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ON A GENERALIZED CASCADING FAILURE MODEL

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Abstract

In this paper we present a new approach of the generalized cascading failure model due to Lefèvre (2006). We derive some corresponding lower bounds.

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1 Introduction

Cascading failure models are intensely studied in literature, especially due to their multiple applications in electrical engineering. Dobson et al. (2005) have proposed a basic model which describes the occurrence of cascading failures for a system of n components which is subject to a disturbance.

In this model, each component i has an initial load $L_i^{(0)}$. Assume that the loads $L_1^{(0)}, \cdots, L_n^{(0)}$ are independent uniform on $(0,1)$ random variables. After the disturbance, the new load of each i is $L_i^{(1)} = L_i^{(0)} + d$, where $d \in (0,1)$.

If, for $j \in \{1, \cdots, n\}$, $L_j^{(1)} > 1$, then:
- the component j fails;
- the failure of j increases with p the load $L_i^{(1)}$ of each $i \neq j$.

Denote $J_1 = \{j \ : \ L_j^{(1)} > 1\}$ and let $n_1 = |J_1|$ be the number of (first) failures. If, for $j \notin J_1$, $L_j^{(2)} = L_j^{(1)} + n_1 p > 1$, then:
- the component j fails;
- the failure of j increases with p the load $L_i^{(2)}$ of each $i \notin J_1 \cup \{j\}$.

Now, for $J_2 = \{j \in \{1, \cdots, n\} \setminus J_1 \ : \ L_j^{(2)} > 1\}$, let $n_2 = |J_2|$ be the number of "second" failures, and so on. Then $N = n_1 + n_2 + \cdots$ is the total number of failures. Assume that $d + np < 1$ (the non-saturation condition). Dobson et al. (2005) proved that the distribution of N is quasi-binomial (in the sense of Consul (1974))

$$\mathbb{P}\{N = k\} = \binom{n}{k} d(d+kp)^{k-1}(1-d-kp)^{n-k}, \qquad (1)$$

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for $k = 0, 1, \cdots, n$.

Mention that, some asymptotic results, in terms of the generalized Poisson distribution, are also given in Dobson et al. (2005). The branching model for cascading failure and the approximation of a loading-dependent cascading failure model with a branching process are discussed, for example, in Dobson et al. (2010) and Kim and Dobson (2010).

Inspired by an epidemic model - SIR schema (see, e.g., Ball and O'Neill (1999)), Lefèvre (2006) defines the following generalized model. Since $\mathbb{P}(L_i = L_j) = 0$ for $i \neq j$, we may consider the sequence $j_1,\ j_2, \cdots$ of successive failed components, such that the failure of j_k increases with $p_{k+1} \in (0,1)$ the load of all functioning components i,

$$i \in \{1, \cdots, n\} \setminus \{j_1, \cdots, j_k\}.$$

Denote: $s_1 = p_1 = d$ and $s_k = p_1 + \cdots + p_k$, for $1 < k \leq n$ (assuming that $s_n < 1$), and $U_i = 1 - L_i,\ i = 1, \cdots, n$. Let $U_{1:n} < U_{2:n} < \cdots < U_{n:n}$ be the order statistics from the sample $(U_1, \cdots, U_n)$. Then, for $k = 1, \cdots, n-1$,

$$\mathbb{P}\{N = k\} = \mathbb{P}\left(U_{1:n} < s_1, \cdots, U_{k:n} < s_k,\ U_{k+1:n} > s_{k+1}\right). \tag{2}$$

Lefèvre (2006) proves that the probabilities of the events $\{N = j\}$ are the solutions of the linear system

$$\sum_{j=0}^{k} \binom{n-j}{k-j} \mathbb{P}\{N = j\} \frac{1}{(1 - s_{1+j})^{n-k}} = \binom{n}{k}, \tag{3}$$

for $k = 0, 1, \cdots, n$. The same author gives some bounds for the distribution of N.

In this paper we propose an alternative method to obtain these probabilities.

2 An appropriate method for the Lefèvre's model

The distribution of the random variable N can be obtained by solving the triangular system (3). More precisely, $\mathbb{P}\{N = k\}$ depends on $\mathbb{P}\{N = i\},\ i < k$. We propose an alternative method to compute $\mathbb{P}\{N = k\}$. Our method is based on the recurring construction of an appropriate sequence of polynomials.

Let us consider the sequence of polynomials

$$g_0(x_0), g_1(x_0, x_1), \cdots, g_k(x_0, x_1, \cdots, x_k), \cdots$$

defined by $g_0(x_0) = 1,\ g_1(x_0, x_1) = x_1 - x_0$, and the recurrence relation

$$g_{k+1}(x_0, x_1, \cdots, x_{k+1}) = \int_{x_0}^{x_1} g_k(t, x_2, \cdots, x_{k+1})dt, \text{ for } k = 1, 2, \cdots.$$

Theorem 1. *Assume* $0 < s_1 < \cdots < s_n < 1$. *Then, for* $k = 1, \cdots, n-1$, *we have*

$$\mathbb{P}\{N = k\} = \frac{n!}{(n-k)!} g_k(0, s_1, \cdots, s_k)(1 - s_{k+1})^{n-k}.$$

Proof. As above, let $U_1, U_2, \cdots, U_n$ be a sequence of independent uniform on $(0,1)$ random variables. Suppose $0 \le s_0 < s_1$. We will proceed by induction to prove that the polynomials g_k have the following meaning

$$g_k(s_0, s_1, \cdots, s_k) = \mathbb{P}\{s_0 < U_1 < s_1;\ U_{i-1} < U_i < s_i,\ 2 \le i \le k\}, \tag{4}$$

for $k = 1, 2, \cdots, n$. Clearly, for $k = 1$, $\mathbb{P}\{s_0 < U_1 < s_1\} = s_1 - s_0 = g_1(s_0, s_1)$. Assume that (4) holds for $k \in \{1, \cdots, n-1\}$. Then

$$\mathbb{P}\{s_0 < U_1 < s_1;\ U_{i-1} < U_i < s_i,\ 2 \le i \le k+1\}$$

$$= \int_{s_0}^{s_1} \mathbb{P}\{t < U_2 < s_2;\ U_{i-1} < U_i < s_i,\ 3 \le i \le k+1\} d(\mathbb{P}\{U_1 < t\})$$

$$= \int_{s_0}^{s_1} g_k(t, s_2, \cdots, s_{k+1}) dt = g_{k+1}(s_0, s_1, \cdots, s_{k+1}).$$

From (4) we obtain

$$g_k(s_0, s_1, \cdots, s_k) = \mathbb{P}\{s_0 < U_{j_1} < s_1;\ \ U_{j_{i-1}} < U_{j_i} < s_i,\ 2 \le i \le k\},$$

for any ordered sequence $(j_1, \cdots, j_k)$ of the set $\{1, \cdots, n\}$. Therefore, using (2),

$$\mathbb{P}\{N = k\}$$

$$= \sum_{(j_1, \cdots, j_k)} \mathbb{P}\{U_{j_1} < s_1;\ \ U_{j_{i-1}} < U_{j_i} < s_i,\ 2 \le i \le k\} \mathbb{P}\{U_j > s_{k+1},\ j \notin \{j_1, \cdots, j_k\}\}$$

$$= \frac{n!}{(n-k)!} g_k(0, s_1, \cdots, s_k)(1 - s_{k+1})^{n-k},$$

for $1 \le k \le n-1$. □

We easily verify

$$g_k(x, d, d+p, \cdots, d+(k-1)p) = \frac{(d-x)(d+kp-x)^{k-1}}{k!},$$

and we check the well-known result (1).

Finally, we will prove two immediate consequences of the above theorem.

Corollary 1. *If* $0 \le x_0 < x_1 < \cdots < x_k \le 1$ $(k \ge 1)$ *then*

$$g_k(x_0, x_1, \cdots, x_k) \ge g_j(x_0, \cdots, x_j) g_{k-j}(x_j, \cdots, x_k),\ j \in \{0, 1, \cdots, k\}, \tag{5}$$

and

$$g_k(x_0, x_1, \cdots, x_k) \ge \prod_{i=1}^{k} (x_i - x_{i-1}). \tag{6}$$

Proof. For $j \in \{0, k\}$, we have equality in (5). Suppose $0 < j < k$ (with $k > 1$). Let $(U_1, \cdots, U_k)$ be a sample of size k of independent uniform on $(0,1)$ random variables. Let us consider the events $A = \{x_0 < U_1 < x_1;\ U_{i-1} < U_i < x_i,\ 2 \le i \le j\}, B = \{x_j < U_{j+1} < x_{j+1};\ U_{i-1} < U_i < x_i,\ j+2 \le i \le k\}$ and $C = \{x_0 < U_1 < x_1;\ U_{i-1} < U_i < x_i,\ 2 \le i \le k\}$. Obviously, $\{U_j < x_j\} \cap \{U_{j+1} > x_j\} \subset \{U_j < U_{j+1};\ U_j < x_j\}$. Hence $A \cap B \subset C$. In addition, A and B are independent events. The inequality (5) is then a consequence of the probabilistic meaning of the polynomials g_i. By induction, we get $g_k(x_0, x_1, \cdots, x_k) \ge \prod_{i=1}^k g_1(x_{i-1}, x_i) = \prod_{i=1}^k (x_i - x_{i-1})$. □

Corollary 2. *If* $0 \le x_0 < x_1 < \cdots < x_k \le 1$ $(k \ge 1)$ *then*

$$g_k(x_0, x_1, \cdots, x_k) \ge \max_{1 \le j \le k-1} \left\{ \frac{1}{j!(k-j)!} \prod_{i=1}^{j} (x_i - x_0) \prod_{i=j+1}^{k} (x_i - x_j) \right\}.$$

Proof. For $0 \le p \le k-1$, $1 \le i \le k-p$ and a sample $(U_1, \cdots, U_i)$ of independent uniform on $(0,1)$ random variables, we easily observe that $i! g_i(x_p, x_{p+1}, \cdots, x_{p+i}) = \mathbb{P}\{U_{j:i} \in (x_p, x_{p+j}),\ 1 \le j \le i\} \ge \mathbb{P}\{U_j \in (x_p, x_{p+j}),\ 1 \le j \le i\} = \prod_{j=1}^i (x_{p+j} - x_p)$. Then we apply the inequality (5). □

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