, then the HK-Stieltjes primitive (HKS) $\int_a^\cdot f(s)dg(s)$ is ACG^* and its derivative equals to fg' almost everywhere.*

3 Main results

Let $\mathbb{T}$ be a bounded time scale contained in the real interval $\mathbb{T}^* = [a, b]$ and $f : \mathbb{R}^n \times \mathbb{T} \to \mathbb{R}^n$ satisfy the following hypotheses:

H1) for every regulated function $x : \mathbb{T} \to \mathbb{R}^n$, the map $f(x(\cdot), \cdot)$ is Henstock-Kurzweil-Δ-integrable.

H2) for every $R > 0$, the collection

$$\{f(x(\cdot), \cdot), x \in C(\mathbb{T}, \mathbb{R}^n), \|x\|_C \le R\}$$

is uniformly Henstock-Kurzweil-Δ-integrable.

The main result of the paper will state that, under these assumptions, the dynamic equation

$$x^\Delta(t) = f(x(t), t),\ \Delta - a.e.\ t \in \mathbb{T}$$

can be seen as a generalized differential equation. In order to prove it, we need several auxiliary properties.

Proposition 4. *Let $k : \mathbb{T} \to \mathbb{R}^n$ be a HK-Δ-integrable function. Then the function $k^* : \mathbb{T}^* \to \mathbb{R}^n$ is HK-Stieltjes integrable with respect to $g : \mathbb{T}^* \to \mathbb{R}$ defined by $g(s) = s^*$. Moreover, if we denote by*

$$F_2 : \mathbb{T}^* \to \mathbb{R}^n, \;\; F_2(t) = (\text{HKS}) \int_a^t k^*(s)dg(s)$$

and by

$$F_1 : \mathbb{T} \to \mathbb{R}^n, \;\; F_1(t) = (\text{HK}) \int_a^t k(s)\Delta s,$$

then

$$F_2 = F_1^*.$$

Proof. The HK-Stieltjes integrability of k^* with respect to g easily follows from the fact that $g(t)$ equals to t on $\mathbb{T}$ and it is constant on any interval $(t_i, \sigma(t_i))$, where $t_i \in R_{\mathbb{T}}$.
Concerning the requested equality, as in [24], it suffices to prove that $F_1 = F_2$ on $\mathbb{T}$ and that F_2 is constant on any interval $(t_i, \sigma(t_i))$, where $t_i \in R_{\mathbb{T}}$. The second assertion is easy to check since on such intervals g is constant.
As for the first one, by the properties of HK-integral on time scales, the function F_1 is Δ-a.e. differentiable and its Δ-derivative equals to k Δ-a.e. Also, by Proposition 3, F_2 is ACG^* and its derivative a.e. equals to kg'. If $t \in \mathbb{T}$ is a right-dense point then

$$g'(t) = \lim_{h \to 0, h > 0} \frac{g(t+h) - g(t)}{h} = \lim_{h \to 0, h > 0} \frac{t + h - t}{h} = 1$$

and so, $F_2^{\Delta}(t) = k(t)$. If t is right-scattered then in the same way as in the proof of Theorem 5 in [24], $F_2^{\Delta}(t) = k(t)$. So, $F_1^{\Delta}(t) = F_2^{\Delta}(t)$ Δ-a.e. This and the fact that $F_1(a) = F_2(a) = 0$ imply, thanks to Proposition 2, that

$$F_1(t) = F_2(t), \; \forall \, t \in \mathbb{T}$$

and so, the equality is proved. □

Lemma 1. *Under the hypotheses H1), H2), the function $F : \mathbb{R}^n \times [a, b] \to \mathbb{R}^n$ given by*

$$F(x, t) = (\text{HKS}) \int_a^t f^*(x, s)dg(s)$$

has the property that for every regulated function $x : [a, b] \to \mathbb{R}^n$,

$$\int_a^b DF(x(\tau), t) = (\text{HKS}) \int_a^b f^*(x(s), s)dg(s).$$

Proof. Case I. The function x is constant: $x(t) = c$, for every $t \in [a, b]$.
The left-hand side of the requested equality is defined as follows: take a partition of

the interval $[a,b]$, a system of intermediary points and take then the limit of the sum $\sum_{j=1}^{n}(F(\tau_j,\alpha_j)-F(\tau_j,\alpha_{j-1}))$. In our situation,

$$
\begin{aligned}
&\sum_{j=1}^{n}\left(F\left(\tau_j,\alpha_j\right)-F\left(\tau_j,\alpha_{j-1}\right)\right)\\
= &\sum_{j=1}^{n}\left((\mathrm{HKS})\int_a^{\alpha_j} f^*(c,s)dg(s)-(\mathrm{HKS})\int_a^{\alpha_{j-1}} f^*(c,s)dg(s)\right)\\
= &\sum_{j=1}^{n}(\mathrm{HKS})\int_{\alpha_{j-1}}^{\alpha_j} f^*(c,s)dg(s)\\
= &(\mathrm{HKS})\int_a^b f^*(x(s),s)dg(s).
\end{aligned}
$$

Case II. x is a step function. Then there exists a partition of the whole interval such that, on each interval of the partition, x is constant. On each such interval the two integrals are, following the previous discussion, equal and, by the additivity of the generalized integral and of HK-Stieltjes integral, we get the requested equality.
Case III. x is regulated. It is known that any regulated function is a uniform limit of step functions, so one can find a sequence $(x_n)_n$ of step functions uniformly convergent to x. Obviously, x is bounded and we are able to choose $R>0$ such that $\max\{\|x_n\|_C,\|x\|_C,n\in\mathbb{N}\}\le R$. So, the sequence $(x_n)_n$ satisfies the hypotheses of Theorem 1 (where the set E is the empty set), whence

$$(\mathrm{HK})\int_a^{\cdot} f(x_n(s),s)\Delta s\to(\mathrm{HK})\int_a^{\cdot} f(x(s),s)\Delta s$$

uniformly on $[a,b]_{\mathbb{T}}$. This, together with Proposition 4, gives that

$$(\mathrm{HKS})\int_a^b f^*(x(t),t)dg(t)=\lim_{n\to\infty}(\mathrm{HKS})\int_a^b f^*(x_n(t),t)dg(t).$$

On the other hand, by a convergence result for generalized Kurzweil integral (e.g. Lemma A.7 in [1]),

$$\int_a^b DF(x(\tau),t)=\lim_{n\to\infty}\int_a^b DF(x_n(\tau),t).$$

As a consequence of Case II, we obtain that

$$\int_a^b DF(x(\tau),t)=(\mathrm{HKS})\int_a^b f^*(x(s),s)dg(s).$$

□

The notion of solution for a generalized differential equation is presented in the sequel ([24]):

Definition 6. *Let $F : \mathbb{R}^n \times [a,b] \to \mathbb{R}^n$. A function $x : [a,b] \to \mathbb{R}^n$ is a solution of the generalized differential equation*

$$\frac{dx}{d\tau} = DF(x,t)$$

if for any $a \leq t_1 \leq t_2 \leq b$:

$$x(t_2) - x(t_1) = \int_{t_1}^{t_2} DF(x(\tau),t).$$

We proceed now to give the main result of the paper.

Theorem 2. *Let $f : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfy the hypotheses H1) and H2).*
i) If $x : \mathbb{T} \to \mathbb{R}^n$ is a solution of dynamic equation

$$x^{\Delta}(t) = f(x(t),t),\ \Delta - a.e.\ t \in \mathbb{T}$$

then $x^* : \mathbb{T}^* \to \mathbb{R}^n$ is a solution of the generalized differential equation

$$\frac{dx}{d\tau} = DF(x,t), t \in \mathbb{T}^*,$$

where $F : \mathbb{R}^n \times \mathbb{T}^* \to \mathbb{R}^n$ is defined by

$$F(x,t) = (\text{HKS}) \int_a^t f^*(x,s)dg(s).$$

ii) Reciprocally, every solution $y : \mathbb{T}^* \to \mathbb{R}^n$ of the above generalized differential equation is of the form $y = x^*$, where $x : \mathbb{T} \to \mathbb{R}^n$ is a solution of the preceding dynamic equation.

Proof. i) Let $x : \mathbb{T} \to \mathbb{R}^n$ be a solution of dynamic equation

$$x^{\Delta}(t) = f(x(t),t), \Delta - a.e.\ t \in \mathbb{T}.$$

Then

$$x(t) = x(a) + (\text{HK}) \int_a^t f(x(s),s)\Delta s,\ \forall t \in \mathbb{T},$$

whence

$$x(t^*) = x(a) + (\text{HK}) \int_a^{t^*} f(x(s),s)\Delta s$$

and, by Proposition 4, we get

$$x^*(t) = x^*(a) + (\text{HKS}) \int_a^t f^*(x^*(s),s)dg(s),\ \forall t \in \mathbb{T}^*.$$

Using Lemma 1 we can rewrite this equality as

$$x^*(t) = x^*(a) + \int_a^t DF(x^*(\tau),s)$$

ad so, x^* is a solution of this generalized differential equation.
ii) Let now $y : \mathbb{T}^* \to \mathbb{R}^n$ be a solution of the previously mentioned generalized differential equation. Then

$$y(t) = y(a) + \int_a^t DF(y(\tau), s),\ \forall t \in \mathbb{T}^*.$$

In the same way as in the proof of Theorem 12 in [24] it follows that y is regulated and so Lemma 1 can be applied in order to get

$$y(t) = y(a) + (\text{HKS}) \int_a^t f^*(y(s), s)dg(s),\ \forall t \in \mathbb{T}^*.$$

Since the function g is constant on any interval $(t_i, \sigma(t_i))$, where $t_i \in R_{\mathbb{T}}$, the function y is constant on any such interval and thus denoting by $x : \mathbb{T} \to \mathbb{R}^n$ the restriction of y to $\mathbb{T}$ we have $y = x^*$. Consequently

$$x^*(t) = x^*(a) + (\text{HKS}) \int_a^t f^*(x^*(s), s)dg(s),\ \forall t \in \mathbb{T}^*,$$

therefore

$$x(t) = x(a) + (\text{HK}) \int_a^t f(x(s), s)\Delta s,\ \forall t \in \mathbb{T},$$

and, finally,

$$x^{\Delta}(t) = f(x(t), t),\ \Delta - a.e.\ t \in \mathbb{T}.$$

□

Remark 1. *Our theorem could be used to obtain new results for dynamic equations on time scales applying some known results from the theory of generalized differential equations (as in [24], where stability properties were obtained).*

Let us finally notice that the same connection was proved in [24] under more restrictive assumptions, namely:

(C1) for every continuous $x : \mathbb{T} \to \mathbb{R}^n$, $t \to f(x(t), t)$ is rd-continuous, i.e. it is continuous at right-dense points and its left-sided limits exist (and it is finite) at all left-dense points;

(C2) there exists a regulated function $m : \mathbb{T} \to \mathbb{R}_+$ such that $\|f(x, t)\| \leq m(t)$ for all $x \in \mathbb{R}^n$ and $t \in \mathbb{T}$;

(C3) there exists a regulated function $l : \mathbb{T} \to \mathbb{R}_+$ and a continuous increasing function $\omega : [0, \infty) \to \mathbb{R}_+$ such that $\omega(0) = 0$ and

$$\|f(x, t) - f(y, t)\| \leq l(t)\omega(\|x - y\|),\ \forall x, y \in \mathbb{R}^n,\ t \in \mathbb{T}.$$

In particular, hypotheses (C1)-(C3) imply that the involved integral is the Lebesgue integral (more precisely, Lebesgue-Δ-integral, resp. Lebesgue-Stieltjes integral).

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