Bulletin of the *Transilvania* University of Braşov • Vol 5(54), No. 2 - 2012
Series III: Mathematics, Informatics, Physics, 75-88

ABOUT THE GROUP OF TRANSFORMATIONS OF METRICAL SEMISYMMETRIC $N-$LINEAR CONNECTIONS ON A GENERALIZED HAMILTON SPACE OF ORDER TWO

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Abstract

In the present paper we obtain in a generalized Hamilton space of order two the transformation laws of the torsion and curvature tensor fields, with respect to the transformations of the group $\mathcal{T}_N$ of the transformations of $N-$linear connections having the same nonlinear connection N.

We also determine in a generalized Hamilton space of order two the set of all metrical semisymmetric $N-$linear connections, in the case when the nonlinear connection is fixed and prove that this set, $\overset{ms}{\mathcal{T}}_N$, of the transformations of metrical semisymmetric $N-$linear connections, having the same nonlinear connection N, together with the composition of mappings, is a group. We obtain some important invariants of the group $\overset{ms}{\mathcal{T}}_N$ and give their properties. We also study the transformations laws of the torsion $d-$tensor fields with respect to the transformation of the group $\overset{ms}{\mathcal{T}}_N$.

2000 *Mathematics Subject Classification:* 53B05

Key words: second order cotangent bundle, generalized Hamilton space of order two, nonlinear connection, $N-$linear connection, metrical $N-$linear connection, metrical semisymmetric $N-$linear connection, transformations group, subgroup, torsion, curvature, invariants.

1 Introduction

Differential geometry of the second order cotangent bundle $\left(T^{*2}M, \pi^{*2}, M\right)$ was introduced and studied by R. Miron [6] , R. Miron, H. Shimada, D. Hrimiuc, V.S. Sabău, [8] and Gh. Atanasiu and M. Târnoveanu [1] .

This geometry is based on the differential geometry of the cotangent bundle (see also: Gh. Atanasiu [2] , S. Ianuş [3] , R. Miron [5] , C. Udrişte [10]).

In the present section we keep the general setting from R. Miron, H. Shimada, D. Hrimiuc, V.S. Sabău, [8] , and subsequently we recall only some needed notions. For more details see [8] .

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Let M be a real $n-$dimensional manifold and let $\left(T^{*2}M, \pi^{*2}, M\right)$ be the dual of the $2-$tangent bundle, or $2-$cotangent bundle. A point $u \in T^{*2}M$ can be written in the form $u = (x, y, p)$, having the local coordinates $\left(x^i, y^i, p_i\right), (i = 1, 2, ..., \ n)$.

A change of local coordinates on the $3n$ dimensional manifold $T^{*2}M$ is

$$\begin{cases} \bar{x}^i = \bar{x}^i \left(x^1, ..., x^n\right), \det\left(\frac{\partial \bar{x}^i}{\partial x^j}\right) \neq 0, \\ \bar{y}^i = \frac{\partial \bar{x}^i}{\partial x^j} \cdot y^j, \\ \bar{p}_i = \frac{\partial x^j}{\partial \bar{x}^i} \cdot p_j, (i, j = 1, 2, ..., n). \end{cases} \tag{1.1}$$

We denote by $T^{*2}M = T^{*2}M \setminus \{0\}$, where $0 : M \longrightarrow T^{*2}M$ is the null section of the projection π^{*2}.

Let us consider the tangent bundle of the differentiable manifold $T^{*2}M$, $\left(TT^{*2}M, \tau^{*2}, T^{*2}M\right)$, where τ^{*2} is the canonical projection and the vertical distribution $V : u \in T^{*2}M \longrightarrow V(u) \subset T_uT^{*2}M$, locally generated by the vector fields $\left\{ \frac{\partial}{\partial y^i}\Big|_u, \frac{\partial}{\partial p_i}\Big|_u \right\}, \forall u \in T^{*2}M$.

The following $\mathcal{F}\left(T^{*2}M\right)-$ linear mapping $J : \chi\left(T^{*2}M\right) \longrightarrow \chi\left(\widetilde{T^{*2}M}\right)$, defined by:

$$J\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^i}, J\left(\frac{\partial}{\partial y^i}\right) = 0, J\left(\frac{\partial}{\partial p_i}\right) = 0, \forall u \in \widetilde{T^{*2}M} \tag{1.2}$$

is a tangent structure on $T^{*2}M$.

We denote with N a nonlinear connection on the manifold $T^{*2}M$, with the local coefficients $\left(N^j{}_i(x, y, p), N_{ij}(x, y, p)\right), (i, j = 1, 2, ..., n)$. Hence, the tangent space of $T^{*2}M$ in the point $u \in T^{*2}M$ is given by the direct sum of vector spaces:

$$T_uT^{*2}M = N(u) \oplus W_1(u) \oplus W_2(u), \forall u \in T^{*2}M. \tag{1.3}$$

A local adapted basis to the direct decomposition (1.3) is given by:

$$\left\{ \frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}, \frac{\partial}{\partial p_i} \right\}, (i = 1, 2, ..., n), \tag{1.4}$$

where:

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j{}_i \frac{\partial}{\partial y^j} + N_{ij} \frac{\partial}{\partial p_j}. \tag{1.5}$$

With respect to the coordinates transformations (1.1), we have the rules:

$$\frac{\delta}{\delta x^i} = \frac{\partial \bar{x}^j}{\partial x^i} \frac{\delta}{\delta \bar{x}^j}; \quad \frac{\partial}{\partial y^i} = \frac{\partial \bar{x}^j}{\partial x^i} \cdot \frac{\partial}{\partial \bar{y}^j}; \quad \frac{\partial}{\partial p_i} = \frac{\partial x^i}{\partial \bar{x}^j} \cdot \frac{\partial}{\partial \bar{p}_j}. \tag{1.5'}$$

The dual basis of the adapted basis (1.4) is given by:

$$\left\{\delta x^i, \delta y^i, \delta p_i\right\}, \tag{1.6}$$

where:

$$\delta x^i = dx^i, \ \delta y^i = dy^i + N^i{}_j dx^j, \ \delta p_i = dp_i - N_{ji} dx^j. \tag{1.6'}$$

With respect to (1.1), the covector fields (1.6) are transformed by the rules:

$$\delta\bar{x}^i = \frac{\partial\bar{x}^i}{\partial x^j}\delta x^j,\ \delta\bar{y}^i = \frac{\partial\bar{x}^i}{\partial x^j}\delta y^j,\ \delta\bar{p}_i = \frac{\partial x^j}{\partial\bar{x}^i}\delta p_j. \tag{1.6"}$$

Let D be an $N-$linear connection on $T^{*2}M$, with the local coefficients in the adapted basis: $D\Gamma(N) = \left(H^i{}_{jk}, C^i{}_{jk}, C_i{}^{jk}\right)$.

An $N-$ linear connection D is uniquely represented, in the adapted basis (1.4) in the following form:

$$\begin{cases} D_{\frac{\delta}{\delta x^j}}\frac{\delta}{\delta x^i} = H^k{}_{ij}\frac{\delta}{\delta x^k}, D_{\frac{\delta}{\delta x^j}}\frac{\partial}{\partial y^i} = H^k{}_{ij}\frac{\partial}{\partial y^k}, D_{\frac{\delta}{\delta x^j}}\frac{\partial}{\partial p_i} = -H^i{}_{kj}\frac{\partial}{\partial p_k}, \\ D_{\frac{\partial}{\partial y^j}}\frac{\delta}{\delta x^i} = C^k{}_{ij}\frac{\delta}{\delta x^k}, D_{\frac{\partial}{\partial y^j}}\frac{\partial}{\partial y^i} = C^k{}_{ij}\frac{\partial}{\partial y^k}, D_{\frac{\partial}{\partial y^j}}\frac{\partial}{\partial p_i} = -C^i{}_{kj}\frac{\partial}{\partial p_k}, \\ D_{\frac{\partial}{\partial p_j}}\frac{\delta}{\delta x^i} = C_i{}^{kj}\frac{\delta}{\delta x^k}, D_{\frac{\partial}{\partial p_j}}\frac{\partial}{\partial y^i} = C_i{}^{kj}\frac{\partial}{\partial y^k}, D_{\frac{\partial}{\partial p_j}}\frac{\partial}{\partial p_i} = -C_k{}^{ij}\frac{\partial}{\partial p_k}. \end{cases} \tag{1.7}$$

2 The transformations of the $d-$tensors of torsion and curvature

In the following, we shall study the Abelian group $\mathcal{T}_N$. Its elements are the transformations $t : D\Gamma(N) \to D\bar{\Gamma}(N)$ given by (see[9]):

$$\begin{cases} \bar{N}^i{}_j = N^i{}_j, \\ \bar{N}_{ij} = N_{ij}, \\ \bar{H}^k{}_{ij} = H^k{}_{ij} - B^k{}_{ij}, \\ \bar{C}^k{}_{ij} = C^k{}_{ij} - D^k{}_{ij}, \\ \bar{C}_i{}^{kj} = C_i{}^{kj} - D_i{}^{kj}, (i,j,k = 1,2,...,n). \end{cases} \tag{2.1}$$

Firstly, we shall study the transformations of the $d-$tensors of torsion of $D\Gamma(N)$.

Proposition 2.1. *The transformations of the Abelian group $\mathcal{T}_N$, given by(2.1) lead to the transformations of the $d-$tensors of torsion in the following way:*

$$\underset{(1)}{\bar{R}}{}^i{}_{jk} = \underset{(1)}{R}{}^i{}_{jk},\ \underset{(2)}{\bar{R}}{}_{ijk} = \underset{(2)}{R}{}_{ijk}, \underset{(1)}{\bar{B}}{}_j{}^{ik} = \underset{(1)}{B}{}_j{}^{ik},\ \underset{(2)}{\bar{B}}{}_{ijk} = \underset{(2)}{B}{}_{ijk}, \tag{2.2}$$

$$\bar{T}^i{}_{jk} = T^i{}_{jk} + \left(B^i{}_{kj} - B^i{}_{jk}\right), \tag{2.3}$$

$$\bar{S}^i{}_{jk} = S^i{}_{jk} + \left(D^i{}_{kj} - D^i{}_{jk}\right), \bar{S}_i{}^{jk} = S_i{}^{jk} + \left(D_i{}^{kj} - D_i{}^{jk}\right), \tag{2.4}$$

$$\underset{(1)}{\bar{P}}{}^i{}_{jk} = \underset{(1)}{P}{}^i{}_{jk} + B^i{}_{kj}, \underset{(2)}{\bar{P}}{}^i{}_{jk} = \underset{(2)}{P}{}^i{}_{jk} - B^i{}_{jk}. \tag{2.5}$$

Proof. Using $(7.2)-p.256, [8], (6.3), (6.3)', (6.3)''-p.273, [8]$ and (2.1) we have the results. □

Now, we shall study the transformations of the $d-$tensors of curvature of $D\Gamma(N)$ (see, (6.4), -p.274, [8] and (5.2)'-p.270, [8]) by a transformation (2.1). We get:

Proposition 2.2. *The transformations of the Abelian group $\mathcal{T}_N$, given by* (2.1) *lead to the transformations of the $d-$tensors of curvature in the following way:*

$$\begin{aligned}\bar{R}_h{}^i{}_{jk} = R_h{}^i{}_{jk} - D^i{}_{hm}\underset{(1)}{R}{}^m{}_{jk} - D_h{}^{im}\underset{(2)}{R}{}_{mjk} - B^i{}_{hm}T^m{}_{jk} + \\ + \mathcal{A}_{jk}\left\{B^m{}_{hj}\cdot B^i{}_{mk} - B^i{}_{hj|k}\right\},\end{aligned} \tag{2.6}$$

$$\begin{aligned}\bar{P}_h{}^i{}_{jk} = P_h{}^i{}_{jk} - D^i{}_{hm}\underset{(1)}{P}{}^m{}_{jk} - D_h{}^{im}\underset{(2)}{B}{}_{mjk} - B^i{}_{hm}C^m{}_{jk} + \\ + B^m{}_{hj}D^i{}_{mk} - D^m{}_{hk}B^i{}_{mj} - B^i{}_{hj}\mid_k + D^i{}_{hk|j},\end{aligned} \tag{2.7}$$

$$\begin{aligned}\bar{P}_h{}^i{}_j{}^k = P_h{}^i{}_j{}^k - D_h{}^{im}\underset{(2)}{P}{}^k{}_{mj} - D^i{}_{hm}\underset{(1)}{B}{}_j{}^{mk} - B^i{}_{hm}C_j{}^{mk} + \\ + B^m{}_{hj}D_m{}^{ik} - D_h{}^{mk}B^i{}_{mj} - B^i{}_{hj}\mid^k + D_h{}^{ik}{}_{|j},\end{aligned} \tag{2.8}$$

$$\bar{S}_h{}^i{}_{jk} = S_h{}^i{}_{jk} - D^i{}_{hm}S^m{}_{jk} + \mathcal{A}_{jk}\{-D^i{}_{hj}\mid_k + D^m{}_{hj}D^i{}_{mk}\}, \tag{2.9}$$

$$\begin{aligned}\bar{S}_h{}^i{}_j{}^k = S_h{}^i{}_j{}^k - D^i{}_{hj}\mid^k + D_h{}^{ik}\mid_j - C_j{}^{mk}D^i{}_{hm} - \\ - C^k{}_{mj}D_h{}^{im} + D^m{}_{hj}D_m{}^{ik} - D_h{}^{mk}D^i{}_{mj},\end{aligned} \tag{2.10}$$

$$\bar{S}_h{}^{ijk} = S_h{}^{ijk} + D_h{}^{im}S_m{}^{jk} + \mathcal{A}_{jk}\left\{-D_h{}^{ij}\mid^k + D_h{}^{mj}D_m{}^{ik}\right\}, \tag{2.11}$$

where $\mathcal{A}_{ij}$ denotes the alternate summation and $_{|m}$, $|_m$ and $|^m$ denote the h-covariant derivative, the w_1-covariant derivative and the w_2-covariant derivative with respect to $D\Gamma(N)$ respectively.

We shall consider the tensor fields:

$$\mathcal{K}_h{}^i{}_{jk} = R_h{}^i{}_{jk} - C^i{}_{hm}\underset{(1)}{R}{}^m{}_{jk} - C_h{}^{im}\underset{(2)}{R}{}_{mjk}, \tag{2.12}$$

$$\mathcal{P}_h{}^i{}_{jk} = \mathcal{A}_{jk}\left\{P_h{}^i{}_{jk} - C^i{}_{hm}\frac{\partial N^m{}_j}{\partial y^k} + C_h{}^{im}\frac{\partial N_{jm}}{\partial p_k}\right\}, \tag{2.13}$$

$$\mathcal{P}_h{}^i{}_j{}^k = \mathcal{A}_{jk}\left\{\mathcal{P}_h{}^i{}_j{}^k - C^i{}_{hm}\frac{\partial N^m{}_j}{\partial p_k} + C_h{}^{im}\frac{\partial N_{jm}}{\partial p_k}\right\}. \tag{2.14}$$

Proposition 2.3. *By a transformation of the Abelian group $\mathcal{T}_N$, given by* (2.1) *,the tensor fields $\mathcal{K}_h{}^i{}_{jk}, \mathcal{P}_h{}^i{}_{jk}, \mathcal{P}_h{}^i{}_j{}^k$ are transformed according to the following laws:*

$$\bar{\mathcal{K}}_h{}^i{}_{jk} = \mathcal{K}_h{}^i{}_{jk} - B^i{}_{hm}T^m{}_{jk} + \mathcal{A}_{jk}\left\{B^m{}_{hj}B^i{}_{mk} - B^i{}_{hj|k}\right\}, \tag{2.15}$$

$$\begin{aligned}&\bar{\mathcal{P}}_h{}^i{}_{jk}=\mathcal{P}_h{}^i{}_{jk}-D^i{}_{hm}T^m{}_{jk}-B^i{}_{hm}S^m{}_{jk}+\\&+\mathcal{A}_{jk}\left\{-B^i{}_{hj}\,|_k+D^i{}_{hk|j}+B^m{}_{hj}D^i{}_{mk}-D^m{}_{hk}B^i{}_{mj}\right\},\end{aligned}\tag{2.16}$$

$$\begin{aligned}&\bar{\mathcal{P}}_h{}^i{}_j{}^k=\mathcal{P}_h{}^i{}_j{}^k+\mathcal{A}_{jk}\left\{-B^i{}_{hj}\,|^k+D_h{}^{ik}{}_{|j}-D_h{}^{im}H^k{}_{mj}-\right.\\&\left.-B^i{}_{hm}C_j{}^{mk}+B^m{}_{hj}D_m{}^{ik}-D_h{}^{mk}B^i{}_{mj}\right\}.\end{aligned}\tag{2.17}$$

Proof. From (2.7) we get:

$$\begin{aligned}&\mathcal{A}_{jk}\left\{\bar{P}_h{}^i{}_{jk}\right\}=\mathcal{A}_{jk}\left\{P_h{}^i{}_{jk}\right\}+\mathcal{A}_{jk}\left\{-D^i{}_{hm}\underset{(1)}{P}{}^m{}_{jk}-D_h{}^{im}\underset{(2)}{B}{}_{mjk}\right\}-\\&-\mathcal{A}_{jk}\left\{B^i{}_{hm}C^m{}_{jk}\right\}+\mathcal{A}_{jk}\left\{B^m{}_{hj}D^i{}_{mk}-D^m{}_{hk}B^i{}_{mj}-B^i{}_{hj}\,|_k+D^i{}_{hk|j}\right\}.\end{aligned}$$

Using $(7.2)-p.256,[8],(6.3)',(6.3)'',-p.273,[8]$ and (2.18) we have:

$$\begin{aligned}&\mathcal{A}_{jk}\left\{\bar{P}_h{}^i{}_{jk}\right\}=\mathcal{A}_{jk}\left\{P_h{}^i{}_{jk}\right\}+\mathcal{A}_{jk}\left\{\left(\bar{C}^i{}_{hm}-C^i{}_{hm}\right)\left(\tfrac{\partial N^m{}_j}{\partial y^k}-H^m{}_{kj}\right)+\right.\\&\left.+\left(\bar{C}_h{}^{im}-C_h{}^{im}\right)\left(-\tfrac{\partial N_{jm}}{\partial p_k}\right)\right\}-B^i{}_{hm}S^m{}_{jk}+\\&+\mathcal{A}_{jk}\left\{-B^i{}_{hj}\,|_k+D^i{}_{hk|j}+B^m{}_{hj}D^i{}_{mk}-D^m{}_{hk}B^i{}_{mj}\right\}.\end{aligned}$$

If we separate the terms we get:

$$\begin{aligned}&\mathcal{A}_{jk}\left\{\bar{P}_h{}^i{}_{jk}-\bar{C}^i{}_{hm}\frac{\partial\bar{N}^m{}_j}{\partial y^k}+\bar{C}_h{}^{im}\frac{\partial\bar{N}_{jm}}{\partial p_k}\right\}=\\&=\mathcal{A}_{jk}\left\{P_h{}^i{}_{jk}-C^i{}_{hm}\frac{\partial N^m{}_j}{\partial y^k}+C_h{}^{im}\frac{\partial N_{jm}}{\partial p_k}\right\}-D^i{}_{hm}T^m{}_{jk}-B^i{}_{hm}S^m{}_{jk}+\\&+\mathcal{A}_{jk}\left\{-B^i{}_{hj}\,|_k+D^i{}_{hk\,|j}+B^m{}_{hj}D^i{}_{mk}-D^m{}_{hk}B^i{}_{mj}\right\}.\end{aligned}$$

Using (2.13) we obtain: (2.16). Analogous we obtain the other formulas. □

3 Metrical semisymmetric $N-$linear connections in $GH^{(2)n}-$spaces

Definition 3.1. ([8]) A generalized Hamilton space of order two is a pair $GH^{(2)n}=\left(M,g^{ij}\left(x,y,p\right)\right)$, where:

1° g^{ij} is a $d-$ tensor field of type $(2,0)$, symmetric and nondegenerate on the manifold $T^{*2}M$.

2° The quadratic form $g^{ij}X_iX_j$ has a constant signature on $T^{*2}M$.

g^{ij} is called the fundamental tensor or metric tensor of space $GH^{(2)n}$.

In the case when $T^{*2}M$ is a paracompact manifold then on $T^{*2}M$ the metric tensors $g^{ij}(x,y,p)$ exist positively defined such that $\left(M,g^{ij}\right)$ is a generalized Hamilton space.

Definition 3.2. ([8]) A generalized Hamilton metric $g^{ij}(x,y,p)$ of order two (on short $GH-$metric) is called reductible to an Hamilton metric ($H-$metric) of order two if there exists a function $H(x,y,p)$ on $T^{*2}M$ such that:

$$g^{ij}=\frac{1}{2}\frac{\partial^2 H}{\partial p_i \partial p_j}. \tag{3.1}$$

The covariant tensor field g_{ij} is obtained from the equations

$$g_{ij}g^{jk}=\delta_i^k. \tag{3.2}$$

g_{ij} is a symmetric, nondegenerate and covariant of order two, $d-$tensor field.

If a nonlinear connection N, with the coefficients $\left(N_i^j(x,y,p), N_{ij}(x,y,p)\right)$, is a priori given, let us consider the direct decomposition (1.3) and the adapted basis to it, (1.4), where (1.5) hold. The dual adapted basis is (1.6), where $(1.6)'$ hold. An $N-$linear connection: $D\Gamma(N)=\left(H^i{}_{jk}, C^i{}_{jk}, C_i^{jk}\right)$ determines the $h-, w_1-, w_2-$covariant derivatives in the tensor algebra of $d-$tensor fields.

Definition 3.3. ([8]) An $N-$linear connection $D\Gamma(N)$ is called metrical with respect to $GH-$metric g^{ij} if g^{ij} is covariant constant (or absolute parallel) with respect to $D\Gamma(N)$, *i.e.*

$$g^{ij}{}_{|k}=0,\quad g^{ij}\mid_k=0,\quad g^{ij}\mid^k=0. \tag{3.3}$$

The tensorial equations (3.3) imply:

$$g_{ij|k}=0,\quad g_{ij}\mid_k=0,\quad g_{ij}\mid^k=0. \tag{3.4}$$

Theorem 3.1. ([8]) *1. There is a unique $N-$linear connection $D\bar{\Gamma}(N)=$ $=\left(\bar{H}^i{}_{jk}, \bar{C}^i{}_{jk}, \bar{C}_i^{jk}\right)$ having the properties:*

1°. The nonlinear connection is a priori given.

2°. $D\bar{\Gamma}(N)$ is metrical with respect to $GH-$metric g^{ij} i.e.(3.3) are verified.

3°. The torsion tensors $\bar{T}^i{}_{jk}, \bar{S}^i{}_{jk}$, and $\bar{S}_i^{jk}$ vanish.

2. The previous connection has the coefficients $\bar{C}^i{}_{jk}$ and $\bar{C}_i^{jk}$ given by

$$\begin{aligned}\bar{C}^i{}_{jk}&=\tfrac{1}{2}g^{im}\left(\tfrac{\partial g_{mk}}{\partial y^j}+\tfrac{\partial g_{jm}}{\partial y^k}-\tfrac{\partial g_{jk}}{\partial y^m}\right),\\ \bar{C}_i^{jk}&=\tfrac{1}{2}g_{im}\left(\tfrac{\partial g^{mk}}{\partial p_j}+\tfrac{\partial g^{jm}}{\partial p_k}-\tfrac{\partial g^{jk}}{\partial p_m}\right),\end{aligned} \tag{3.5}$$

and $\bar{H}^i{}_{jk}$ are generalized Christoffel symbols:

$$\bar{H}^i{}_{jk}=\frac{1}{2}g^{im}\left(\frac{\delta g_{mk}}{\delta x^j}+\frac{\delta g_{jm}}{\delta x^k}-\frac{\delta g_{jk}}{\delta x^m}\right). \tag{3.6}$$

The Obata's operators, are given by:

$$\Omega_{hk}^{ij} = \frac{1}{2}\left(\delta_h^i\delta_k^j - g_{hk}g^{ij}\right),\ \Omega^{*ij}_{\ \ hk} = \frac{1}{2}\left(\delta_h^i\delta_k^j + g_{hk}g^{ij}\right). \tag{3.7}$$

There is inferred:

Proposition 3.1. *The Obata's operators have the following properties:*

$$\Omega_{sj}^{ir} + \Omega^{*ir}_{\ \ sj} = \delta_s^i\delta_j^r, \tag{3.8}$$

$$\Omega_{sj}^{ir}\Omega_{mr}^{sn} = \Omega_{mj}^{in},\ \Omega^{*ir}_{\ \ sj}\Omega^{*sn}_{\ \ mr} = \Omega^{*in}_{\ \ mj},\ \Omega_{sj}^{ir}\Omega^{*sn}_{\ \ mr} = \Omega^{*ir}_{\ \ sj}\Omega_{mr}^{sn} = 0, \tag{3.9}$$

$$\Omega_{rj}^{ir} = \Omega_{si}^{ir} = 0,\ \Omega_{ij}^{ir} = \frac{1}{2}(n-1)\,\delta_j^r,\ \Omega^{*ir}_{\ \ ij} = \frac{1}{2}(n+1)\,\delta_j^r. \tag{3.10}$$

Theorem 3.2. ([8]) *There is a unique metrical connection* $\bar{D}\Gamma(N) = \left(\bar{H}^i{}_{jk}, \bar{C}^i{}_{jk}, \bar{C}_i{}^{jk}\right)$ *with respect to GH−metric* g^{ij}*, having the torsion d−tensor fields* $T^i{}_{jk}, S^i{}_{jk}, S_i^{jk}$ *a priori given. The coefficients of* $\bar{D}\Gamma(N)$ *are given by the following formulas:*

$$\begin{array}{l} \bar{H}^i{}_{jk} = \frac{1}{2}g^{im}\left(\frac{\delta g_{mk}}{\delta x^j} + \frac{\delta g_{jm}}{\delta x^k} - \frac{\delta g_{jk}}{\delta x^m}\right) + \\ +\frac{1}{2}g^{im}\left(g_{mh}T^h{}_{jk} - g_{jh}T^h{}_{mk} + g_{kh}T^h{}_{jm}\right), \\ \bar{C}^i{}_{jk} = \frac{1}{2}g^{im}\left(\frac{\partial g_{mk}}{\partial y^j} + \frac{\partial g_{jm}}{\partial y^k} - \frac{\partial g_{jk}}{\partial y^m}\right) + \\ +\frac{1}{2}g^{im}\left(g_{mh}S^h{}_{jk} - g_{jh}S^h{}_{mk} + g_{kh}S^h{}_{jm}\right), \\ \bar{C}_i{}^{jk} = \frac{1}{2}g_{im}\left(\frac{\partial g^{mk}}{\partial p_j} + \frac{\partial g^{jm}}{\partial p_k} - \frac{\partial g^{jk}}{\partial p_m}\right) - \\ -\frac{1}{2}g_{im}\left(g^{mh}S_h{}^{jk} - g^{jh}S_h{}^{mk} + g^{kh}S_h{}^{jm}\right). \end{array} \tag{3.11}$$

Definition 3.4. ([1]) An $N-$linear connection on $T^{*2}M$ is called semisymmetric if:

$$T^i{}_{jk} = \frac{1}{2}\left(-\delta_j^i\sigma_k + \delta_k^i\sigma_j\right), S^i{}_{jk} = \frac{1}{2}\left(-\delta_j^i\tau_k + \delta_k^i\tau_j\right), S_i^{jk} = -\frac{1}{2}\left(-\delta_i^j v^k + \delta_i^k v^j\right), \tag{3.12}$$

where $\sigma, \tau \in \chi^*\left(T^{*2}M\right)$ and $v \in \chi\left(T^{*2}M\right)$.

Theorem 3.3. *The set of all metrical semisymmetric N−linear connections with local coefficients* $D\Gamma(N) = \left(H^i{}_{jk}, C^i{}_{jk}, C_i{}^{jk}\right)$ *is given by:*

$$\left\{\begin{array}{lll} H^i{}_{jk} & = & \bar{H}^i{}_{jk} + \frac{1}{2}\left(-g_{jk}g^{im}\sigma_m + \sigma_j\delta_k^i\right), \\ C^i{}_{jk} & = & \bar{C}^i{}_{jk} + \frac{1}{2}\left(-g_{jk}g^{im}\tau_m + \tau_j\delta_k^i\right), \\ C_i^{jk} & = & \bar{C}_i^{jk} + \frac{1}{2}\left(-g^{jk}g_{im}v^m + v^j\delta_i^k\right), \end{array}\right. \tag{3.13}$$

where $D\bar{\Gamma}(N) = \left(\bar{H}^i{}_{jk}, \bar{C}^i{}_{jk}, \bar{C}_i{}^{jk}\right)$ *are the local coefficients* (3.6) *and* (3.5) *of the metrical N−linear connection given in Theorem 3.1 and* $\sigma, \tau \in \chi^*\left(T^{*2}M\right)$ *and* $v \in \chi\left(T^{*2}M\right)$.

Proof. Using Theorem 3.2 and Definition 3.4 we obtain the results by direct calculation. □

4 The group of transformations of metrical semisymmetric $N-$linear connections

Let N be a given nonlinear connection on $T^{*2}M$. Then any metrical semisymmetric $N-$linear connection with local coefficients $\bar{D}\Gamma(N) = \left(\bar{H}^i{}_{jk}, \bar{C}^i{}_{jk}, \bar{C}_i{}^{jk}\right)$ is given by (3.11) with (3.12). From Theorem 3.3 we have:

Theorem 4.1. *Two metrical semisymmetric $N-$linear connections:D and $\bar{D}$, with local coefficients: $D\Gamma(N) = \left(H^i{}_{jk}, C^i{}_{jk}, C_i{}^{jk}\right)$ and $\bar{D}\Gamma(N) =$*
$= \left(\bar{H}^i{}_{jk}, \bar{C}^i{}_{jk}, \bar{C}_i{}^{jk}\right)$ are related as follows:

$$\begin{cases} \bar{H}^i{}_{jk} = H^i{}_{jk} + \frac{1}{2}\left(-g_{jk}g^{im}\sigma_m + \sigma_j\delta^i_k\right), \\ \bar{C}^i{}_{jk} = C^i{}_{jk} + \frac{1}{2}\left(-g_{jk}g^{im}\tau_m + \tau_j\delta^i_k\right), \\ \bar{C}_i{}^{jk} = C_i{}^{jk} + \frac{1}{2}\left(-g^{jk}g_{im}v^m + v^j\delta^k_i\right), \end{cases} \tag{4.1}$$

where $\sigma, \tau \in \chi^*\left(T^{*2}M\right)$ and $v \in \chi\left(T^{*2}M\right)$.

Conversely, given $\sigma, \tau \in \chi^*\left(T^{*2}M\right)$ and $v \in \chi\left(T^{*2}M\right)$ the above (4.1) is thought to be a transformation of a metrical semisymmetric $N-$linear connection D, with local coefficients $D\Gamma(N) = \left(H^i{}_{jk}, C^i{}_{jk}, C_i{}^{jk}\right)$, to a metrical semisymmetric $N-$linear connection $\bar{D}$, with local coefficients $\bar{D}\Gamma(N) = \left(\bar{H}^i{}_{jk}, \bar{C}^i{}_{jk}, \bar{C}_i{}^{jk}\right)$.

We shall denote this transformation by: $t(\sigma, \tau, v)$. Thus we have:

Theorem 4.2. *The set $\overset{ms}{\mathcal{T}}_N$ of all transformations $t(\sigma, \tau, v) : D\Gamma(N) \longrightarrow \bar{D}\Gamma(N)$ of the metrical semisymmetric $N-$linear connections, given by (4.1) is an Abelian group, together with the mapping product.*

This group acts on the set of all metrical semisymmetric $N-$linear connections, corresponding to the same nonlinear connection N, transitively.

Theorem 4.3. *By means of transformation (4.1), the tensor fields:*
$\mathcal{K}_h{}^i{}_{jk}, \mathcal{P}_h{}^i{}_{jk}, \mathcal{P}_h{}^i{}_j{}^k, \mathcal{S}_h{}^i{}_{jk}$ and $\mathcal{S}_h{}^{ijk}$ are changed by the laws:

$$\bar{\mathcal{K}}_h{}^i{}_{jk} = \mathcal{K}_h{}^i{}_{jk} + \mathcal{A}_{jk}\left\{\Omega^{ir}_{jh}\sigma_{rk}\right\}, \tag{4.2}$$

$$\bar{\mathcal{P}}_h{}^i{}_{jk} = \mathcal{P}_h{}^i{}_{jk} + \mathcal{A}_{jk}\left\{\Omega^{ir}_{jh}\gamma_{rk}\right\}, \tag{4.3}$$

$$\bar{\mathcal{P}}_h{}^i{}_j{}^k = \mathcal{P}_h{}^i{}_{jk} + \mathcal{A}_{jk}\left\{\Omega^{ir}_{jh}\sigma_r\ |^k + \Omega^{ij}_{rh}v^r{}_{|k} + \Omega^{im}_{rh}\left(H^k{}_{mj}v^r + C_j{}^{rk}\sigma_m\right) + \right. \tag{4.4}$$
$$\left. + \frac{1}{2}\Omega^{ik}_{hj}\sigma_r v^r + \frac{1}{4}\delta^k_h g_{jr}g^{is}\sigma_s v^r + \frac{1}{4}\delta^i_j g_{rh}g^{ks}\sigma_s v^r - \frac{1}{4}g_{js}g^{ik}\sigma_h v^s - \frac{1}{4}g_{jh}g^{rk}\sigma_r v^i\right\}$$

$$\bar{S}_h{}^i{}_{jk} = S_h{}^i{}_{jk} + \mathcal{A}_{jk}\left\{\Omega^{ir}_{jh}\tau_{rk}\right\}, \tag{4.5}$$

$$\bar{S}_h{}^{ijk} = S_h{}^{ijk} + \mathcal{A}_{jk}\left\{\Omega^{ij}_{rh}v^{rk}\right\}, \tag{4.6}$$

where:

$$\sigma_{rk} = -\sigma_r\sigma_k + \sigma_{r|k} + \frac{1}{4}g_{rk}\cdot\sigma, (\sigma = g^{rm}\sigma_r\sigma_m), \tag{4.7}$$

$$\gamma_{rk} = -(\sigma_k\tau_r + \sigma_r\tau_k) + \sigma_r\,|_k + \tau_{r|k} + \frac{1}{4}g_{rk}\gamma, (\gamma = g^{rm}(\sigma_r\tau_m + \sigma_m\tau_r)), \tag{4.8}$$

$$\tau_{rk} = -\tau_r\tau_k + \tau_r\,|_k + \frac{1}{4}g_{rk}\tau,\ (\tau = g^{rs}\tau_r\tau_s), \tag{4.9}$$

$$v^{rk} = v^r v^k + v^r\,|^k - \frac{1}{4}g^{rk}v,\ (v = g_{rs}v^r v^s). \tag{4.10}$$

Proof. Using (2.1) and (4.1) we get:

$$B^i{}_{jk} = \frac{1}{2}\left(-\sigma_j\delta^i_k + g_{jk}g^{im}\sigma_m\right) = -\Omega^{im}_{kj}\sigma_m, D^i{}_{jk} = \frac{1}{2}\left(-\tau_j\delta^i_k + g_{jk}g^{im}\tau_m\right) = \\ -\Omega^{im}_{kj}\tau_m, D_i^{jk} = \frac{1}{2}\left(-v^j\delta^k_i + g^{jk}g_{im}v^m\right) = -\Omega^{jk}_{mi}v^m. \tag{4.11}$$

By applying Proposition 2.3, relations (3.12) and (4.11) we obtain the results. □

Using these results we can determine some invariants of the group $\overset{ms}{\mathcal{T}}_N$. To this aim we eliminate $\sigma_{ij}, \gamma_{ij}, \tau_{ij}$ and v^{ij} from (4.2), (4.3), (4.4) and (4.5) and we obtain:

Theorem 4.4. *For $n > 2$ the following tensor fields: $H_h{}^i{}_{jk}, N_h{}^i{}_{jk}, M_h{}^i{}_{jk}M_h{}^{ijk}$, of metrical semisymmetric $N-$linear connections on $T^{*2}M$, are invariants of the group $\overset{ms}{\mathcal{T}}_N$:*

$$H_h{}^i{}_{jk} = \mathcal{K}_h{}^i{}_{jk} + \frac{2}{n-2}\mathcal{A}_{jk}\left\{\Omega^{ir}_{jh}\left(\mathcal{K}_{rk} - \frac{g_{rk}\mathcal{K}}{2(n-1)}\right)\right\}, \tag{4.12}$$

$$N_h{}^i{}_{jk} = \mathcal{P}_h{}^i{}_{jk} + \frac{2}{n-2}\mathcal{A}_{jk}\left\{\Omega^{ir}_{jh}\left(\mathcal{P}_{rk} - \frac{g_{rk}\mathcal{P}}{2(n-1)}\right)\right\}, \tag{4.13}$$

$$M_h{}^i{}_{jk} = S_h{}^i{}_{jk} + \frac{2}{n-2}\mathcal{A}_{jk}\left\{\Omega^{ir}_{jh}\left(S_{rk} - \frac{g_{rk}S}{2(n-1)}\right)\right\}, \tag{4.14}$$

$$M_h{}^{ijk} = S_h{}^{ijk} + \frac{2}{n-2}\mathcal{A}_{jk}\left\{\Omega^{ij}_{rh}\left(S^{rk} - \frac{g^{rk}S'}{2(n-1)}\right)\right\}, \tag{4.15}$$

where:

$$\begin{aligned} \mathcal{K}_{hj} &= \mathcal{K}_h{}^i{}_{ji}, \mathcal{K} = g^{hj}\mathcal{K}_{hj}, \mathcal{P}_{hj} = \mathcal{P}_h{}^i{}_{ji}, \mathcal{P} = g^{hj}\mathcal{P}_{hj}, \\ S_{hj} &= S_h{}^i{}_{ji}, S = g^{hj}S_{hj}, S^{ij} = S_h{}^{ijh}, S' = g_{ij}S^{ij}. \end{aligned}$$

In order to find other invariants of the group $\overset{ms}{\mathcal{T}}_N$, let us consider the transformation formulas of the torsion $d-$tensor fields by a transformation $t(\sigma,\tau,v) : D\Gamma(N) \longrightarrow \bar{D}\Gamma(N)$ of metrical semisymmetric $N-$linear connections on $T^{*2}M$, corresponding to the same nonlinear connection N, given by (4.1). Using Proposition 2.1 and transformation (4.1), by direct calculation we obtain:

Proposition 4.1. *By a transformation* (4.1) *of metrical semisymmetric* $N-$ *linear connections, corresponding to the same nonlinear connection* N :
$t(\sigma,\tau,v) : D\Gamma(N) \longrightarrow \bar{D}\Gamma(N)$, *the torsion tensor fields:* $\underset{(1)}{R}{}^i{}_{jk}$, $\underset{(2)}{R}{}_{ijk}$, $\underset{(1)}{B}{}_j{}^{ik}$, $\underset{(1)}{B}{}^i{}_{jk}$, $\underset{(2)}{B}{}_{ijk}$, $T^i{}_{jk}$, $S^i{}_{jk}$, $S_i{}^{jk}$, $\underset{(1)}{P}{}^i{}_{jk}$, $\underset{(2)}{P}{}^i{}_{jk}$ *are transformed as follows:*

$$\begin{cases} \underset{(1)}{\bar{R}}{}^i{}_{jk} = \underset{(1)}{R}{}^i{}_{jk},\ \underset{(2)}{\bar{R}}{}_{ijk} = \underset{(2)}{R}{}_{ijk}, \\ \underset{(1)}{\bar{B}}{}^i{}_{jk} = \underset{(1)}{B}{}^i{}_{jk},\ \underset{(2)}{\bar{B}}{}^k{}_{ij} = \underset{(2)}{B}{}^k{}_{ij}, \\ \underset{(1)}{\bar{B}}{}^i{}_j{}^k = \underset{(1)}{B}{}^i{}_j{}^k,\ \underset{(2)}{\bar{B}}{}_{ijk} = \underset{(2)}{B}{}_{ijk}, \\ \bar{T}^i{}_{jk} = T^i{}_{jk} + \frac{1}{2}\mathcal{A}_{jk}\left\{\sigma_j\delta^i_k\right\}, \\ \bar{S}^i{}_{jk} = S^i{}_{jk} + \frac{1}{2}\mathcal{A}_{jk}\left\{\tau_j\delta^i_k\right\}, \\ \bar{S}_i{}^{jk} = S_i{}^{jk} + \frac{1}{2}\mathcal{A}_{jk}\left\{v^j\delta^k_i\right\}, \\ \underset{(1)}{\bar{P}}{}^i{}_{jk} = \underset{(1)}{P}{}^i{}_{jk} + \frac{1}{2}\left(-\sigma_k\delta^i_j + g_{jk}g^{im}\sigma_m\right), \\ \underset{(2)}{\bar{P}}{}^i{}_{jk} = \underset{(2)}{P}{}^i{}_{jk} - \frac{1}{2}\left(-\sigma_j\delta^i_k + g_{jk}g^{im}\sigma_m\right). \end{cases} \tag{4.16}$$

We denote with:

$$\underset{(1)}{t}{}^i{}_{jk} = \mathcal{A}_{jk}\left\{\frac{\partial N^i{}_j}{\partial y^k}\right\},\ \underset{(2)}{t}{}_j{}^{ik} = \mathcal{A}_{jk}\left\{\frac{\partial N^i{}_j}{\partial p_k}\right\},\ \underset{(3)}{t}{}^i{}_{jk} = \mathcal{A}_{jk}\left\{\frac{\partial N_{jk}}{\partial p_i}\right\}, \tag{4.17}$$

and with:

$$\begin{cases} \underset{(1)}{t}{}^*{}_{ijk} = \Sigma_{ijk}\left\{g_{im}\underset{(1)}{t}{}^m{}_{jk}\right\}, \\ \underset{(2)}{t}{}^*{}_{ij}{}^k = \Sigma_{ijk}\left\{g_{im}\underset{(2)}{t}{}_j{}^{mk}\right\}, \\ \underset{(3)}{t}{}^*{}_{ijk} = \Sigma_{ijk}\left\{g_{im}\underset{(3)}{t}{}^m{}_{jk}\right\}, \\ T^*{}_{ijk} = \Sigma_{ijk}\left\{g_{im}T^m{}_{jk}\right\}, \\ \underset{(1)}{R}{}^*{}_{ijk} = \Sigma_{ijk}\left\{g_{im}\underset{(1)}{R}{}^m{}_{jk}\right\}, \\ C^*{}_{ijk} = \Sigma_{ijk}\left\{g_{im}C^m{}_{jk}\right\}, \end{cases} \tag{4.18}$$

$$\begin{cases} \underset{(1)}{P}{}^{*}{}_{ijk} = \Sigma_{ijk}\left\{ g_{im}\underset{(1)}{P}{}^{m}{}_{jk}\right\}, \\ \underset{(2)}{P}{}^{*}{}_{ijk} = \Sigma_{ijk}\left\{ g_{im}\underset{(2)}{P}{}^{m}{}_{jk}\right\}, \\ S^{*}{}_{ijk} = \Sigma_{ijk}\left\{ g_{im}S^{m}{}_{jk}\right\}, \\ \underset{(1)}{B}{}^{*}{}_{ijk} = \Sigma_{ijk}\left\{ g_{im}\underset{(1)}{B}{}^{m}{}_{jk}\right\}, \\ \overset{1}{B}{}^{*}{}_{ijk} = \Sigma_{ijk}\left\{ g_{im}\mathcal{A}_{jk}\left\{\underset{(1)}{B}{}^{m}{}_{jk}\right\}\right\}, \\ \overset{2}{B}{}^{*}{}_{ijk} = \Sigma_{ijk}\left\{ g_{im}\mathcal{A}_{jk}\left\{\underset{(2)}{B}{}^{m}{}_{jk}\right\}\right\}, \end{cases} \tag{4.19}$$

where $\Sigma_{ijk}\{...\}$ *denotes the cyclic summation and with:*

$$\begin{cases} \underset{(1)}{\overset{1}{K}}{}_{ijk} = -g_{km}T^{m}{}_{ij} + \mathcal{A}_{ij}\left\{ g_{im}\underset{(1)}{P}{}^{m}{}_{jk}\right\}, \\ \underset{(2)}{\overset{1}{K}}{}_{ijk} = g_{im}T^{m}{}_{jk} - \mathcal{A}_{jk}\left\{ g_{km}H^{m}{}_{ij}\right\}, \\ \overset{2}{K}{}_{ijk} = g_{im}S^{m}{}_{jk} - \mathcal{A}_{jk}\left\{ g_{km}C^{m}{}_{ij}\right\}, \\ \overset{3}{K}{}_{ijk} = \mathcal{A}_{jk}\left\{ g_{km}\left(\frac{1}{2}\underset{(1)}{P}{}^{m}{}_{ij} + \underset{(2)}{P}{}^{m}{}_{ij}\right)\right\}, \\ \overset{4}{K}{}_{ijk} = g_{jm}C^{m}{}_{ik} + g_{im}C^{m}{}_{jk}, \\ \overset{1}{\varphi}{}_{ijk} = \mathcal{A}_{ij}\left\{ g_{im}\underset{(1)}{B}{}^{m}{}_{jk}\right\} \\ \overset{2}{\varphi}{}_{ijk} = \mathcal{A}_{jk}\left\{ g_{jm}\mathcal{A}_{ik}\left\{\underset{(1)}{B}{}^{m}{}_{ik}\right\}\right\}, \\ \overset{3}{\varphi}{}_{ijk} = \mathcal{A}_{ik}\left\{ -g_{jm}\underset{(2)}{P}{}^{m}{}_{ik} - g_{km}\underset{(1)}{P}{}^{m}{}_{ij}\right\}. \end{cases} \tag{4.20}$$

Remark 4.1. It is noted that $\underset{(1)}{t}{}^{*}{}_{ijk}$, $\underset{(2)}{t}{}^{*}{}_{ij}{}^{k}$, $\underset{(3)}{t}{}^{*}{}_{ijk}, T^{*}{}_{ijk}, \underset{(1)}{R}{}^{*}{}_{ijk}, S^{*}{}_{ijk}$, $\overset{1}{B}{}^{*}{}_{ijk}, \overset{2}{B}{}^{*}{}_{ijk}$ are alternate, $\underset{(1)}{\overset{1}{K}}{}_{ijk}, \overset{1}{\varphi}{}_{ijk}$ are alternate with respect to i, j, $\underset{(2)}{\overset{1}{K}}{}_{ijk}$, $\overset{2}{K}{}_{ijk}, \overset{3}{K}{}_{ijk}, \overset{2}{\varphi}{}_{ijk}$ are alternate with respect to j, k and $\overset{3}{\varphi}{}_{ijk}$ is alternate with respect to i, k.

Theorem 4.5. *The tensor fields:* $\underset{(1)}{R}{}^{i}{}_{jk}$, $\underset{(2)}{R}{}_{ijk}$, $\underset{(1)}{B}{}_{j}{}^{ik}$, $\underset{(2)}{B}{}_{ijk}$, $\underset{(1)}{t}{}^{i}{}_{jk}$, $\underset{(2)}{t}{}_{j}{}^{ik}$, $\underset{(3)}{t}{}^{i}{}_{jk}$, $\underset{(1)}{t}{}^{*}{}_{ijk}$, $\underset{(2)}{t}{}^{*}{}_{ij}{}^{k}$, $\underset{(3)}{t}{}^{*}{}_{ijk}, T^{*}{}_{ijk}$, $\underset{(1)}{R}{}^{*}{}_{ijk}, C^{*}{}_{ijk}$, $\underset{(1)}{P}{}^{*}{}_{ijk}$, $\underset{(2)}{P}{}^{*}{}_{ijk}, S^{*}{}_{ijk}$, $\underset{(1)}{B}{}^{*}{}_{ijk}, \overset{1}{B}{}^{*}{}_{ijk}$, $\overset{2}{B}{}^{*}{}_{ijk}$, $\underset{(1)}{\overset{1}{K}}{}_{ijk}$, $\underset{(2)}{\overset{1}{K}}{}_{ijk}$, $\overset{2}{K}{}_{ijk}, \overset{3}{K}{}_{ijk}$, $\overset{4}{K}{}_{ijk}, \overset{1}{\varphi}{}_{ijk}$, $\overset{2}{\varphi}{}_{ijk}$, $\overset{3}{\varphi}{}_{ijk}$ *are invariants of the group* $\overset{ms}{\mathcal{T}}{}_{N}$.

Proof. By means of transformations of the torsion given in (4.16) and using the notations: (4.17), (4.18), (4.19) (4.20) by direct calculation, from (4.1) we obtain the results. □

Theorem 4.6. *Between the invariants in Theorem 4.5 the following relations exist:*

$$\Sigma_{ijk}\left\{\underset{(1)}{\overset{1}{K}}{}_{ijk}\right\} = -T^{*}{}_{ijk} + \mathcal{A}_{ij}\left\{\underset{(1)}{P}{}^{*}{}_{ijk}\right\} = \underset{(1)}{t}{}^{*}{}_{ijk}, \tag{4.21}$$

$$\Sigma_{ijk}\left\{\underset{(2)}{\overset{1}{K}}{}_{ijk}\right\} = 0, \tag{4.22}$$

$$\Sigma_{ijk}\left\{\overset{2}{K}{}_{ijk}\right\} = 0, \tag{4.23}$$

$$\Sigma_{ijk}\left\{\overset{3}{K}{}_{ijk}\right\} = \frac{3}{2}T^{*}{}_{ijk} + \frac{1}{2}\underset{(1)}{t}{}^{*}{}_{ijk} + \underset{(3)}{t}{}^{*}{}_{ijk}, \tag{4.24}$$

$$\Sigma_{ijk}\left\{\overset{4}{K}{}_{ijk}\right\} = C^{*}{}_{ijk} + C^{*}{}_{ikj},\ \mathcal{A}_{ij}\left\{\overset{4}{K}{}_{ijk}\right\} = 0, \tag{4.25}$$

$$\Sigma_{ijk}\left\{\overset{1}{\varphi}{}_{ijk}\right\} = \underset{(1)}{B}{}^{*}{}_{ijk} - \underset{(1)}{B}{}^{*}{}_{ikj} = \mathcal{A}_{jk}\left\{\underset{(1)}{B}{}^{*}{}_{ijk}\right\}, \tag{4.26}$$

$$\Sigma_{ijk}\left\{\overset{2}{\varphi}{}_{ijk}\right\} = -2\left(\underset{(1)}{B}{}^{*}{}_{ijk} - \underset{(1)}{B}{}^{*}{}_{ikj}\right) = -2\mathcal{A}_{jk}\left\{\underset{(1)}{B}{}^{*}{}_{ijk}\right\}, \tag{4.27}$$

$$\Sigma_{ijk}\left\{\overset{3}{\varphi}{}_{ijk}\right\} = \underset{(2)}{P}{}^{*}{}_{ijk} - \underset{(1)}{P}{}^{*}{}_{ijk} + \underset{(1)}{P}{}^{*}{}_{ikj} - \underset{(2)}{P}{}^{*}{}_{ikj} = -\left(\overset{1}{B}{}^{*}{}_{ijk} + \overset{2}{B}{}^{*}{}_{ijk}\right). \tag{4.28}$$

Proof. Using notations (4.17) , (4.18) , (4.19), (4.20) Remark 4.1 and the definitions of the torsion $d-$tensor fields given in [8] - p.256 and - 273, by direct calculations we obtain the results. □

References

[1] Atanasiu, Gh. and Târnoveanu, M., *New aspects in the differential geometry of the second order cotangent bundle*, Univ. de Vest din Timişoara, **90** (2005), 1-64.

[2] Atanasiu, Gh., *The invariant expression of Hamilton geometry*, Tensor N.S., Japonia, **47** (1988), 23-32.

[3] Ianuş, S., *On differential geometry of the dual of a vector bundle*, The Proc. of the Fifth National Sem. of Finsler and Lagrange Spaces, Univ. din Braşov, 1988, 173-180.

[4] Matsumoto, M., *The theory of Finsler connections*, Publ. of the Study Group of Geometry 5, Depart. Math. Okayama Univ., 1970, XV + 220pp.

[5] Miron, R., *Hamilton geometry*, Seminarul de Mecanică , Univ. Timişoara **3** (1987), 1-54.

[6] Miron, R., *Hamilton spaces of order k grater than or equal to 1*, Int. Journal of Theoretical Phys. **39** (2000), no. 9, 2327-2336.

[7] Miron, R., Ianuş, S. and Anastasiei, M., *The geometry of the dual of a vector bundle,* Publ. de l'Inst. Math., **46(60)** (1989), 145-162.

[8] Miron, R., Hrimiuc, D., Shimada, H. and Sabău, V.S., *The geometry of Hamilton and Lagrange spaces*, Kluwer Academic Publisher, FTPH, 118, 2001.

[9] Purcaru, M.A.P. and Târnoveanu, M., *On transformation group of N-linear connections one second order cotangent bundle* (to appear).

[10] Saunders, D.J., *The geometry of jet bundles*, Cambridge Univ. Press, 1989.

[11] Udrişte, C., Şandru, O., *Dual nonlinear connections*, Proc. of 22^{nd} Conference Differential Geometry and Topology, Polytechnic Institute of Bucharest, Romania, sept., 1991.

[12] Yano, K., Ishihara, S., *Tangent and cotangent bundles*. Differential Geometry. M. Dekker, Inc., New-York, 1973.