EINSTEIN EQUATIONS IN A WEAKLY GRAVITATIONAL COMPLEX FINSLER SPACE

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Abstract

In this paper we give a two-dimensional complex Finsler model for the real gravitation space-time. The study of the weakly gravitational field leads to some interesting geometrical and physical aspects, such as the study of curvature invariants with respect to complex Berwald frame.

A generalization of the Einstein equations is proposed for a two-dimensional complex Finsler space and these are written with respect to the directions of the complex Berwald frame attached. Some conditions are required so that the energy of the space should be conservative. In all these results the curvature invariants play a special meaning.

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1 Introduction

We already know in literature many paths in the study of General Relativity by using the methods of complex geometry. These are mainly in connection with the electromagnetic theories, with unification theories of physics fields or with spinorial techniques. The research of electromagnetic phenomena in complex coordinates has already a long history since 1907s' by L. Silberstein and continued with papers of P. Dirac, E. Schroedinger and P. Weiss, the latter two used the complex vector field $\mathbf{F} = \mathbf{E} + i\mathbf{H}$ in the framework of Born-Infeld theory.

The idea of using the Hermitian metrics in Gravitation only comes to A. Einstein' mind, [13, 14], who in 1945 attempted to establish a new relativistic theory of unified gravitation and electromagnetism. Since the gravitational tensor G_{ij} is one symmetric, while the electromagnetic field F_{ij} corresponding to the Maxwell equations is one antisymmetric, Einstein studied the real space-time manifold endowed with the metric

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$g_{j\bar{k}} = G_{jk} + \sqrt{-1}\, F_{jk}$, which is Hermitian (i.e. $\overline{g_{j\bar{k}}} = g_{k\bar{j}}$) with respect to the local natural complex structure.

In the present paper we extend Einstein's idea to the complex Finsler spaces proceeding rather the other way round. It is well known (see for instance [30], p. 170) that a n-dimensional complex Hermitian metric g with matrix $(g_{j\bar{k}}) = A + \sqrt{-1}B$ is coming from a real $2n$-dimensional Hermitian metric G iff its matrix is $(G_{\alpha\beta}) = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$, where A is symmetric and B is an antisymmetric matrix of the Kähler form.

Consider now a two-dimensional complex Finsler space (M, F), with the fundamental metric tensor $g_{j\bar{k}}$, $j, k = 1, 2$. The geometry of two-dimensional complex Finsler spaces was intensively studied by us in a recent paper, [3], using a special frame, the so-called complex Berwald frame. The Sasaki type lift of $g_{j\bar{k}}$ induces in real coordinates a Sasaki lift of a four-dimensional real Hermitian metric $G_{\alpha\beta}$ and, conversely. Accordingly, the four-dimensional space-time endowed with a real Hermitian metric $G_{\alpha\beta}$ could be embedded in a two dimensional complex Finsler space if some special circumstances are required. Such conditions are obtained in Proposition 3.1, for the particular case of weakly gravitational field. Therefore, we apply to the complex weakly gravitational field, the general settings from the geometry of two dimensional complex Finsler spaces. So that, in Theorem 3.2 we find the expressions of the curvature invariants. These will play an essential role in the expressions of complex Einstein equations which we do in the last part of this section.

2 Preliminaries

For the beginning we shall survey some results about 2 - dimensional complex Finsler geometry with Chern-Finsler complex linear connection and the local complex Berwald frames. Here we set the basic notions and terminology. For more details, see [1, 23, 3].

2.1 Complex Finsler spaces

Let M be a complex manifold of complex dimension two. We consider $z \in M$, and so $z = (z^1, z^2)$ are the complex coordinates in a local chart. Since $z^k = x^k + \sqrt{-1}x^{k+2}$, $k = \overline{1,2}$, the complex coordinates induce the real coordinates $\{x^1, x^2, x^3, x^4\}$ on M. Let $T_R M$ be the real tangent bundle. Its complexified tangent bundle $T_C M$ splits into the sum of holomorphic tangent bundle $T'M$ and its conjugate $T''M$, under the action of the natural complex structure J on M. The holomorphic tangent bundle $T'M$ is itself a complex manifold, and the coordinates in a local chart will be denoted by $u = (z^k, \eta^k)_{k=1,2}$, with $\eta^k = y^k + \sqrt{-1}y^{k+2}$, $k = 1, 2$.

Everywhere in this paper the indices $i, j, k, \dots$ run over $\{1, 2\}$.

A two dimensional *complex Finsler space* is a pair (M, F), where $F : T'M \to \mathbb{R}^+$ is a continuous function satisfying the conditions:

i) $L := F^2$ is smooth on $\widetilde{T'M} := T'M \backslash \{0\}$;

ii) $F(z, \eta) \geq 0$, the equality holds if and only if $\eta = 0$;

iii) $F(z, \lambda\eta) = |\lambda|F(z, \eta)$ for $\forall \lambda \in \mathbb{C}$;

iv) the Hermitian matrix $\left(g_{j\bar{k}}(z,\eta)\right)_{j,k=1,2} = \begin{pmatrix} g_{1\bar{1}} & g_{1\bar{2}} \\ g_{2\bar{1}} & g_{2\bar{2}} \end{pmatrix}$, (i.e. $\overline{g_{j\bar{k}}} = g_{k\bar{j}}$) is positive defined, where $g_{j\bar{k}} := \frac{\partial^2 L}{\partial \eta^j \partial \bar{\eta}^k}$ and $L := F^2$. Equivalently, it means that the indicatrix is strongly pseudo-convex.

Then, $g_{i\bar{j}}$ is called the *fundamental metric tensor* of the complex Finsler space. Consequently, from *iii)* we have $\frac{\partial L}{\partial \eta^k}\eta^k = \frac{\partial L}{\partial \bar{\eta}^k}\bar{\eta}^k = L$, $\frac{\partial g_{i\bar{j}}}{\partial \eta^k}\eta^k = \frac{\partial g_{i\bar{j}}}{\partial \bar{\eta}^k}\bar{\eta}^k = 0$ and $L = g_{i\bar{j}}\eta^i\bar{\eta}^j$.

Consider the sections of the complexified tangent bundle of $T'M$. Let $VT'M \subset T'(T'M)$ be the vertical bundle, locally spanned by $\{\frac{\partial}{\partial \eta^k}\}$, and $VT''M$ its conjugate. The idea of complex nonlinear connection, briefly (*c.n.c.*), is an instrument in the 'linearization' of the geometry of the manifold $T'M$. A (*c.n.c.*) is a supplementary complex subbundle to $VT'M$ in $T'(T'M)$, i.e. $T'(T'M) = HT'M \oplus VT'M$. The horizontal distribution $H_u T'M$ is locally spanned by $\{\frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N_k^j \frac{\partial}{\partial \eta^j}\}$, where $N_k^j(z,\eta)$ are the coefficients of the (*c.n.c.*). The pair $\{\delta_k := \frac{\delta}{\delta z^k}, \dot{\partial}_k := \frac{\partial}{\partial \eta^k}\}$ will be called the adapted frame of the (*c.n.c.*), which obeys the change rules $\delta_k = \frac{\partial z'^j}{\partial z^k}\delta'_j$ and $\dot{\partial}_k = \frac{\partial z'^j}{\partial z^k}\dot{\partial}'_j$. By conjugation everywhere we obtain an adapted frame $\{\delta_{\bar{k}}, \dot{\partial}_{\bar{k}}\}$ on $T''_u(T'M)$. The dual adapted bases are $\{dz^k, \delta\eta^k\}$ and $\{d\bar{z}^k, \delta\bar{\eta}^k\}$.

Next, let us consider the *Sasaki* type lift of the metric tensor $g_{j\bar{k}}$,

$$\mathcal{G} = g_{j\bar{k}}dz^j \otimes d\bar{z}^k + g_{j\bar{k}}\delta\eta^j \otimes \delta\bar{\eta}^k, \tag{2.1}$$

which passed in the real coordinates has the form, as we have proved in [23], p. 117,

$$G_{\alpha\beta}dx^\alpha \otimes dx^\beta + G_{\alpha\beta}\delta y^\alpha \otimes \delta y^\beta \; ; \; \alpha,\beta = \overline{1,4}, \tag{2.2}$$

where

$$(G_{\alpha\beta})_{\alpha,\beta=\overline{1,4}} = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}, \tag{2.3}$$

with

$$A := \begin{pmatrix} g_{1\bar{1}} & \mathrm{Re}g_{1\bar{2}} \\ \mathrm{Re}g_{1\bar{2}} & g_{2\bar{2}} \end{pmatrix} \text{ and } B := \begin{pmatrix} 0 & \mathrm{Im}g_{1\bar{2}} \\ -\mathrm{Im}g_{1\bar{2}} & 0 \end{pmatrix}.$$

We note that A is symmetric, while B is antisymmetric, so that $G_{\alpha\beta}$ is symmetric, i.e. $G_{\alpha\beta} = G_{\beta\alpha}$.

There exists an unique Hermitian connection D, of $(1,0)-$ type, which satisfies in addition $D_{JX}Y = JD_X Y$, for all X horizontal vectors, called the Chern-Finsler connection (cf. [1]), in brief $C - F$, which have a special meaning in complex Finsler geometry. The $C - F$ connection is locally given by the following coefficients (cf. [23]):

$$N_j^k = g^{\overline{m}k}\frac{\partial g_{l\overline{m}}}{\partial z^j}\eta^l = L_{lj}^k\eta^l \; ; \; L_{jk}^h = g^{\bar{l}h}\delta_k g_{j\bar{l}} \; ; \; C_{jk}^h = g^{\bar{l}h}\dot{\partial}_k g_{j\bar{l}}, \tag{2.4}$$

where $D_{\delta_k}\delta_j = L_{jk}^h\delta_h$, $D_{\dot{\partial}_k}\dot{\partial}_j = C_{jk}^h\dot{\partial}_h$, etc. Further on, the $C - F$ connection is the main tool in this study.

The nonzero curvatures of the $C - F$ connection are denoted by

$$
\begin{aligned}
R(\delta_h, \delta_{\bar{k}})\delta_j &= R^i_{j\bar{k}h}\delta_i \ ; \ R(\dot{\partial}_h, \delta_{\bar{k}})\delta_j = \Xi^i_{j\bar{k}h}\delta_i \ ; \ R(\delta_h, \dot{\partial}_{\bar{k}})\delta_j = P^i_{j\bar{k}h}\delta_i; \\
R(\delta_h, \delta_{\bar{k}})\dot{\partial}_j &= R^i_{j\bar{k}h}\dot{\partial}_i \ ; \ R(\dot{\partial}_h, \delta_{\bar{k}})\dot{\partial}_j = \Xi^i_{j\bar{k}h}\dot{\partial}_i \ ; \ R(\delta_h, \dot{\partial}_{\bar{k}})\dot{\partial}_j = P^i_{j\bar{k}h}\dot{\partial}_i; \\
R(\dot{\partial}_h, \dot{\partial}_{\bar{k}})\delta_j &= S^i_{j\bar{k}h}\delta_i \ ; \ R(\dot{\partial}_h, \dot{\partial}_{\bar{k}})\dot{\partial}_j = S^i_{j\bar{k}h}\dot{\partial}_i,
\end{aligned} \tag{2.5}
$$

where

$$
\begin{aligned}
R^i_{j\bar{h}k} &= -\delta_{\bar{h}}L^i_{jk} - \delta_{\bar{h}}(N^l_k)C^i_{jl} \ ; \ \Xi^i_{j\bar{h}k} = -\delta_{\bar{h}}C^i_{jk} = \Xi^i_{k\bar{h}j}; \\
P^i_{j\bar{h}k} &= -\dot{\partial}_{\bar{h}}L^i_{jk} - \dot{\partial}_{\bar{h}}(N^l_k)C^i_{jl} \ ; \ S^i_{j\bar{h}k} = -\dot{\partial}_{\bar{h}}C^i_{jk} = S^i_{k\bar{h}j}.
\end{aligned} \tag{2.6}
$$

The Riemann tensor

$$
\begin{aligned}
\mathbf{R}(W, \overline{Z}, X, \overline{Y}) \ &: \ = G(R(X, \overline{Y})W, \overline{Z}), \\
\mathbf{R}(W, \overline{Z}, X, \overline{Y}) &= \overline{\mathbf{R}(Z, \overline{W}, Y, \overline{X})}
\end{aligned} \tag{2.7}
$$

for $W, X, \ \overline{Z}, \overline{Y}$ horizontal or vertical vectors, has the following $h\bar{h}-, \ h\bar{v}-, \ v\bar{h}-, \ v\bar{v}-$ components: $R_{\bar{j}i\bar{h}k} := g_{l\bar{j}}R^l_{i\bar{h}k}; \ P_{\bar{j}i\bar{h}k} := g_{l\bar{j}}P^l_{i\bar{h}k}; \ \Xi_{\bar{j}i\bar{h}k} := g_{l\bar{j}}\Xi^l_{i\bar{h}k}; \ S_{\bar{j}i\bar{h}k} =: g_{l\bar{j}}S^l_{i\bar{h}k}$, which have properties $R_{i\bar{j}k\bar{h}} = R_{\bar{j}i\bar{h}k} \ ; \ \Xi_{i\bar{j}k\bar{h}} = P_{\bar{j}i\bar{h}k}; \ P_{i\bar{j}k\bar{h}} = \Xi_{\bar{j}i\bar{h}k} \ ; \ S_{i\bar{j}k\bar{h}} = S_{\bar{j}i\bar{h}k} = S_{\bar{h}i\bar{j}k}$, where $R_{i\bar{j}k\bar{h}} := \overline{R_{\bar{i}j\bar{k}h}}$, etc., (see [23], p. 77). Further, the Ricci curvature tensors are $R_{\bar{h}k} = g^{\bar{j}i}R_{\bar{j}i\bar{h}k} \ , \ P_{\bar{h}k} = g^{\bar{j}i}P_{\bar{j}i\bar{h}k} \ , \ \Xi_{\bar{h}k} = g^{\bar{j}i}\Xi_{\bar{j}i\bar{h}k}$ and $S_{\bar{h}k} = g^{\bar{j}i}S_{\bar{j}i\bar{h}k}$, from which we obtain the following Ricci scalars $r := g^{\bar{h}k}R_{\bar{h}k}$ and $p := g^{\bar{h}k}P_{\bar{h}k}, \ q := g^{\bar{h}k}\Xi_{\bar{h}k}, \ s := g^{\bar{h}k}S_{\bar{h}k}$. Since $P_{i\bar{j}k\bar{h}} = \Xi_{\bar{j}i\bar{h}k} = \overline{\Xi_{j\bar{i}k\bar{h}}}$ and $g^{i\bar{j}} = \overline{g^{\bar{j}i}}$, we deduce $P_{\bar{h}k} = \overline{\Xi_{\bar{k}h}}$ and $p = \bar{q}$.

2.2   The local complex Berwald frames

In [3] we introduced the local complex Berwald frames and by means of these, an exhaustive study of 2 - dimensional complex Finsler spaces is made. Here, we shall summarize some basic results. We set $l := l^i\dot{\partial}_i$ and its dual form is $\omega = l_i\delta\eta^i$, where

$$
l^i = \frac{1}{F}\eta^i \ \text{ and } \ l_i = \frac{1}{F}g_{i\bar{j}}\bar{\eta}^j = g_{i\bar{j}}l^{\bar{j}}. \tag{2.8}
$$

Our aim was to construct an orthonormal frame in the vertical bundle $VT'M$, which is 2 - dimensional in any point. Therefore, it is decomposed into $VT'M = \{l\} \oplus \{l\}^\perp$, where $\{l\}^\perp$ is spanned by a complex vector m. Requiring the orthogonality condition $\mathcal{G}(l, \bar{m}) = 0$ and $\mathcal{G}(m, \bar{m}) = 1$, we find

$$
m = \frac{1}{\sqrt{g}}(-l_2\dot{\partial}_1 + l_1\dot{\partial}_2), \tag{2.9}
$$

in a fixed chart. Then $\{l, m, \bar{l}, \bar{m}\}$, with m given by (2.9) is called the *local complex Berwald frame* of the space. Indeed, since the local frame is orthonormal we have: $l^il_i = m^im_i = 1$ and $l^im_i = l_im^i = 0$, where $l_i = g_{i\bar{j}}l^{\bar{j}}$ and $m_i = g_{i\bar{j}}m^{\bar{j}}$.

We specify that (2.9) provides only a local frame, as we point out in [3].

With respect to the local complex Berwald frame, $\dot{\partial}_k$ and $g_{i\bar{j}}$ are decomposed as follows $\dot{\partial}_i = l_i l + m_i m$ and hence, $g_{i\bar{j}} = l_i l_{\bar{j}} + m_i m_{\bar{j}}$. Decomposing $g^{\bar{j}k}$ from $l^j l^{\bar{k}}$, $l^j m^{\bar{k}}$, $m^j l^{\bar{k}}$ and $m^j m^{\bar{k}}$, from $g_{i\bar{j}} g^{\bar{j}k} = \delta_i^k$ we obtain $g^{\bar{j}k} = l^j l^{\bar{k}} + m^j m^{\bar{k}}$ and $l_i l^k + m_i m^k = \delta_i^k$.

From here is deduced

$$C_{jk}^i = g^{\bar{m}i} \dot{\partial}_k g_{j\bar{m}} = A l^i m_k m_j + B m^i m_k m_j, \tag{2.10}$$

where $A := m^j m^k l_h C_{kj}^h$ and $B := m_h m^k m^j C_{jk}^h$.

Now, via the natural isomorphism between the bundles $VT'M$ and $T'M$, composed with the horizontal lift of $HT'M$, we obtain the following orthonormal local frame on $H_C T'M$, $\{\lambda := l^i \delta_i, \ \mu = m^i \delta_i, \ \bar{\lambda} := l^{\bar{\imath}} \delta_{\bar{\imath}}, \ \bar{\mu} = m^{\bar{\imath}} \delta_{\bar{\imath}}\}$.

Using $L_{jk}^i = g^{\bar{m}i} \delta_k g_{j\bar{m}}$ it results

$$\begin{aligned} L_{jk}^i &= J l^i l_j l_k + U l^i m_j l_k + V l^i l_j m_k + X l^i m_j m_k \\ &\quad + O m^i l_j l_k + Y m^i m_j l_k + E m^i l_j m_k + H m^i m_j m_k, \end{aligned} \tag{2.11}$$

where $J := l^j l^k l_i L_{jk}^i$, $U := m^j l^k l_i L_{jk}^i$, $V := l^j m^k l_i L_{jk}^i$, $X := m^j m^k l_i L_{jk}^i$, $O := l^j l^k m_i L_{jk}^i$, $Y := m^j l^k m_i L_{jk}^i$, $E := l^j m^k m_i L_{jk}^i$, $H := m^j m^k m_i L_{jk}^i$.

The local complex Berwald frames also satisfy important properties. We mention here only some, which are needed in our study, (for more, see [3]),

$$\begin{aligned} m(l^i) &= \frac{1}{F} m^i \ ; \ \bar{m}(l^i) = 0 \ ; \ l(g) = 0 \ ; \ m(g) = Bg \ ; \tag{2.12} \\ \bar{\lambda}(l_i) &= \bar{\lambda}(l^i) = 0 \ ; \ \lambda(l^i) = -J l^i - O m^i = -\frac{2}{L} G^i \ ; \\ \bar{\mu}(l_i) &= \bar{\mu}(l^i) = 0 \ ; \ \mu(l^i) = -V l^i - E m^i \ ; \ \lambda(L) = \mu(L) = 0 \ ; \\ \bar{\mu}(m^i) &= -\frac{1}{2}(\bar{V} + \bar{H}) m^i \ ; \ \lambda(g) = (J + Y) g \ ; \ \mu(g) = (V + H) g. \end{aligned}$$

In [3] we speak about two horizontal holomorphic sectional curvatures: one in direction λ and other in direction μ, defined as

$$K_{F,\lambda}^h(z, \eta) := 2\mathbf{R}(\lambda, \bar{\lambda}, \lambda, \bar{\lambda}) = 2\mathbf{K} \ ; \ K_{F,\mu}^h(z, \eta) = 2\mathbf{R}(\mu, \bar{\mu}, \mu, \bar{\mu}) = 2\mathbf{W}, \tag{2.13}$$

where $\mathbf{K} := -\bar{\lambda}(J)$ and $\mathbf{W} := -\bar{\mu}(H) - \frac{1}{2} H(\bar{V} + \bar{H}) - BF\bar{\mu}(E)$ are called the *horizontal curvature invariants*.

Therefore, there are two vertical holomorphic sectional curvatures, one in the direction l and other in the direction m, defined as

$$K_{F,l}^v(z, \eta) := 2\mathbf{R}(l, \bar{l}, l, \bar{l}) = 0 \ ; \ K_{F,m}^v(z, \eta) = 2\mathbf{R}(m, \bar{m}, m, \bar{m}) = 2\mathbf{I}, \tag{2.14}$$

where $\mathbf{I} := -\bar{m}(B) - \frac{B\bar{B}}{2}$ is the *vertical curvature invariant*.

Moreover, the *horizontal holomorphic bisectional curvature* in directions $\{\lambda, \mu\}$ is

$$B_{F,\lambda,\mu}^h(z, \eta) := 2\mathbf{R}(\lambda, \bar{\lambda}, \mu, \bar{\mu}) + 2\mathbf{R}(\mu, \bar{\mu}, \lambda, \bar{\lambda}) = 2\mathbf{Z}, \tag{2.15}$$

where $\mathbf{Z} := -\bar{\mu}(V) - \frac{1}{2}V(\bar{V} + \bar{H}) - \bar{\lambda}(Y) - FB\bar{\lambda}(O) + \frac{1}{2}FBO(\bar{J} + \bar{Y})$ and, the *vertical holomorphic bisectional curvature* in directions $\{l, m\}$ is

$$B^v_{F,l,m}(z, \eta) := 2\mathbf{R}(l, \bar{l}, m, \bar{m}) + 2\mathbf{R}(m, \bar{m}, l, \bar{l}) = 0. \tag{2.16}$$

In this way, the study of the holomorphic sectional and bisectional curvatures of a 2 - dimensional complex Finsler space is reduced to the study of the curvature invariants $\mathbf{I}$, $\mathbf{K}$, $\mathbf{W}$ and $\mathbf{Z}$.

The vertical covariant derivatives of $l, m, \bar{l}$ and $\bar{m}$ with respect to the $C - F$ connection are given by:

$$
\begin{aligned}
l_i|_j &= \frac{-1}{2F}l_i l_j \ ; \ \ l_i|_{\bar{j}} = \frac{1}{2F}l_i l_{\bar{j}} + \frac{1}{F}m_i m_{\bar{j}}; \tag{2.17}\\
m_i|_j &= \frac{1}{2F}m_i l_j - \frac{1}{F}l_i m_j - \frac{B}{2}m_i m_j \ ; \ \ m_i|_{\bar{j}} = \frac{-1}{2F}m_i l_{\bar{j}} + \frac{\bar{B}}{2}m_i m_{\bar{j}};\\
l^i|_j &= \frac{1}{F}\delta^i_j - \frac{1}{2F}l_j l^i \ ; \ \ l^i|_{\bar{j}} = \frac{-1}{2F}l_{\bar{j}} l^i \ ; \ \ F|_j = \frac{1}{2}l_j;\\
m^i|_j &= \frac{-1}{2F}l_j m^i + \frac{B}{2}m_j m^i \ ; \ \ m^i|_{\bar{j}} = \frac{1}{2F}l_{\bar{j}} m^i - \frac{1}{F}m_{\bar{j}} l^i - \frac{\bar{B}}{2}m_{\bar{j}} m^i,
\end{aligned}
$$

and the horizontal derivatives are:

$$
\begin{aligned}
l_{i|j} &= l_{i|\bar{j}} = l^i_{|j} = l^i_{|\bar{j}} = 0; \ \ m^i_{|j} = \frac{1}{2}[(J + Y)l_j + (V + H)m_j]m^i; \tag{2.18}\\
m_{i|\bar{j}} &= \frac{1}{2}[(\bar{J} + \bar{Y})l_{\bar{j}} + (\bar{V} + \bar{H})m_{\bar{j}}]m_i;\\
m_{i|j} &= -\frac{1}{2}[(J + Y)l_j + (V + H)m_j]m_i;\\
m^i_{|\bar{j}} &= -\frac{1}{2}[(\bar{J} + \bar{Y})l_{\bar{j}} + (\bar{V} + \bar{H})m_{\bar{j}}]m^i
\end{aligned}
$$

Also, in [3] we prove that $A|_{\bar{j}}m^{\bar{j}} = -A\bar{B} + \frac{B}{F}$ and if the metric is pure Hermitian, i.e. $g_{i\bar{j}}(z)$, then $A = B = 0$.

The local expressions of the Riemann tensors with respect to the components of the local Berwald frame are as follow:

$$
\begin{aligned}
S_{\bar{j}i\bar{h}k} &= \mathbf{I} \, m_{\bar{h}} m_{\bar{r}} m_j m_k \ ; \tag{2.19}\\
\Xi_{\bar{r}j\bar{h}k} &= -[A_{|\bar{h}}l_{\bar{r}} + A(\bar{J} + \bar{Y})l_{\bar{r}}l_{\bar{h}} + A(\bar{V} + \bar{H})l_{\bar{r}}m_{\bar{h}} + B_{|\bar{h}}m_{\bar{r}}\\
&\quad + \frac{B}{2}(\bar{J} + \bar{Y})m_{\bar{r}}l_{\bar{h}} + \frac{B}{2}(\bar{V} + \bar{H})m_{\bar{r}}m_{\bar{h}}]m_j m_k = \overline{P_{\bar{j}r\bar{k}h}} \ ;
\end{aligned}
$$

and

$$
\begin{aligned}
R_{\bar{r}j\bar{h}k} &= \mathbf{K}l_{\bar{r}}l_j l_{\bar{h}}l_k + \mathbf{W}m_{\bar{r}}m_j m_{\bar{h}}m_k - \bar{J}_{|s}m^s l_{\bar{r}}l_j l_{\bar{h}}m_k - J_{|\bar{s}}m^{\bar{s}}l_{\bar{r}}l_j m_{\bar{h}}l_k \\
&\quad -[\frac{1}{F}\bar{O}_{|0} - \frac{1}{2}\bar{O}(J+Y)]l_{\bar{r}}m_j l_{\bar{h}}l_k - [\frac{1}{F}O_{|\bar{0}} - \frac{1}{2}O(\bar{J}+\bar{Y})]m_{\bar{r}}l_j l_{\bar{h}}l_k \\
&\quad -[V_{|\bar{s}}m^{\bar{s}} + \frac{1}{2}V(\bar{V}+\bar{H})]l_{\bar{r}}l_j m_{\bar{h}}m_k - \frac{1}{F}\bar{E}_{|0}m_{\bar{r}}l_j l_{\bar{h}}m_k \\
&\quad -\frac{1}{F}E_{|\bar{0}}l_{\bar{r}}m_j m_{\bar{h}}l_k - [\frac{1}{F}Y_{|\bar{0}} + BO_{|\bar{0}} - \frac{1}{2}FBO(\bar{J}+\bar{Y})]m_{\bar{r}}m_j l_{\bar{h}}l_k \\
&\quad -E_{|\bar{s}}m^{\bar{s}}m_{\bar{r}}l_j m_{\bar{h}}m_k - [\frac{1}{F}\bar{H}_{|0} + \frac{1}{2}\bar{H}(J+Y) + \bar{B}\bar{E}_{|0}]m_{\bar{r}}m_j m_{\bar{h}}l_k \\
&\quad -\bar{E}_{|s}m^s l_{\bar{r}}m_j m_{\bar{h}}m_k - [\frac{1}{F}H_{|\bar{0}} + \frac{1}{2}H(\bar{J}+\bar{Y}) + BE_{|\bar{0}}]m_{\bar{r}}m_j l_{\bar{h}}m_k \\
&\quad -[O_{|\bar{s}}m^{\bar{s}} - \frac{1}{2}O(\bar{V}+\bar{H})]m_{\bar{r}}l_j m_{\bar{h}}l_k \\
&\quad -[\bar{O}_{|s}m^s - \frac{1}{2}\bar{O}(V+H)]l_{\bar{r}}m_j l_{\bar{h}}m_k.
\end{aligned}
\tag{2.20}
$$

Now, by taking into account that $g^{\bar{r}j} = l^j l^{\bar{r}} + m^j m^{\bar{r}}$ and the orthogonality conditions of the Berwald frame, the local expressions of the Ricci tensors become:

$$
\begin{aligned}
S_{\bar{h}k} &= \mathbf{I}\, m_{\bar{h}}m_k \\
\Xi_{\bar{h}k} &= -[B_{|\bar{h}} + \frac{B}{2}(\bar{J}+\bar{Y})l_{\bar{h}} + \frac{B}{2}(\bar{V}+\bar{H})m_{\bar{h}}]m_k = \overline{P_{\bar{k}h}} \\
R_{\bar{h}k} &= \{\mathbf{K} - [\frac{1}{F}Y_{|\bar{0}} + BO_{|\bar{0}} - \frac{1}{2}FBO(\bar{J}+\bar{Y})]\}l_{\bar{h}}l_k \\
&\quad +\{\mathbf{W} - [V_{|\bar{s}}m^{\bar{s}} + \frac{1}{2}V(\bar{V}+\bar{H})]\}m_{\bar{h}}m_k \\
&\quad -\{\bar{J}_{|s}m^s + [\frac{1}{F}H_{|\bar{0}} + \frac{1}{2}H(\bar{J}+\bar{Y}) + BE_{|\bar{0}}]\}l_{\bar{h}}m_k \\
&\quad -\{J_{|\bar{s}}m^{\bar{s}} + [\frac{1}{F}\bar{H}_{|0} + \frac{1}{2}\bar{H}(J+Y) + \bar{B}\bar{E}_{|0}]\}m_{\bar{h}}l_k.
\end{aligned}
\tag{2.21}
$$

Consequently, we find the following writing for the complex Ricci scalars:

$$
s = \mathbf{I} \;\; ; \;\; q = -B_{|\bar{h}}m^{\bar{h}} - \frac{B}{2}(\bar{V}+\bar{H}) = \bar{p} \;\; ; \;\; r = \mathbf{K} + \mathbf{W} + \mathbf{Z}.
\tag{2.22}
$$

3 The complex Einstein equations with respect to Berwald frame

With this preparation we are able to write a generalization of the classical Einstein equations for a two-dimensional complex Finsler space. For real Finsler spaces such an eqations was considered by G. Asanov, [5, 4], and in a more general context of $n-$dimensional real Finsler and Lagrange spaces by R. Miron, [20].

Anyway, in such generalizations to the Finsler spaces, we to take need two motifs into account. First, the Chern-Finsler connection (or Cartan connection in real case) has

more curvatures and therefore will obtain a large set of Einstein equations. Secondly, this connection is with torsions and then to be in the classical context of Einstein theory, one solution is to require the divergence of the obtained Einstein tensors vanish. Also, the energy of the space must be conservative with respect to the geometry of the space.

Let us say that the Einstein tensors $\mathbf{E}_{\bar{h}k} := \mathbf{R}_{\bar{h}k} - \frac{1}{2}\rho g_{k\bar{h}}$ are proportional to the energy-momentum tensors $\mathbf{T}_{\bar{h}k}$, where $\mathbf{R}_{\bar{h}k}$ denotes the components of the Ricci tensors and ρ are the Ricci scalars. Obvious, the unknowns are complex Hermitian potentials $g_{k\bar{h}}$. Then, we have

$$\mathbf{R}_{\bar{h}k} - \frac{1}{2}\rho g_{k\bar{h}} = \chi \mathbf{T}_{\bar{h}k} \ . \tag{3.1}$$

We saw that the Einstein tensor could be decomposed in a two dimensional Finsler space by the components of the Berwald frame. Let us assume that $\mathbf{T}_{\bar{h}k}$, which represents the matter aspects is decomposed by the same Berwald frame,

$$\mathbf{T}_{\bar{h}k} = \overset{\bar{1}1}{T} l_{\bar{h}}l_k + \overset{\bar{1}2}{T} l_{\bar{h}}m_k + \overset{\bar{2}1}{T} m_{\bar{h}}l_k + \overset{\bar{2}2}{T} m_{\bar{h}}m_k \ . \tag{3.2}$$

Now, separating the $v\bar{v}-$, $h\bar{v}-$, $v\bar{h}-$, $h\bar{h}-$ components in (3.1), we obtain in order the set of equations:

$$\frac{1}{2}\mathbf{I} = \chi \overset{\bar{1}1}{T} = -\chi \overset{\bar{2}2}{T} \ ; \quad \overset{\bar{1}2}{T} = \overset{\bar{2}1}{T} = 0 \ ;$$

$$\frac{1}{2}[B_{|\bar{h}}m^{\bar{h}} + \frac{B}{2}(\bar{V} + \bar{H})] = \chi \overset{\bar{1}1}{T} = -\chi \overset{\bar{2}2}{T} \ ; \quad \frac{1}{F}B_{|\bar{0}} + \frac{B}{2}(\bar{J} + \bar{Y}) = -\chi \overset{\bar{1}2}{T} \ ; \quad \overset{\bar{2}1}{T} = 0 \ ;$$

$$\frac{1}{2}[\mathbf{K} - \mathbf{W} + \mathbf{Z} + 2V_{|\bar{s}}m^{\bar{s}} + V(\bar{V} + \bar{H})] \ = \ \chi \overset{\bar{1}1}{T} = -\chi \overset{\bar{2}2}{T} \ ;$$

$$\bar{J}_{|s}m^s + [\frac{1}{F}H_{|\bar{0}} + \frac{1}{2}H(\bar{J} + \bar{Y}) + BE_{|\bar{0}}] \ = \ -\chi \overset{\bar{1}2}{T} \ ;$$

$$J_{|\bar{s}}m^{\bar{s}} + [\frac{1}{F}\bar{H}_{|0} + \frac{1}{2}\bar{H}(J + Y) + \bar{B}\bar{E}_{|0}] \ = \ -\chi \overset{\bar{2}1}{T},$$

from which we conclude,

Theorem 3.1. *The complex Einstein equations of a two-dimensional complex Finsler space are given by*

$$\mathbf{I} = \mathbf{K} - \mathbf{W} + \mathbf{Z} + 2V_{|\bar{s}}m^{\bar{s}} + V(\bar{V} + \bar{H}) = B_{|\bar{h}}m^{\bar{h}} + \frac{B}{2}(\bar{V} + \bar{H}) = 2\chi \overset{\bar{1}1}{T} = -2\chi \overset{\bar{2}2}{T} \ ; \tag{3.3}$$

$$\overset{\bar{1}2}{T} = \overset{\bar{2}1}{T} = 0 \ ; \quad J_{|\bar{s}}m^{\bar{s}} + [\frac{1}{F}\bar{H}_{|0} + \frac{1}{2}\bar{H}(J + Y) + \bar{B}\bar{E}_{|0}] = \frac{1}{F}B_{|\bar{0}} + \frac{B}{2}(\bar{J} + \bar{Y}) = 0 \ ;$$

and their conjugates.

Due to the Einstein equations, some Ricci tensors are reduced to

$$\Xi_{\bar{h}k} \ = \ -\mathbf{I}\, m_{\bar{h}}m_k \ ;$$

$$R_{\bar{h}k} \ = \ [\mathbf{K} + \mathbf{Z} + V_{|\bar{s}}m^{\bar{s}} + \frac{1}{2}V(\bar{V} + \bar{H})]l_{\bar{h}}l_k$$

$$+ [\mathbf{W} - V_{|\bar{s}}m^{\bar{s}} - \frac{1}{2}V(\bar{V} + \bar{H})]m_{\bar{h}}m_k.$$

By rising the indices we obtain the energy-momentum tensors $\mathbf{T}_k^j = \overset{\bar{1}1}{T} l^j l_k + \overset{\bar{1}2}{T} l^j m_k + \overset{\bar{2}1}{T}$ $m^j l_k + \overset{\bar{2}2}{T} m^j m_k$, which in view of (3.3) are reduced to $\mathbf{T}_k^j = \frac{1}{2\chi}\mathbf{I}(l^j l_k - m^j m_k) = \frac{1}{2\chi}\mathbf{I}(2l^j l_k - \delta_k^j)$, and it is conservative iff the C-F connection performs $D_{\delta_j}\mathbf{T}_k^j = D_{\mathring{\partial}_j}\mathbf{T}_k^j = 0$, that is $2(\mathbf{I}l^j l_k)_{|\ j} = \mathbf{I}_{|\ k}$ and $2(\mathbf{I}l^j l_k)\ |_j = \mathbf{I}\ |_k$. Then, owing (2.17)-(2.18), the energy is conservative iff

$$2\mathbf{I}_{|j} l^j l_k = \mathbf{I}_{|k} \quad \text{and} \quad 2\mathbf{I}\ |_j l^j l_k = \mathbf{I}|_k. \tag{3.4}$$

Next, the rising of the indices for $\mathbf{R}_{\bar{h}k}$ gets:

$$S_k^j = \mathbf{I}\ m^j m_k \ ; \ \Xi_k^j = -\mathbf{I}\ m^j m_k;$$
$$R_k^j = [\mathbf{K} + \mathbf{Z} + V_{|\bar{s}}m^{\bar{s}} + \frac{1}{2}V(\bar{V} + \bar{H})]\ l^j l_k$$
$$+[\mathbf{W} - V_{|\bar{s}}m^{\bar{s}} - \frac{1}{2}V(\bar{V} + \bar{H})]\ m^j m_k.$$

Then the conservation law yields

$$2(\mathbf{I}\ m^j m_k)_{|j} = \mathbf{I}_{|k}\ ; \ 2(\mathbf{I}m^j m_k)|_j = \mathbf{I}|_k\ ; \tag{3.5}$$
$$2\Xi_{k|j}^j = -\mathbf{I}_{|k}\ ; \ 2\Xi_k^j\ |_j = -\mathbf{I}|_k\ ;$$

$$2\{[\mathbf{K} + \mathbf{Z} + V_{|\bar{s}}m^{\bar{s}} + \frac{1}{2}V(\bar{V} + \bar{H})]l_{\bar{h}}l_k + [\mathbf{W} - V_{|\bar{s}}m^{\bar{s}} - \frac{1}{2}V(\bar{V} + \bar{H})]m_{\bar{h}}m_k\}_{|j}$$
$$= (\mathbf{K} + \mathbf{W} + \mathbf{Z})_{|k}\ ;$$
$$2\{[\mathbf{K} + \mathbf{Z} + V_{|\bar{s}}m^{\bar{s}} + \frac{1}{2}V(\bar{V} + \bar{H})]l_{\bar{h}}l_k + [\mathbf{W} - V_{|\bar{s}}m^{\bar{s}} - \frac{1}{2}V(\bar{V} + \bar{H})]m_{\bar{h}}m_k\}|_j$$
$$= (\mathbf{K} + \mathbf{W} + \mathbf{Z})|_k.$$

The first conditions $2(\mathbf{I}\ m^j m_k)_{|j} = \mathbf{I}_{|k}$ can be rewriten $2(\mathbf{I}\ \delta_k^j - \mathbf{I}\ l^j l_k)_{|j} = \mathbf{I}_{|k}$, which is exactly the first condition (3.4). Similar computation gets $2(\mathbf{I}m^j m_k)|_j = \mathbf{I}\ |_k$, which is equivalent to $2\mathbf{I}|_j l^j l_k = \mathbf{I}|_k$.

One particular case is when the metric is purely Hermitian, $g_{i\bar{j}}(z)$, and the space is empty. Then $A = B = \mathbf{I} = 0$ and the Einstein equations reduce merely to $\mathbf{K} - \mathbf{W} + \mathbf{Z} + 2V_{|\bar{s}}m^{\bar{s}} + V(\bar{V} + \bar{H}) = 0$. In this case, the horizontal Ricci tensor is proportional with the fundamental metric tensor, $R_{\bar{h}k} = \frac{1}{2}(\mathbf{K} + \mathbf{W} + \mathbf{Z})g_{k\bar{h}} = \frac{1}{2}rg_{k\bar{h}}$, i.e. the space is Einstein.

4 A complex approximation of the weakly gravitational fields

In this section our goal is to give a complex Finsler version for the study of Gravity. We start from the real metrics which frequently appear in the relativistic theories of weakly gravitational fields ([11, 26]),

$$\tilde{g}_{\alpha\beta}(x, y) = \eta_{\alpha\beta} + p_{\alpha\beta}(x, y)\ ; \ \alpha, \beta = \overline{1, 4},$$

where $\eta_{\alpha\beta}$ is the Minkowski metric and $p_{\alpha\beta}(x, y) := \frac{2\Phi}{c^2}\delta_\alpha^\beta$ is a small perturbation of $\eta_{\alpha\beta}$, where Φ has the meaning of a gravitational potential.

Now, the form (2.3) and [11], p.148, suggest us to consider the real metrics

$$(G_{\alpha\beta})_{\alpha,\beta=\overline{1,4}} = \begin{pmatrix} A & B \\ -B & A \end{pmatrix},$$

in which we choose

$$A := \begin{pmatrix} 1+\frac{2\Phi}{c^2} & 0 \\ 0 & -(1-\frac{2\Phi}{c^2}) \end{pmatrix} \text{ and } B := \begin{pmatrix} 0 & -(1-\frac{2\Phi}{c^2}) \\ 1-\frac{2\Phi}{c^2} & 0 \end{pmatrix},$$

where Φ is here a smooth function on $T'M$, real valued, $\Phi \neq \frac{c^2}{2}$, and $c \in \mathbf{R}$, $c \neq 0$. Consequently, in the complex coordinates $(z^k, \eta^k)_{k=1,2}$, $(G_{\alpha\beta})_{\alpha,\beta=\overline{1,4}}$ can be rewritten as a Hermitian metric

$$\left(g_{j\bar{k}}(z,\eta)\right)_{j,k=1,2} = \begin{pmatrix} 1+\frac{2\Phi}{c^2} & -i(1-\frac{2\Phi}{c^2}) \\ i(1-\frac{2\Phi}{c^2}) & -(1-\frac{2\Phi}{c^2}) \end{pmatrix}, \text{ with } i := \sqrt{-1} \tag{4.1}$$

and the inverse matrix of (4.1) is

$$\left(g^{\bar{k}j}\right)_{j,k=1,2} = \begin{pmatrix} \frac{1}{2} & -\frac{i}{2} \\ \frac{i}{2} & -\frac{1+\frac{2\Phi}{c^2}}{2(1-\frac{2\Phi}{c^2})} \end{pmatrix}. \tag{4.2}$$

Let us point out that in general the Hermitian metric (4.1) may not come from a complex Finsler function.

Proposition 4.1. *The following conditions are all necessary for the metric (4.1) to be derived from the fundamental metric tensor of a complex Finsler space:*
i) $\Phi > \frac{c^2}{2}$;
ii) Φ is a homogeneous function with respect to η;
iii) $i\Phi_{\cdot 2} = \Phi_{\cdot 1}$, where $\Phi_{\cdot h} := \frac{\partial \Phi}{\partial \eta^h}$, $h = 1,2$.

The proof is obvious. Condition i) assures that $g := \det(g_{i\bar{j}}) = -2\left(1-\frac{2\Phi}{c^2}\right) > 0$, but for the pseudo-complex Finsler case the inequality from i) can be replaced with the condition of being nondegenerate, i.e. $\Phi \neq \frac{c^2}{2}$. Then, immediate computations get

$$\dot{\partial}_h g_{k\bar{k}} = \frac{2}{c^2}\Phi_{\cdot h} \; ; \; \dot{\partial}_h g_{1\bar{2}} = \frac{2i}{c^2}\Phi_{\cdot h} \; ; \; \dot{\partial}_h g_{2\bar{1}} = -\frac{2i}{c^2}\Phi_{\cdot h}. \tag{4.3}$$

So that, $\dot{\partial}_h g_{j\bar{k}} = \dot{\partial}_j g_{h\bar{k}}$ iff $i\Phi_{\cdot 2} = \Phi_{\cdot 1}$ and $(\dot{\partial}_h g_{j\bar{k}})\eta^h = (\dot{\partial}_{\bar{h}} g_{j\bar{k}})\bar{\eta}^h = 0$ iff $\Phi_{\cdot h}\eta^h = \Phi_{\cdot\bar{h}}\bar{\eta}^h = 0$, i.e. the condition ii).

Moreover, the above requirements ii) and iii) imply

$$\Phi_{\cdot 1}(\eta^1 - i\eta^2) = i\Phi_{\cdot 2}(\eta^1 - i\eta^2) = 0, \tag{4.4}$$

and their conjugates.

Since

$$g_{j\bar{k}}(z,\eta) = \eta_{j\bar{k}} + p_{j\bar{k}},$$

where $(\eta_{j\bar{k}}) := \begin{pmatrix} 1 & -i \\ i & -1 \end{pmatrix}$ and $(p_{j\bar{k}}) := \begin{pmatrix} \frac{2\Phi}{c^2} & i\frac{2\Phi}{c^2} \\ -i\frac{2\Phi}{c^2} & \frac{2\Phi}{c^2} \end{pmatrix}$, the metric (4.1) under assumptions i), ii) and iii) can be used in the study of the weakly gravitational fields in the complex Finsler space (M, L), with

$$L = (1+\frac{2\Phi}{c^2})\eta^1\bar{\eta}^1 - i(1-\frac{2\Phi}{c^2})\eta^1\bar{\eta}^2 + i(1-\frac{2\Phi}{c^2})\eta^2\bar{\eta}^1 - (1-\frac{2\Phi}{c^2})\eta^2\bar{\eta}^2. \tag{4.5}$$

Note that if function Φ depends only on the position $z \in M$, then the space (M, L) is purely Hermitian and if Φ depends only on the direction η, then the space (M, L) is locally Minkowski.

Once the metric tensor of the complex Finsler space (4.5) obtained, it is a technical computation to give the expression of the $C - F$ connection. Certainly, it involves some trivial calculus which lead to

$$N_k^1 = 0 \; ; \; N_k^2 = -\frac{2i}{c^2(1-\frac{2\Phi}{c^2})}(\eta^1 - i\eta^2)\Phi_k \; ; \tag{4.6}$$

$$L_{jk}^1 = 0 \; ; \; L_{1k}^2 = -\frac{2i}{c^2(1-\frac{2\Phi}{c^2})}\Phi_k = iL_{2k}^2 \; , \; \text{where} \; \Phi_k := \frac{\partial\Phi}{\partial z^k} \; ;$$

$$C_{jk}^1 = 0 \; ; \; C_{1k}^2 = -\frac{2i}{c^2(1-\frac{2\Phi}{c^2})}\Phi_{\cdot k} = iC_{2k}^2 \; , \; \text{for} \; j, k = 1, 2.$$

Secondly, since (M, L) with the metric (4.5) is two-dimensional, we can study this space with respect to the locally complex Berwald frames, verifying incessantly the global validity of the obtained results. Corresponding to the metric (4.5) the terms of the locally complex Berwald frames are

$$g = -2(1-\frac{2\Phi}{c^2}) \; ; \; l^k = \frac{1}{F}\eta^k \qquad , \; k = 1, 2 \; ;$$

$$l_1 = \frac{g}{2}\left[(1+\frac{4}{g})\bar{l}^1 + i\bar{l}^2\right] \; ; \; l_2 = -\frac{ig}{2}(\bar{l}^1 + i\bar{l}^2) \; ;$$

$$m^1 = -\frac{l_2}{\sqrt{g}} \; ; \; m^2 = \frac{l_1}{\sqrt{g}} \; ; \; m_1 = -\sqrt{g}l^2 \; ; \; m_2 = \sqrt{g}l^1;$$

and

$$B = \frac{-16i}{c^2g\sqrt{g}}\bar{l}^1|l^1|^2\Phi_{\cdot 1} \; ; \; J = \frac{2}{c^2}|l^1 - il^2|^2\Phi_j l^j \; ; \tag{4.7}$$

$$V = \frac{2}{c^2}|l^1 - il^2|^2\Phi_j m^j \; ; \; O = \frac{4i}{c^2\sqrt{g}}l^1(l^1 - il^2)\Phi_j l^j \; ;$$

$$Y = \frac{8}{c^2g}|l^1|^2\Phi_j l^j \; ; \; E = \frac{4i}{c^2\sqrt{g}}l^1(l^1 - il^2)\Phi_j m^j \; ; \; H = \frac{8}{c^2g}|l^1|^2\Phi_j m^j.$$

By means of the locally complex Berwald frames the study of the vertical or horizontal holomorphic sectional curvatures of the space (M, L) with the metric (4.5) is reduced to the study of the complex curvatures invariants **I**, **K**, **W** and **Z** of the $C - F$ connection. Corresponding to the metric (4.5), using (4.6), after some tedious computations we obtain,

Theorem 4.1. *Let (M, L) be a complex Finsler space, where L is the metric (4.5). Then,*

$$\mathbf{I} = -\frac{32}{c^2 g^2}|l^1|^4\left(-\frac{8}{c^2 g}|l^1|^2\Phi_{\cdot 1}\Phi_{\cdot \bar{1}} + \Phi_{\cdot 1 \cdot \bar{1}}\right) ; \tag{4.8}$$

$$\mathbf{K} = -\frac{4}{c^2}|l^1 - il^2|^2\left(-\frac{4}{c^2 g}\Phi_j\Phi_{\bar{k}} + \Phi_{j\bar{k}}\right)l^j \bar{l}^k ;$$

$$\mathbf{W} = -\frac{8}{c^2 g}|l^1|^2\left[-\frac{2}{c^2}\left(\frac{4}{g}|l^1|^2 + |l^1 - il^2|^2\right)\Phi_j\Phi_{\bar{k}} + \Phi_{j\bar{k}}\right]m^j \bar{m}^k ;$$

$$\mathbf{Z} = -\frac{8}{c^2 g}|l^1|^2\left[-\frac{2}{c^2}\left(\frac{4}{g}|l^1|^2 + |l^1 - il^2|^2\right)\Phi_j\Phi_{\bar{k}} + \Phi_{j\bar{k}}\right]l^j \bar{l}^k$$

$$\quad -\frac{2}{c^2}|l^1 - il^2|^2\left[-\frac{2}{c^2}\left(\frac{4}{g}|l^1|^2 + |l^1 - il^2|^2\right)\Phi_j\Phi_{\bar{k}} + \Phi_{j\bar{k}}\right]m^j \bar{m}^k$$

$$\quad = \frac{2|l^1|^2}{g|l^1 - il^2|^2}\mathbf{K} + \frac{g|l^1 - il^2|^2}{4|l^1|^2}\mathbf{W} + \frac{16|l^1|^2}{c^4 g^2}(4|l^1|^2 + g|l^1 - il^2|^2 - 2).$$

The Einstein equations can be obtained easily by replacing these quantities in (3.3).

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