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PROLONGATION OF HERMITIAN STRUCTURES TO $J^{(2,0)}M$ HOLOMORPHIC JETS BUNDLE

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Abstract

In the previous papers [14, 15], we made a study of the connections in the $J^{(2,0)}M$ jet bundle.

In the present paper some results are applied in order to obtain adapted frames on $J^{(2,0)}M$ for the prolongation of the Hermitian and complex Finsler structures on the base complex manifold M. Such prolongations are not in unique ways, but if (M, J_M, g) is a Kähler manifold, then there exists an unique nonlinear complex connection such that $(J^{(2,0)}M\,, J, G)$ is the prolongation of the (M, J, g).

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1 Introduction

Let M be a complex manifold, $dim_{\mathbf{C}}M = n$, and (z^i) be complex coordinates in a local chart. The complexified tangent bundle $T_{\mathbf{C}}M$ admits the classical decomposition $T_{\mathbf{C}}M = T'M \oplus T''M$, where $T'M$ is the holomorphic vector bundle over M and its conjugate $T''M$ is the anti-holomorphic tangent bundle. In any point T'_zM is spanned by $\{\frac{\partial}{\partial z^i}\}$ and T''_zM by its conjugate $\{\frac{\partial}{\partial \bar{z}^i}\}$.The complex manifold M has a natural complex structure $J_M : \chi(M) \to \chi(M)$, $J_M^2 = -I$, whose action can be extended to the sections of $T_{\mathbf{C}}M$ by $J_M(\frac{\partial}{\partial z^k}) = i\frac{\partial}{\partial z^k}$ and $J_M(\frac{\partial}{\partial \bar{z}^k}) = -i\frac{\partial}{\partial \bar{z}^k}$, where $i = \sqrt{-1}$.

The holomorphic bundle of $k-$th *order jets differential* was introduced by Green and Griffiths in [6] as the sheaf of germs of holomorphic curves $\{f : \Delta_r \to M,\ f \in \mathcal{H}_{z_0},\ f(0) = z_0\}$ depending on a complex parameter θ.

By denoting $f^i = z^i \circ f$, $\forall i = \overline{1,n}$, $f \in \mathcal{H}_{z_0}$, according to [12], $f, g \in \mathcal{H}_{z_0}$ are said to *be* $k-$*equivalent*, $f \overset{k}{\sim} g$, iff $f^i(0) = g^i(0)$ and $\dfrac{d^p f^i}{d\theta^p}(0) = \dfrac{d^p g^i}{d\theta^p}(0)$, $\forall i = \overline{1,n},\ p = \overline{1,k}$. The class of f is $[f]_{\underset{\sim}{k}}$ and the set of all classes is $J^{(k,0)}M = \cup_{z_0\in M}\mathcal{H}_{z_0}/_{\underset{\sim}{k}}$. By $j^k f(0) = \Big(f(0), \frac{df}{d\theta}(0), ..., \frac{d^p f}{d\theta^p}(0)\Big)$ we denote the $k-$jet of $f \in [f]_{\underset{\sim}{k}}$.

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Let $\pi : J^{(k,0)}M \to M$ be the canonical projection. Then we check immediately that $\left(J^{(k,0)}M, \pi\right)$ has a fibre bundle structure which is one holomorphic, named in [12] the restricted $k-$jet bundle.

Further on we call $J^{(k,0)}M$ simply the *holomorphic k-jets bundle*. Note that $J^{(k,0)}M$ does not have a vector bundle structure, aside from $k = 1$, when it is identified with $T'M$, the holomorphic tangent bundle.

For the sake of simplicity, in case $k = 2$ we have made recently an intensive approach of this holomorphic jets bundle, [13, 15, 14].

Further on, in this paper we will resume our study to the second order jets manifold $J^{(2,0)}M$. In a local chart, the coordinates are denoted by $u = (z^i, \eta^i, \zeta^i)$, $i = \overline{1, n}$, and at changes of local charts on M will transform as follows:

$$\begin{aligned} z'^i &= z'^i(z); \\ \eta'^i &= \frac{\partial z'^i}{\partial z^j}\eta^j; \\ 2\zeta'^i &= \frac{\partial \eta'^i}{\partial z^j}\eta^j + 2\frac{\partial \eta'^i}{\partial \eta^j}\zeta^j \end{aligned} \tag{1.1}$$

and that $\frac{\partial z'^i}{\partial z^j} = \frac{\partial \eta'^i}{\partial \eta^j} = \frac{\partial \zeta'^i}{\partial \zeta^j}$; $\frac{\partial \eta'^i}{\partial z^j} = \frac{\partial \zeta'^i}{\partial \eta^j}$. A local base in the holomorphic bundle $T'(J^{(2,0)}M)$ is $\left\{\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \eta^i}, \frac{\partial}{\partial \zeta^i}\right\}$ and in $T''(J^{(2,0)}M)$ is obtained by conjugation. The changes of the local basis are made according to the following rules:

$$\begin{aligned} \frac{\partial}{\partial z^j} &= \frac{\partial z'^i}{\partial z^j}\frac{\partial}{\partial z'^i} + \frac{\partial \eta'^i}{\partial z^j}\frac{\partial}{\partial \eta'^i} + \frac{\partial \zeta'^i}{\partial z^j}\frac{\partial}{\partial \zeta'^i}; \\ \frac{\partial}{\partial \eta^j} &= \frac{\partial \eta'^i}{\partial \eta^j}\frac{\partial}{\partial \eta'^i} + \frac{\partial \zeta'^i}{\partial \eta^j}\frac{\partial}{\partial \zeta'^i}; \\ \frac{\partial}{\partial \zeta^j} &= \frac{\partial \zeta'^i}{\partial \zeta^j}\frac{\partial}{\partial \zeta'^i} \end{aligned} \tag{1.2}$$

and similarly for the conjugate basis that corresponds in $T_z''(J^{(2,0)}M)$.

Two structures play a special role in defining the linear and nonlinear connection on $J^{(2,0)}M$:

- the natural complex structure J , which acts on the sections of $T_C(J^{(2,0)}M)$ by $J(\frac{\partial}{\partial z^k}) = i\frac{\partial}{\partial z^k}$; $J(\frac{\partial}{\partial \eta^k}) = i\frac{\partial}{\partial \eta^k}$; $J(\frac{\partial}{\partial \zeta^k}) = i\frac{\partial}{\partial \zeta^k}$ and $J(\frac{\partial}{\partial \bar{z}^k}) = -i\frac{\partial}{\partial \bar{z}^k}$; $J(\frac{\partial}{\partial \bar{\eta}^k}) = -i\frac{\partial}{\partial \bar{\eta}^k}$; $J(\frac{\partial}{\partial \bar{\zeta}^k}) = -i\frac{\partial}{\partial \bar{\zeta}^k}$. We note that the sections of $T_C(J^{(2,0)}M)$ project to the sections of $T_C M$ and so, we can see the complex structure J as an extension of J_M structure.

- the almost second order tangent structure F, (see [13]), defined locally by $F(\frac{\partial}{\partial z^k}) = \frac{\partial}{\partial \eta^k}$; $F(\frac{\partial}{\partial \eta^k}) = \frac{\partial}{\partial \zeta^k}$; $F(\frac{\partial}{\partial \zeta^k}) = 0$ and $F(\bar{X}) = \overline{F(X)}$.

With $V(J^{(2,0)}M)$ we denote the vertical bundle, locally spanned by $\left\{\frac{\partial}{\partial \zeta^j}\right\}$, and denote by $W_u(J^{(2,0)}M)$ the distribution spanned by $\left\{\frac{\partial}{\partial \eta^j}, \frac{\partial}{\partial \zeta^j}\right\}$ in a local chart. A complex nonlinear connection, (c.n.c.) in brief, is given by the subbundle $H(J^{(2,0)}M)$ which is supplementary to $W(J^{(2,0)}M)$ in $T'(J^{(2,0)}M)$. A local base in $H_u(J^{(2,0)}M)$ is called adapted base of the (c.n.c.), and it is written as $\frac{\delta}{\delta z^j} = \frac{\partial}{\partial z^j} - \overset{(1)}{N}{}^i_j \frac{\partial}{\partial \eta^i} - \overset{(2)}{N}{}^i_j \frac{\partial}{\partial \zeta^i}$, iff $\frac{\delta}{\delta z^j} = \frac{\partial z'^i}{\partial z^j} \frac{\delta}{\delta z'^i}$. Then $F(\frac{\delta}{\delta z^j}) =: \frac{\delta}{\delta \eta^j} = \frac{\partial}{\partial \eta^j} - \overset{(1)}{N}{}^i_j \frac{\partial}{\partial \zeta^i}$ span a local adapted base in $W_u(J^{(2,0)}M)$. By conjugation, we obtain the decomposition for $T_C(J^{(2,0)}M)$. The changes (1.1) of coordinates on $J^{(2,0)}M$ produce the changes of the coefficients $\overset{(1)}{N}{}^i_j$ and $\overset{(2)}{N}{}^i_j$ of the (c.n.c.) in the form:

$$\begin{aligned} \overset{(1)}{N'}{}^i_k \frac{\partial z'^k}{\partial z^j} &= \frac{\partial z'^i}{\partial z^k} \overset{(1)}{N}{}^k_j - \frac{\partial \eta'^i}{\partial z^j}; \\ \overset{(2)}{N'}{}^i_k \frac{\partial z'^k}{\partial z^j} &= \frac{\partial z'^i}{\partial z^k} \overset{(2)}{N}{}^k_j + \frac{\partial \eta'^i}{\partial z^k} \overset{(1)}{N}{}^k_j - \frac{\partial \zeta'^i}{\partial z^j}. \end{aligned} \tag{1.3}$$

The adapted basis will change as follows: $\frac{\delta}{\delta z^j} = \frac{\partial z'^i}{\partial z^j} \frac{\delta}{\delta z'^i}$ and $\frac{\delta}{\delta \eta^j} = \frac{\partial z'^i}{\partial z^j} \frac{\delta}{\delta \eta'^i}$. Obviously, $\frac{\delta}{\delta \zeta^j} = \frac{\partial z'^i}{\partial z^j} \frac{\delta}{\delta \zeta'^i}$ and so these fields are changing as those on the base manifold M. Generally, the geometrical objects which are changed by $\frac{\partial z'^i}{\partial z^j}$ or by their conjugates $\frac{\partial \bar{z}'^i}{\partial \bar{z}^j}$, are called *d-tensor fields*. The corresponding adapted basis on $T''(J^{(2,0)}M)$ are obtained by conjugation everywhere. The relation between the dual cobasis $\{dz^i, \delta\eta^i = d\eta^i + \overset{(1)}{M}{}^i_j dz^j, \delta\zeta^i = d\zeta^i + \overset{(1)}{M}{}^i_j d\eta^j + \overset{(2)}{M}{}^i_j dz^j\}$ and the adapted basis is given by the rules:

$$\overset{(1)}{M}{}^i_j = \overset{(1)}{N}{}^i_j \;\; ; \quad \overset{(2)}{M}{}^i_j = \overset{(2)}{N}{}^i_j + \overset{(1)}{N}{}^i_k \overset{(1)}{N}{}^k_j,$$

where $\overset{(1)}{M}{}^i_j$ and $\overset{(2)}{M}{}^i_j$ are changing by the following rules (see [13]):

$$\begin{aligned} \frac{\partial z'^i}{\partial z^k} \overset{(1)}{M}{}^k_j &= \overset{(1)}{M'}{}^i_k \frac{\partial z'^k}{\partial z^j} + \frac{\partial \eta'^i}{\partial z^j}; \\ \frac{\partial z'^i}{\partial z^k} \overset{(2)}{M}{}^k_j &= \overset{(2)}{M'}{}^i_k \frac{\partial z'^k}{\partial z^j} + \overset{(1)}{M'}{}^i_k \frac{\partial \eta'^k}{\partial z^j} + \frac{\partial \zeta'^i}{\partial z^j}. \end{aligned} \tag{1.4}$$

The notion of complex nonlinear connection is connected with the *complex spray* notion, which is defined as a field $S \in T'(J^{(2,0)}M)$ with property $F \circ S = \overset{(2)}{\mathcal{L}}$, where $\overset{(2)}{\mathcal{L}} = \eta^i \frac{\partial}{\partial \eta^i} + 2\zeta^i \frac{\partial}{\partial \zeta^i}$ is the Liouville field. The spray S has the coefficients G^i, thus $S = \eta^i \frac{\partial}{\partial z^i} + 2\zeta^i \frac{\partial}{\partial \eta^i} - 3G^i(z, \eta, \zeta) \frac{\partial}{\partial \zeta^i}$, and they are transformed by the rule:

$$3G'^i = 3\frac{\partial z'^i}{\partial z^j} G^j - (\eta^j \frac{\partial \zeta'^i}{\partial z^j} + 2\zeta^j \frac{\partial \zeta'^i}{\partial \eta^j}). \tag{1.5}$$

In short, a normal complex nonlinear connection, *N-(c.l.c.)*, is a derivative law on $T_{\mathbf{C}}(J^{(2,0)}M)$ with respect to adapted frames, which preserves the distributions and is well defined by the set of coefficients $D\Gamma = (L^i_{jk}, L^{\bar{i}}_{\bar{j}k}, F^i_{jk}, F^{\bar{i}}_{\bar{j}k}, C^i_{jk}, C^{\bar{i}}_{\bar{j}k})$ which are changing as follows:

$$L'^i_{jk} = \frac{\partial z'^i}{\partial z^r} \frac{\partial z^p}{\partial z'^j} \frac{\partial z^q}{\partial z'^k} L^r_{pq} + \frac{\partial z'^i}{\partial z^p} \frac{\partial^2 z^p}{\partial z'^j \partial z'^k} \tag{1.6}$$

and the others are d -tensors. For details see [13].

In [15] we have proved the following results.

Proposition 1.1. *If* $\overset{(1)}{M}{}^i_j$ *and* $\overset{(2)}{M}{}^i_j$ *are the dual coefficients of a (c.n.c.) on* $J^{(2,0)}M$*, then a complex spray is given by:*

$$3G^i = \overset{(2)}{M}{}^i_j \eta^j + 2\overset{(1)}{M}{}^i_j \zeta^j. \tag{1.7}$$

Conversely, any complex spray determines a (c.n.c):

Proposition 1.2. *If* S *is a complex spray with coefficients* G^i *which are changing by the rule (1.5), then*

$$\overset{(1)}{M}{}^i_j = \frac{\partial G^i}{\partial \zeta^j} \quad , \quad \overset{(2)}{M}{}^i_j = \frac{\partial G^i}{\partial \eta^j} \tag{1.8}$$

determine a (c.n.c) with dual coefficients $\overset{(1)}{M}{}^i_j$ *and* $\overset{(2)}{M}{}^i_j$.

Therefore, the problem of determining a (c.n.c.) on $J^{(2,0)}M$ is closely related with the problem of determining a complex spray. Such complex spray results on a complex Lagrange space.

Definition 1.1. *A complex second order Lagrange space is a pair* (M, L)*, where* $L : J^{(2,0)}M \to \mathbf{R}$ *is a smooth function of order at least two, where the Hermitian matrix*

$$g_{i\bar{j}} = \frac{\partial^2 L}{\partial \zeta^i \partial \bar{\zeta}^j} \tag{1.9}$$

is non-degenerated.

Theorem 1.1. *The pair* $\overset{(1)}{M}{}^i_j$, $\overset{(2)}{M}{}^i_j$ *determines the dual coefficients of a (c.n.c.), named Chern-Lagrange connection, where*

$$\overset{(1)}{M}{}^i_j = g^{\overline{m}i} \frac{\partial^2 L}{\partial \eta^j \partial \bar{\zeta}^m} \quad ; \quad \overset{(2)}{M}{}^i_j = g^{\overline{m}i} \frac{\partial^2 L}{\partial z^j \partial \bar{\zeta}^m}. \tag{1.10}$$

From Proposition 1.1 and the previous Theorem, it results:

Corollary 1.1. *Functions* $\overset{c}{G}{}^i = \frac{1}{3}(\overset{(2)}{M}{}^i_j \eta^j + 2\overset{(1)}{M}{}^i_j \zeta^j)$*, given by (1.10), define a complex spray on* $J^{(2,0)}M$*, called the canonical spray.*

Following Proposition 1.2, we can obtain a sequence of (c.n.c.). Functions $\overset{(1)c}{M}{}^i_j = \frac{\partial \overset{c}{G}{}^i}{\partial \zeta^j}$ and $\overset{(2)c}{M}{}^i_j = \frac{\partial \overset{c}{G}{}^i}{\partial \eta^j}$ will be called the coefficients of the canonical (c.n.c.).

The terminologies of the complex Chern-Lagrange nonlinear connection and the canonical one, used here, are purely formal and they were introduced by analogy with those from complex Lagrange spaces (of first order), [10].

2 Prolongation of the Hermitian metrics to $J^{(2,0)}M$

Let us consider a metric Hermitian structure $g(z)$ with respect to J_M on the base manifold M, that mince that g is bilinear on the sections of $T_C M$ and $\overline{g(X,\overline{Y})} = g(Y,\bar{X})$. Locally, if we consider the chart (U, z^k), then $g = g_{i\bar{j}}(z)dz^i \otimes d\bar{z}^j$ is well defined by $g_{i\bar{j}}(z) := g(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j})$ with $\overline{g_{i\bar{j}}} = g_{j\bar{\imath}}$. Because g defines a metric structure, the Hermitian matrix $(g_{i\bar{j}})$ has $\det(g_{i\bar{j}}) \neq 0$, with the inverse $(g^{\bar{j}i})$ such that $g_{i\bar{j}}g^{\bar{j}k} = \delta_i^k$.

The Hermitian metric tensor $g_{i\bar{j}}$ could be considered as a $d-$tensor on $J^{(2,0)}M$, still denoted by $g_{i\bar{j}}$, such that $(g_{i\bar{j}} \circ \pi)(u) = g_{i\bar{j}}(z)$, where $\pi(u) = z$.

The prolongation problem of the Hermitian metric $g(z)$ from M to $J^{(2,0)}M$ consists of determining a Hermitian metric G acting on the sections of $T_C(J^{(2,0)}M)$ such that $(G \circ \pi)(u) = g(z)$. Indeed, we have $G(JX, JY) = G(X,Y)$ for any sections of $T_C(J^{(2,0)}M)$.

This problem has same similitude to that of prolongation of Riemannian metrics to $Osc^2 M$ osculator bundle, discussed in [7, 8], which solve a difficult old problem, [9].

If we consider $g_{i\bar{j}}(u)$ a Hermitian $d-$tensor on $J^{(2,0)}M$, then the problem of the prolongation of g is rather simple if an adapted frame $\{\frac{\delta}{\delta z^j}, \frac{\delta}{\delta \eta^j}, \frac{\partial}{\partial \zeta^i}\}$, with its dual base $\{dz^i, \delta\eta^i, \delta\zeta^i\}$, could be determined. Then certainly,

$$G(u) = g_{i\bar{j}}dz^i \otimes d\bar{z}^j + g_{i\bar{j}}\delta\eta^i \otimes \delta\bar{\eta}^j + g_{i\bar{j}}\delta\zeta^i \otimes \delta\bar{\zeta}^j \tag{2.1}$$

defines a Hermitian metric on $J^{(2,0)}M$ called, by analogy to real osculator bundle, the Sasaki type lift of $g(z)$.

Indeed the adapted frame must be function only on $g(z)$ and its classical derivatives.

One, like for real $Osc^2 M$, let us consider the first Cristoffel symbol of the metric $g(z)$,

$$\begin{aligned} \Gamma^i_{jk} \ &: = \frac{1}{2}g^{\bar{m}i}\{\frac{\partial g_{k\bar{m}}}{\partial z^j} + \frac{\partial g_{j\bar{m}}}{\partial z^k}\} = \Gamma^i_{kj}; \\ \Gamma^i_{j\bar{k}} \ &: = \frac{1}{2}g^{\bar{m}i}\{\frac{\partial g_{j\bar{m}}}{\partial \bar{z}^k} - \frac{\partial g_{j\bar{k}}}{\partial \bar{z}^m}\} \ ; \ \Gamma^i_{\bar{j}k} := \frac{1}{2}g^{\bar{m}i}\{\frac{\partial g_{k\bar{m}}}{\partial \bar{z}^j} - \frac{\partial g_{k\bar{j}}}{\partial \bar{z}^m}\}, \end{aligned} \tag{2.2}$$

and their conjugates, because the Hermitian structure is integrable. Coefficients Γ^i_{jk} transform by the rule $\Gamma'^i_{rs}\frac{\partial z'^r}{\partial z^j}\frac{\partial z'^s}{\partial z^k} = \Gamma^r_{jk}\frac{\partial z'^i}{\partial z^r} - \frac{\partial^2 z'^i}{\partial z^j \partial z^k}$ and the others are d -tensors. Moreover, in the Kähler case, we have $\Gamma^i_{jk} = g^{\bar{m}i}\frac{\partial g_{k\bar{m}}}{\partial z^j}$ and $\Gamma^i_{j\bar{k}} = \Gamma^i_{\bar{j}k} = 0$.

Now let us consider the following transformation of coordinates:

$$(z^i, \eta^i, \zeta^i) \to (z^i, \eta^i, \xi^i := \zeta^i + \frac{1}{2}\Gamma^i_{jk}\eta^j\eta^k). \tag{2.3}$$

Taking into account (2.2) and (1.1), (1.2) is a straightforward computation to prove that $\xi'^i = \frac{\partial z'^i}{\partial z^j}\xi^j$, that is ξ^i are the components of one $d-$tensor on $J^{(2,0)}M$.

Let us note that by the same computation, if instead of $\Gamma^i_{jk}(z)$, $\Gamma^i_{jk}(z,\eta)$ is considered, whith the same rule of change as above (which actually is the same with that of L^i_{jk} from (1.6)), then ξ^i remain the components of one $d-$tensor on $J^{(2,0)}M$.

Now let us consider

$$\mathcal{L} = g_{i\bar{j}}(z)\xi^i\bar{\xi}^j \tag{2.4}$$

a Lagrangian function which performs $\mathcal{L}' = g'_{i\bar{j}}(z)\xi'^i\bar{\xi}'^j = g_{i\bar{j}}(z)\xi^i\bar{\xi}^j = \mathcal{L}$ and the metric tensor defined by $\mathcal{L}$ is, indeed owing (2.3), $\frac{\partial^2\mathcal{L}}{\partial\zeta^i\partial\bar{\zeta}^j} = g_{ij}(z)$ which is invertible.

For the Lagrangian function $\mathcal{L}$ from (2.4) we can apply the result from Theorem 1.1 and Corollary 1.1. By (1.10), it follows.

Theorem 2.1. *The associated Chern-Lagrange complex nonlinear connection to the Lagrangian function (2.4) has the dual coefficients:*

$$\overset{1}{M}{}^i_j = \Gamma^i_{js}\eta^s \quad and \quad \overset{2}{M}{}^i_j = \frac{1}{2}\frac{\partial\Gamma^i_{rs}}{\partial z^j}\eta^r\eta^s + g^{\bar{m}i}\frac{\partial g_{k\bar{m}}}{\partial z^j}\xi^k. \tag{2.5}$$

In adapted frame of this Chern-Lagrange (c.n.c) the Hermitiam structure $(J^{(2,0)}M\ , J, G)$ *is the prolongation of the* (M, J, g) *Hermitian structure.*

As we say in the preview section, the Chern-Lagrange (c.n.c) determines a spray $\overset{c}{G}{}^i = \frac{1}{3}(\overset{(2)}{M}{}^i_j\eta^j + 2\overset{(1)}{M}{}^i_j\zeta^j)$, called canonical and further in its turn determines a (c.n.c) with coefficients $\overset{(1)c}{M}{}^i_j = \frac{\partial \overset{c}{G}{}^i}{\partial\zeta^j}$ and $\overset{(2)c}{M}{}^i_j = \frac{\partial \overset{c}{G}{}^i}{\partial\eta^j}$, like we described in Corollary 1.1. One interesting matter is to determine circumstances in which these (c.n.c) coincide and then the sequences of (c.n.c) generated by the same algorithm will be stopped. The answer is a consequence of the following result which is related to the homogeneity of the sprays, [14].

This condition of homogeneity spray is translated in the following requirements to the coefficients of the (c.n.c.):

$$\frac{\partial \overset{(1)}{M}{}^i_j}{\partial\eta^k}\eta^k + 2\frac{\partial \overset{(1)}{M}{}^i_j}{\partial\zeta^k}\zeta^k = \overset{(1)}{M}{}^i_j \text{ and } \frac{\partial \overset{(2)}{M}{}^i_j}{\partial\eta^k}\eta^k + 2\frac{\partial \overset{(2)}{M}{}^i_j}{\partial\zeta^k}\zeta^k = 2\overset{(2)}{M}{}^i_j. \tag{2.6}$$

Now, a direct computation in (2.6) leads to:

Proposition 2.1. *The Chern-Lagrange (c.n.c) and the canonical (c.n.c.) coincides if and only if*

$$\begin{aligned} g^{\bar{m}i}\frac{\partial g_{k\bar{m}}}{\partial z^j}\eta^k &= \Gamma^i_{jk}\eta^k \qquad (2.7)\\ R^i_{jks}\eta^k\eta^s \quad : &= \left(\frac{\partial\Gamma^i_{ks}}{\partial z^j} - \frac{\partial\Gamma^i_{js}}{\partial z^k} + \Gamma^i_{jh}\Gamma^h_{ks} - \Gamma^i_{kh}\Gamma^h_{js}\right)\eta^k\eta^s = 0. \end{aligned}$$

We note that, if the space (M, g) is one Kähler then both conditions (2.7) are identically satisfied. Thus, we have.

Corollary 2.1. *If* (M, J_M, g) *is a Kähler manifold, then there exists only one nonlinear complex connection such that* $(J^{(2,0)}M\ , J, G)$ *is the prolongation of the* (M, J, g)*, where* G *is given by (2.1).*

The unicity of the prolongation results from the unicity of Chern-Lagrange (c.n.c.).

With this assertions, the problem of prolongation of the Hermitian, in particular of Kählerian structures, is solved. In adapted base and cobase of (2.5) nonlinear connection we consider the prolongation $g(z)$ of the metric Hermitian structure, which defines a $d-$tensor on $J^{(2,0)}M$, and further the geometry of $J^{(2,0)}M$ manifold follow the same type as in [14] (the $N-$liner connection, curvatures, torsions, geodesics, etc.).

Other interesting issue, somewhat connected to the above, is the *prolongation of complex Finslerian and Lagrangian structures to the* $J^{(2,0)}M$ *manifold.*

Recall from [1, 10] that a *complex Finsler structure* is a pair (M,F), where $F: T'M \to \mathbb{R}^+$ is a continuous function satisfying the conditions:

i) $L := F^2$ is smooth on $\widetilde{T'M} := T'M \backslash \{0\}$;

ii) $F(z,\eta) \geq 0$, the equality holds if and only if $\eta = 0$;

iii) $F(z,\lambda\eta) = |\lambda| F(z,\eta)$ for $\forall \lambda \in \mathbb{C}$;

iv) the Hermitian matrix $\left(g_{j\bar{k}}(z,\eta)\right)_{j,k=\overline{1,n}}$ (i.e. $\overline{g_{j\bar{k}}} = g_{k\bar{j}}$) is positively defined, where $g_{j\bar{k}} := \frac{\partial^2 L}{\partial\eta^j \partial\bar{\eta}^k}$ and $L := F^2$.

Then, $g_{j\bar{k}}$ is called the *fundamental metric tensor* of the complex Finsler space.

If the requirement *ii)* is neglected and *iv)* is replaced with nondegenerate of the metric tensor, i.e. $\det\left(g_{j\bar{k}}(z,\eta)\right) \neq 0$, then a more general study is obtained, that of *complex Lagrange spaces*, [10].

Further, we will adopt the terminology from [10]. Let us consider the Chern-Lagrange, particularly Finsler, (c.nc.) and then the Chern-Lagrange $N-$linear connection given by

$$N_j^i = g^{\overline{m}i} \frac{\partial g_{l\overline{m}}}{\partial z^j} \eta^l \ ; \ L_{jk}^i = g^{\bar{m}i} \frac{\delta g_{j\overline{m}}}{\delta z^j} \ \ ; \ C_{jk}^i = g^{\bar{m}i} \frac{\partial g_{j\overline{m}}}{\partial \eta^j}, \tag{2.8}$$

where $\frac{\delta}{\delta z^j} = \frac{\partial}{\partial z^j} - N_j^i \frac{\partial}{\partial \eta^i}$ and, obviously, $N_j^i(z,\eta) = g^{\overline{m}i} \frac{\partial g_{l\overline{m}}}{\partial z^j} \eta^l$.

Like above we consider transformation (2.3), $(z^i, \eta^i, \zeta^i) \to (z^i, \eta^i, \xi^i := \zeta^i + \frac{1}{2} L_{jk}^i(z,\eta)\eta^j\eta^k)$. As we have already remarked ξ^i are the components of one $d-$tensor on $J^{(2,0)}M$.

Now let us consider again the Lagrangian function $\mathcal{L}_F(z,\eta) = g_{i\bar{j}}(z,\eta)\xi^i\bar{\xi}^j$ on $J^{(2,0)}M$ and its metric tensor is $\frac{\partial^2 \mathcal{L}_F}{\partial\zeta^i \partial\bar{\zeta}^j} = g_{ij}(z,\eta)$, which is invertible.

For this Lagrangian function $\mathcal{L}_F$ we can apply again the result of Theorem 1.1 and Corollary 1.1. From (1.10) it is easy to check the following result.

Theorem 2.2. *The Chern-Lagrange complex nonlinear connection on* $J^{(2,0)}M$ *associated to the Lagrangian function* $\mathcal{L}_F$ *has the dual coefficients:*

$$\begin{aligned} \overset{1}{M}{}_j^i &= C_{jk}^i \xi^k + \frac{1}{2} \frac{\partial}{\partial \eta^j} \left(L_{rs}^i \eta^r \eta^s \right) \\ \overset{2}{M}{}_j^i &= \left(L_{jk}^i + N_j^h C_{hk}^i \right) \xi^k + \frac{1}{2} \frac{\partial L_{rs}^i}{\partial z^j} \eta^r \eta^s, \end{aligned} \tag{2.9}$$

where N_j^i, L_{jk}^i *and* C_{jk}^i *are the coefficients from (2.8).*

If (M,F) is a (strongly) Kähler-Finsler space, see [1, 10], then $L_{rs}^i \eta^r = N_s^i$ and $\overset{1}{M}{}_j^i$ reduces to $\overset{1}{M}{}_j^i = N_j^i + C_{jk}^i \xi^k$.

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