

## QUATERNION/VECTOR DUAL SPACE ALGEBRAS APPLIED TO THE DIRAC EQUATION AND ITS EXTENSIONS

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### Abstract

The paper re-applies the 64-part algebra discussed by P. Rowlands in a series of (FERT and other) papers in the recent years. It demonstrates that the original introduction of the  $\gamma$  algebra by Dirac to "the quantum theory of the electron" can be interpreted with the help of quaternions: both the  $\alpha$  matrices and the Pauli ( $\sigma$ ) matrices in Dirac's original interpretation can be rewritten in quaternion forms. This allows to construct the Dirac  $\gamma$  matrices in two (quaternion-vector) matrix product representations - in accordance with the double vector algebra introduced by P. Rowlands. The paper attempts to demonstrate that the Dirac equation in its form modified by P. Rowlands, essentially coincides with the original one. The paper shows that one of these representations affects the  $\gamma_4$  and  $\gamma_5$  matrices, but leaves the vector of the Pauli spinors intact; and the other representation leaves the  $\gamma$  matrices intact, while it transforms the spin vector into a quaternion pseudovector. So the paper concludes that the introduction of quaternion formulation in QED does not provide us with additional physical information. These transformations affect all gauge extensions of the Dirac equation, presented by the author earlier, in a similar way. This is demonstrated by the introduction of an additional parameter (named earlier isotopic field-charge spin) and the application of a so-called tau algebra that governs its invariance transformation. The invariance of the extended Dirac equation is subject of the convolution of the Lorentz transformation and this invariance group. It is shown that additional physical information can be expected by extending these investigations to the Finsler-like metric proposed by Dirac in 1962 for the theory of the electron.

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## 1 Vector algebra, quaternion algebra

Several statements in this paper continue certain ideas set forth in recent years publications by P. Rowlands [1, 2, 3, 4].

Vectors can be transformed into quaternions multiplying them by the imaginary unit, and vice versa. When one substitutes vectors with quaternions or quaternions with vectors obtained in this way in physical equations, they will not change the physical meaning of the equations. In other words, they do not bring in extra physical information. (This statement does not concern certainly those quaternions and vectors that have not been obtained in this way.)

We use the following notations:  $\mathbf{i} = \mathbf{j} = \mathbf{k}$  vectors,  $i = j = k$  quaternions,  $\mathbf{i} = i\mathbf{i}$ .

Vectors are subjects of the following algebra:

$$\begin{aligned} \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = 1 & \quad \mathbf{ijk} = i \\ \text{anticommutation rules: } & \{\mathbf{i}, \mathbf{j}\} = \{\mathbf{j}, \mathbf{k}\} = \{\mathbf{k}, \mathbf{i}\} = 0 \\ \text{commutation rules: } & [\mathbf{i}, \mathbf{j}] = 2i\mathbf{k}, [\mathbf{j}, \mathbf{k}] = 2i\mathbf{i}, [\mathbf{k}, \mathbf{i}] = 2ij. \end{aligned}$$

*Quaternions* are subjects of the following algebra

$$i^2 = (-i)^2 i^2 = -1, \quad j^2 = (-i)^2 j^2 = -1, \quad k^2 = (-i)^2 k^2 = -1, \quad \mathbf{ijk} = -i.$$

$$\begin{aligned} \text{anticommutation rules: } & \{i, j\} = \{j, k\} = \{k, i\} = 0 \\ \text{commutation rules: } & [i, j] = 2k, [j, k] = 2i, [k, i] = 2j. \end{aligned}$$

Let's apply this algebra to Pauli's and Dirac's matrices with the following notations (the upper case characters refer to the notations introduced by P. Rowlings):

$$\begin{aligned} \mathbf{i} = \sigma_1 & \quad \mathbf{j} = \sigma_2 & \quad \mathbf{k} = \sigma_3 \\ i = -i\mathbf{i} = -i\mathbf{i}\sigma_1 & \quad j = -i\mathbf{j} = -ij\sigma_2 & \quad k = -i\mathbf{k} = -i\sigma_3 \\ \sigma_i = -i\sigma_i & \quad \sigma_i = i\sigma_i & \quad (\mathbf{i}, \mathbf{j}, \mathbf{k}) \\ \rho_i = -i\rho_i & \quad \sigma_i = i\rho_i & \quad (\mathbf{I}, \mathbf{J}, \mathbf{K}) \end{aligned}$$

Using these definitions, we can interpret Dirac's matrices as a double vector algebra, (presented in vector and quaternion combinations). They can be expressed as products of  $\rho$  and  $\sigma$  matrices. Dirac introduced [5] first matrices marked  $\alpha$ , then the matrices  $\gamma$ , named after him.

$$\begin{aligned} \alpha_i & = \rho_1 \sigma_i = i\rho_1 \sigma_i = i\rho_1 \sigma_i = -i\rho_1 \sigma_i \\ \gamma_i & = \rho_2 \sigma_i = i\rho_2 \sigma_i = i\rho_2 \sigma_i = -i\rho_2 \sigma_i \end{aligned}$$

Observe, that the  $\alpha$  and  $\gamma$  matrices are generated by the help of the  $\rho_1$  and  $\rho_2$  matrices as coefficients of the  $\sigma_i$  matrices. We extend this set of matrices with matrices to be called  $\beta$ , generated by the help of  $\rho_3$  :

$$\beta_i = \rho_3 \sigma_i = i\rho_3 \sigma_i = i\rho_3 \sigma_i = -i\rho_3 \sigma_i.$$

The latter did not appear in the system of matrices introduced by Dirac. Notice, that all the above product combinations of either vectors, quaternions or both provide vectors. The vector and quaternion algebra of all the above listed five types of matrices are summarized in the following table:

$$\begin{array}{l}
 \rho_i: \quad \rho_i^2 = 1 \quad \rho_i \rho_j = -\rho_j \rho_i \quad \rho_i \rho_j = -i \rho_k \quad \rho_i = -i \rho_i \quad \rho_i^2 = -1 \quad \rho_i \rho_j = -\rho_j \rho_i \quad \rho_i \rho_j = \rho_k \\
 \sigma_i: \quad \sigma_i^2 = 1 \quad \sigma_i \sigma_j = -\sigma_j \sigma_i \quad \sigma_i \sigma_j = -i \sigma_k \quad \sigma_i = -i \sigma_i \quad \sigma_i^2 = -1 \quad \sigma_i \sigma_j = -\sigma_j \sigma_i \quad \sigma_i \sigma_j = \sigma_k \\
 \alpha_i = \rho_1 \sigma_i \quad \alpha_i^2 = 1 \quad \alpha_i \alpha_j = -\alpha_j \alpha_i \quad \alpha_i \alpha_j = -i \sigma_k \quad \alpha_i = -i \alpha_i \quad \alpha_i^2 = -1 \quad \alpha_i \alpha_j = -\alpha_j \alpha_i \quad \alpha_i \alpha_j = \sigma_k \\
 \gamma_i = \rho_2 \sigma_i \quad \gamma_i^2 = 1 \quad \gamma_i \gamma_j = -\gamma_j \gamma_i \quad \gamma_i \gamma_j = -i \sigma_k \quad \gamma_i = -i \gamma_i \quad \gamma_i^2 = -1 \quad \gamma_i \gamma_j = -\gamma_j \gamma_i \quad \gamma_i \gamma_j = \sigma_k \\
 \beta_i = \rho_3 \sigma_i \quad \beta_i^2 = 1 \quad \beta_i \beta_j = -\beta_j \beta_i \quad \beta_i \beta_j = -i \sigma_k \quad \beta_i = -i \beta_i \quad \beta_i^2 = -1 \quad \beta_i \beta_j = -\beta_j \beta_i \quad \beta_i \beta_j = \sigma_k
 \end{array}$$

We must note that they are not all independent. These five matrices can be expressed by two of the others (but not in all arbitrary combinations).  $\sigma$  plays a distinguished role what restricts the number of allowed representations. The different pairings provide different dual representations of the Dirac algebra. We see that they can be represented both in vector and quaternion (pseudovector) forms.

Note, that  $\alpha$ ,  $\gamma$  and  $\beta$  do not transform according to the  $\mathbf{i}$ ,  $\mathbf{j}$ ,  $\mathbf{k}$  vector algebra, like  $\rho$  and  $\sigma$  do. Their commutators are not produced from their own set, but they are a component of  $\sigma$ .

Another observation is that the roles of the  $\rho$  and  $\sigma$  matrices show similitude to the roles of  $U$  and  $T$  in the genetic algebra [6]!

The roles of  $\rho$  and  $\sigma$  can be changed. This allows an interesting observation and leads to a third (mixed) algebra. Nevertheless, this allows another set of representations.

For  $\rho$  and  $\sigma$  commute,  $\rho_i \sigma_j = \rho_j \sigma_i$ , let's compose the following combinations:

$$\begin{array}{l}
 \rho_1 \sigma_1 = \sigma_1 \rho_1 = \alpha_1 = \kappa_1 \quad \rho_2 \sigma_1 = \sigma_1 \rho_2 = \gamma_1 = \kappa_2 \quad \rho_3 \sigma_1 = \sigma_1 \rho_3 = \beta_1 = \kappa_3 \\
 \rho_1 \sigma_2 = \sigma_2 \rho_1 = \alpha_2 = \lambda_1 \quad \rho_2 \sigma_2 = \sigma_2 \rho_2 = \gamma_2 = \lambda_2 \quad \rho_3 \sigma_2 = \sigma_2 \rho_3 = \beta_2 = \lambda_3 \\
 \rho_1 \sigma_3 = \sigma_3 \rho_1 = \alpha_3 = \mu_1 \quad \rho_2 \sigma_3 = \sigma_3 \rho_2 = \gamma_3 = \mu_2 \quad \rho_3 \sigma_3 = \sigma_3 \rho_3 = \beta_3 = \mu_3
 \end{array}$$

These equalities serve as definitions of the  $\kappa$ ,  $\lambda$  and  $\mu$  matrices.  $\kappa$ ,  $\lambda$ ,  $\mu$  are defined by a new indexation (row-column change), they provide no new matrices, however, allow to compose new vectors, in the following way:

$$\begin{array}{l}
 \sigma_1 \rho_i = \kappa_i \quad \sigma_2 \rho_i = \lambda_i \quad \sigma_3 \rho_i = \mu_i \\
 \begin{pmatrix} \alpha_1 \\ \gamma_1 \\ \beta_1 \end{pmatrix} = \kappa_i \quad \begin{pmatrix} \alpha_2 \\ \gamma_2 \\ \beta_2 \end{pmatrix} = \lambda_i \quad \begin{pmatrix} \alpha_3 \\ \gamma_3 \\ \beta_3 \end{pmatrix} = \mu_i
 \end{array}$$

They follow the algebra:

$$\begin{array}{l}
 \kappa_i^2 = \lambda_i^2 = \mu_i^2 = 1 \\
 \kappa_i \kappa_j = -\kappa_j \kappa_i \quad \kappa_i \kappa_j = i \rho_k \quad (\text{and the same rules apply for } \lambda \text{ and } \mu \text{ too}).
 \end{array}$$

Expressing the similar equalities with quaternions:  $\kappa_i = -i \kappa_i$ ,  $\lambda_i = -i \lambda_i$ ,  $\mu_i =$

$-i\boldsymbol{\mu}_i$

$$\kappa_i^2 = \lambda_i^2 = \mu_i^2 = -1$$

$$\kappa_i \kappa_j = -\kappa_j \kappa_i \quad \kappa_i \kappa_j = i\rho_k \quad (\text{and the same rules apply for } \lambda \text{ and } \mu \text{ too}).$$

The  $\kappa$ ,  $\lambda$ ,  $\mu$  representation ensures a  $\sigma - \rho$  mirror-symmetric algebra of the  $\alpha$ ,  $\gamma$ ,  $\beta$  representation. This confirms the genetic  $U - T$  nucleotide analogy conjecture (see more about it later in section 3.1). While the generators of the  $\alpha$ ,  $\gamma$ ,  $\beta$  double vector representation were  $\rho$ , the generators of the  $\kappa$ ,  $\lambda$ ,  $\mu$  double vector representation are  $\sigma$ .

Let us recluster the  $\kappa$ ,  $\lambda$ ,  $\mu$  matrices. They can define another (mixed, vector-quaternion-like) Clifford algebra. They define a semi-vector - semi-quaternion algebra. The  $\kappa$ ,  $\lambda$ ,  $\mu$  matrices

- commute;
- square like quaternions; and
- multiple like vectors!

$$\begin{array}{lll} \kappa_i^2 = \lambda_i^2 = \mu_i^2 = 1 & \kappa_i \lambda_j = \lambda_j \kappa_i & \kappa_i \lambda_j = -\mu_k \\ \kappa_i^2 = \lambda_i^2 = \mu_i^2 = -1 & \kappa_i \lambda_j = \lambda_j \kappa_i & \kappa_i \lambda_j = i\mu_k \end{array}$$

## 2 How can they applied to physics?

All the above representations indicate Clifford algebras generated either by vectors or by quaternions over a spinor field. Their specifics is that the roles of the applied vectors, or quaternions, or their mixtures (in the doublets) can be changed, and may serve once as generators of the *field*, once as bases of the (vector, quaternion) *space*. Any of them may serve as generator - at least in formal mathematical terms -, however, there is doubtful, at least at our present day knowledge of physics, whether a definite physical meaning can be assigned to all (formally) so generated fields. (As Wigner noticed in his famous essay on the unreasonable effectiveness of mathematics in the sciences, mathematics allows more than nature can realise. This can be interpreted as underdetermination seen from one side, and overdetermination seen from the other.) Concerning the applicability of these algebras in physics, we have to answer, at least, three questions: having introduced the double vector/quaternion matrices in a physical (e.g., the Dirac) equation

- (1) can we obtain any additional physical information;
- (2) how can we interpret (in physical terms) the results obtained by the changed formalism; (e.g., certainly, not all matrices of the presented algebra can represent spinor spaces), and

(3) can we eliminate imaginary factors from the equations?

I attempt to answer these questions applying the sketched double vector-quaternion algebra for the Dirac equation. Note, QED, in its original formulation by Dirac (1928-29) [5] could not handle highly relativistic situations. It considered invariance of the equation under Lorentz transformation, but did not take into account all relativistic effects (e.g., mass increase far from the rest, or effect of a velocity dependent field). As he wrote in the introduction to [5]: „The resulting theory is therefore still only an approximation, ...” (p. 612). Later (1951) [7], (1962) [8] he made two attempts to extend his „theory of the electron”, but none of them could completely solve the problems left open, (although he applied a Finsler-like metric in the latter extension, without mentioning Finsler).

## 2.1 Dirac equation with no EM field

First, examine the construction of the classical Dirac equation with no electromagnetic field, in Hamiltonian representation. Originally Dirac applied 4 matrix sets:  $\alpha$ ;  $\sigma$  and  $\rho$ ; then  $\gamma$ .

We are free to choose any matrix representation, because they can be transformed into each other. For simplicity, we reduce our scope to Dirac’s original formalism like P. Rowlands did in his cited papers.

$$(\alpha_4 p_4 + \alpha_i \alpha_i p_i + mc)\psi = 0; \quad \text{where } \alpha_i = \rho_1 \sigma_i, \quad \alpha_4 = \rho_3.$$

Multiplying with  $-\rho_1$  :

$$\begin{aligned} (i\rho_2 p_4 + \rho_3 \sigma_i p_i - \rho_1 mc)\psi &= 0 \quad \text{or with quaternions:} \\ (i\rho_2 p_4 + \rho_3 \sigma_i p_i - \rho_1 mc)\psi &= 0. \end{aligned}$$

Why did we keep  $\sigma$  instead of converting it also to  $\sigma$ ?

At first: does a quaternionic sigma provide us with any new information? Probably not.

At second: could we physically interpret  $\sigma$ ? Presumably, not.

At third: is  $\sigma$  less imaginary than  $\sigma$ ? The first and third components of  $\sigma$  consist of real elements, the second component consists of imaginaries. In the case of  $\sigma$  the situation is just the opposite. Thus, both  $\sigma$  and  $\sigma$  include real and imaginary elements either.

Now we can answer questions (1)-(3): during the construction of the classical Dirac equation, we do not obtain any additional physical information when we replace vector-like matrices with quaternionic ones; the interpretation of the results obtained by the changed formalism (at least, in physical terms), not only do not provide extra information, even there is doubtful whether all matrices of the presented algebra can represent spinor spaces; and we demonstrated that we cannot eliminate imaginary factors from the equations by the application of quaternions. The double vector,  $\rho - \sigma$  representations follow Clifford algebras as developed by P. Rowlands in the recent years. The algebraic product of the + and - versions of 8 base units, e.g.,  $1, \mathbf{i}, \mathbf{j}, \mathbf{k}, i, j, k$ , provides 64 terms. They are isomorphic to the  $\gamma$  algebra of the Dirac equation, based on  $4 \times 4$  matrices.

## 2.2 Dirac equation in the presence of EM field

Now, let us investigate the construction of the classical Dirac equation in the presence of electromagnetic field. We extend the momentum components with the contribution of the field components. We demonstrate it both in vector and quaternion forms:

$$\begin{aligned} & \left[ i\rho_2 \left( p_4 + \frac{e}{c} A_4 \right) + \rho_3 \left( \boldsymbol{\sigma}, \mathbf{p} + \frac{e}{c} \mathbf{A} \right) - \rho_1 m c \right] \psi = 0 \\ & \left[ i\rho_2 \left( p_4 + \frac{e}{c} A_4 \right) + \rho_3 \left( \boldsymbol{\sigma}, \mathbf{p} + \frac{e}{c} \mathbf{A} \right) - \rho_1 m c \right] \psi = 0. \end{aligned}$$

It is obvious that, similar to the no-field case, we have not got any new physical information in the latter equation.

## 2.3 Dirac equation in the presence of isotopic field-charges

In the third step, let us investigate the construction of the Dirac equation (from the Klein-Gordon equation) in the presence of isotopic field-charges ([9, 10, 11, 12, 13]). For this reason, we replace the electric and the gravitational field-charges with the respective isotopic electric charge densities ( $\rho_T, \rho_V$ ) and isotopic masses ( $m_T, m_V$ ), in both vector and quaternion forms. ( $V$  refers to the *potential*, and  $T$  to the *kinetic* isotopic field charges.)

$$\begin{aligned} & \left[ -i\rho_2 \left( p_4 + \frac{\rho_T}{c} A_4 \right) + \rho_3 \left( \boldsymbol{\sigma}, \mathbf{p} + \frac{\rho_V}{c} \mathbf{A} \right) - \rho_1 m_V c \right] \cdot \\ & \cdot \left[ i\rho_2 \left( p_4 + \frac{\rho_T}{c} A_4 \right) + \rho_3 \left( \boldsymbol{\sigma}, \mathbf{p} + \frac{\rho_V}{c} \mathbf{A} \right) - \rho_1 m_T c \right] \psi = 0 \\ & \left[ -i\rho_2 \left( p_4 + \frac{\rho_T}{c} A_4 \right) + \rho_3 \left( \boldsymbol{\sigma}, \mathbf{p} + \frac{\rho_V}{c} \mathbf{A} \right) - \rho_1 m_V c \right] \cdot \\ & \cdot \left[ i\rho_2 \left( p_4 + \frac{\rho_T}{c} A_4 \right) + \rho_3 \left( \boldsymbol{\sigma}, \mathbf{p} + \frac{\rho_V}{c} \mathbf{A} \right) - \rho_1 m_T c \right] \psi = 0. \end{aligned}$$

Essentially we haven't got new information due to the quaternion formalism, however, we must notice that the nilpotent character of the equations is lost. This influences the invariance properties of the equation. We will return later to the solution of this problem in subsection 2.5. Let us execute the multiplication of the square brackets to obtain the modified form(s) of the Dirac equation:

$$\begin{aligned} & \left\{ \left[ - \left( p_4 + \frac{\rho_T}{c} A_4 \right)^2 + \rho_3 \left( \boldsymbol{\sigma}, \mathbf{p} + \frac{\rho_V}{c} \mathbf{A} \right)^2 + m_V^2 c^2 + \right. \right. \\ & \left. \left. + \hbar \left( \boldsymbol{\sigma}, \text{rot} \left( \frac{\rho_V}{c} \mathbf{A} \right) \right) - i\hbar \rho_1 \left( \boldsymbol{\sigma}, \text{grad} \left( \frac{\rho_V}{c} A_4 \right) + \frac{1}{c} \frac{\partial}{\partial t} \left( \frac{\rho_V}{c} \mathbf{A} \right) \right) \right] + \right. \\ & \left. + \rho_3 \left[ - \left( p_4 + \frac{\rho_T}{c} A_4 \right) + \rho_1 \left( \boldsymbol{\sigma}, \mathbf{p} + \frac{\rho_V}{c} \mathbf{A} \right) - \rho_3 m_V c \right] (m_T - m_v) \right\} \psi = 0 \end{aligned}$$

This form of the equation is in consistence with the usual  $\gamma_4$  and  $\gamma_5$  representation. See the discussion of this equation in my paper presented at the 2012

FERT conference held in Fryazino (Russia) [10]. The same equation in quaternion form:

$$\begin{aligned} & \left\{ \left[ - \left( p_4 + \frac{\rho_T}{c} A_4 \right)^2 + \rho_3 \left( \boldsymbol{\sigma}, \mathbf{p} + \frac{\rho_V}{c} \mathbf{A} \right)^2 + m_V^2 c^2 + \right. \right. \\ & \left. \left. + \hbar \left( \boldsymbol{\sigma}, \text{rot} \left( \frac{\rho_V}{c} \mathbf{A} \right) \right) + \hbar \rho_1 \left( \boldsymbol{\sigma}, \text{grad} \left( \frac{\rho_V}{c} A_4 \right) + \frac{1}{c} \frac{\partial}{\partial t} \left( \frac{\rho_V}{c} \mathbf{A} \right) \right) \right] + \right. \\ & \left. + \rho_3 \left[ - \left( p_4 + \frac{\rho_T}{c} A_4 \right) + \rho_1 \left( \boldsymbol{\sigma}, \mathbf{p} + \frac{\rho_V}{c} \mathbf{A} \right) - \rho_3 m_V c \right] (m_T - m_v) \right\} \psi = 0 \end{aligned}$$

## 2.4 Dirac equation in the presence of isotopic field charges plus a kinetic field (D)

For this reason, we extend the momentum components introduced in subsection 2.2 with the contribution of the respective kinetic field components and execute again the multiplication with these additional components.

$$\begin{aligned} & \left\{ \left[ - \left( p_4 + \frac{\rho_T}{c} A_4 + \frac{\rho_V}{c} D_4 \right)^2 + \rho_3 \left( \boldsymbol{\sigma}, \mathbf{p} + \frac{\rho_V}{c} \mathbf{A} + \frac{\rho_V}{c} \mathbf{D} \right)^2 + m_V^2 c^2 + \right] \right. \\ & \left. + \hbar \left( \boldsymbol{\sigma}, \text{rot} \left( \frac{\rho_V}{c} \mathbf{A} \right) \right) - i \hbar \rho_1 \left( \boldsymbol{\sigma}, \text{grad} \left( \frac{\rho_V}{c} A_4 \right) + \frac{1}{c} \frac{\partial}{\partial t} \left( \frac{\rho_V}{c} \mathbf{A} \right) \right) \right] + \\ & \left. + \hbar \left( \boldsymbol{\sigma}, \text{rot} \left( \frac{\rho_V}{c} \mathbf{D} \right) \right) + \hbar \rho_1 \left( \boldsymbol{\sigma}, \frac{\rho_V^2}{c^2} [D_j D_k - D_k D_j] \right) - \right. \\ & \left. - i \hbar \rho_1 \left( \boldsymbol{\sigma}, \text{grad} \left( \frac{\rho_V}{c} D_4 \right) \frac{1}{c} \frac{\partial}{\partial t} \left( \frac{\rho_V}{c} \mathbf{D} \right) \right) - \rho_1 \frac{\rho_V^2}{c^2} (\boldsymbol{\sigma}, D_4 \mathbf{D} - \mathbf{D} D_4) + \right. \\ & \left. + \rho_3 \left[ - \left( p_4 + \frac{\rho_T}{c} A_4 + \frac{\rho_V}{c} D_4 \right) - \rho_1 \left( \boldsymbol{\sigma}, \mathbf{p} + \frac{\rho_V}{c} \mathbf{A} + \frac{\rho_V}{c} \mathbf{D} \right) + \rho_3 m_V c \right] \cdot \right. \\ & \left. \cdot (m_T - m_v) c \right\} \psi = 0 \end{aligned}$$

As we already mentioned in the above subsection, the asymmetric role of the isotopic field charges in the presence of a kinetic field, destroys the Lorentz invariance of the equation. This feature became more spectacular by the presence of the kinetic field components.

However, as shown earlier, the invariance of the equation is restored by the conservation of the isotopic field charges and the associated symmetry, as it has been proven in [9]. A more demonstrative form of that invariance is given in [11, 12]. We will specify this invariance restoration starting in subsection 2.5. Until that, for the sake of continuing the quaternion presentation, the same equation takes the following form, although, similar to the previous cases, it does not provide us with additional physical information:

$$\begin{aligned}
& \left\{ \left[ - \left( p_4 + \frac{\rho_T}{c} A_4 + \frac{\rho_V}{c} D_4 \right)^2 + \rho_3 \left( \boldsymbol{\sigma}, \mathbf{p} + \frac{\rho_V}{c} \mathbf{A} + \frac{\rho_V}{c} \mathbf{D} \right)^2 + m_V^2 c^2 + \right] \right. \\
& + \hbar \left( \boldsymbol{\sigma}, \text{rot} \left( \frac{\rho_V}{c} \mathbf{A} \right) \right) + \hbar \rho_1 \left( \boldsymbol{\sigma}, \text{grad} \left( \frac{\rho_V}{c} A_4 \right) + \frac{1}{c} \frac{\partial}{\partial t} \left( \frac{\rho_V}{c} \mathbf{A} \right) \right) \left. \right] + \\
& + \hbar \left( \boldsymbol{\sigma}, \text{rot} \left( \frac{\rho_V}{c} \mathbf{D} \right) \right) + \hbar \rho_1 \left( \boldsymbol{\sigma}, \frac{\rho_V^2}{c^2} [D_j D_k - D_k D_j] \right) + \\
& + \hbar \rho_1 \left( \boldsymbol{\sigma}, \text{grad} \left( \frac{\rho_V}{c} D_4 \right) \frac{1}{c} \frac{\partial}{\partial t} \left( \frac{\rho_V}{c} \mathbf{D} \right) \right) - i \rho_1 \frac{\rho_V^2}{c^2} (\boldsymbol{\sigma}, D_4 \mathbf{D} - \mathbf{D} D_4) + \\
& + \rho_3 \left[ -i \left( p_4 + \frac{\rho_T}{c} A_4 + \frac{\rho_V}{c} D_4 \right) + \rho_1 \left( \boldsymbol{\sigma}, \mathbf{p} + \frac{\rho_V}{c} \mathbf{A} + \frac{\rho_V}{c} \mathbf{D} \right) - \rho_3 m_V c \right] \cdot \\
& \cdot (m_T - m_v) c \} \psi = 0
\end{aligned}$$

On the basis of [11] and [12], one can get ascertained that the curvature of the connection field is also independent of the vector or quaternion representations, since none of the magnetic and electric momenta contain either vector or quaternion matrices.

The full magnetic moment is:

$$\mathbf{M}^{FULL} = \rho_V \text{rot}(\mathbf{A} + \mathbf{D}) + ig C_{jk}^i \frac{\rho_V^2}{c} D_j D_k,$$

and the full electric moment is:

$$\mathbf{N}^{FULL} = \text{grad} \rho_T A_4 + \frac{\rho_V}{c} \frac{\partial}{\partial t} (\mathbf{A} + \mathbf{D}).$$

These  $\mathbf{M}^{FULL}$  and  $\mathbf{N}^{FULL}$  should commute with the Hamiltonian of the interacting two charges.

$$H = \rho_T A_4 - \rho_V D_4 + i \rho_1 (\boldsymbol{\sigma}, ihc \text{grad} - \rho_V \mathbf{A} - \rho_V \mathbf{D}) + i \rho_3 m_V c^2$$

This has been discussed in [12].

## 2.5 Kinetic addition to the transformation of the Lorentz force

In order to determine how one should extend the Lorentz transformation to restore the invariance of the basic equation of QED, first, we examine the roles of the extended magnetic and electric moments on a simple example, on the Lorentz force. The presented  $\mathbf{M}^{FULL}$  and  $\mathbf{N}^{FULL}$  appear in the EM field tensor - in the presence of isotopic field charges and an additional kinetic field:

$$\rho_V F^{\mu\nu} = \begin{bmatrix} 0 & M_3^D & -M_2^D & -i\gamma_5 N_1^D \\ -M_3^D & 0 & M_1^D & -i\gamma_5 N_2^D \\ M_2^D & -M_1^D & 0 & -i\gamma_5 N_3^D \\ i\gamma_5 N_1^D & i\gamma_5 N_2^D & i\gamma_5 N_3^D & 0 \end{bmatrix}$$



It includes the  $i\gamma_5 = -i\rho_1$  components, that can be replaced by their quaternion partners  $\gamma_5$  or  $\rho_1$ . This tensor appears in the  $F^\mu$  Lorentz force:

$$\begin{aligned}
 F^\mu &= F^{\mu\nu} \frac{1}{c} j_\nu = \frac{1}{\rho_V} \begin{bmatrix} 0 & M_3^D & -M_2^D & -i\gamma_5 N_1^D \\ -M_3^D & 0 & M_1^D & -i\gamma_5 N_2^D \\ M_2^D & -M_1^D & 0 & -i\gamma_5 N_3^D \\ i\gamma_5 N_1^D & i\gamma_5 N_2^D & i\gamma_5 N_3^D & 0 \end{bmatrix} \begin{bmatrix} \rho_T \frac{\dot{x}^1}{c} \\ \rho_T \frac{\dot{x}^2}{c} \\ \rho_T \frac{\dot{x}^3}{c} \\ i\rho_V \end{bmatrix} = \\
 &= \frac{1}{\rho_V} \begin{bmatrix} M_3^D \rho_T \frac{\dot{x}^2}{c} - M_2^D \rho_T \frac{\dot{x}^3}{c} + \gamma_5 N_1^D \rho_V \\ -M_3^D \rho_T \frac{\dot{x}^1}{c} + M_1^D \rho_T \frac{\dot{x}^3}{c} + \gamma_5 N_2^D \rho_V \\ M_2^D \rho_T \frac{\dot{x}^1}{c} - M_1^D \rho_T \frac{\dot{x}^2}{c} + \gamma_5 N_3^D \rho_V \\ i\gamma_5 N_1^D \rho_T \frac{\dot{x}^1}{c} + i\gamma_5 N_2^D \rho_T \frac{\dot{x}^2}{c} + i\gamma_5 N_3^D \rho_T \frac{\dot{x}^3}{c} \end{bmatrix}
 \end{aligned}$$

where  $\gamma_5 = -\rho_1 = -i\rho_1$ . This Lorentz force can be further shaped in the quaternion form of  $\rho$ :

$$\begin{aligned}
 F^\mu &= \frac{1}{\rho_V} \begin{bmatrix} M_3^D \rho_T \frac{\dot{x}^2}{c} - M_2^D \rho_T \frac{\dot{x}^3}{c} - i\rho_1 N_1^D \rho_V \\ -M_3^D \rho_T \frac{\dot{x}^1}{c} + M_1^D \rho_T \frac{\dot{x}^3}{c} - i\rho_1 N_2^D \rho_V \\ M_2^D \rho_T \frac{\dot{x}^1}{c} - M_1^D \rho_T \frac{\dot{x}^2}{c} - i\rho_1 N_3^D \rho_V \\ \rho_1 N_1^D \rho_T \frac{\dot{x}^1}{c} + \rho_1 N_2^D \rho_T \frac{\dot{x}^2}{c} + \rho_1 N_3^D \rho_T \frac{\dot{x}^3}{c} \end{bmatrix} = \\
 &= \frac{1}{c\rho_V} \begin{bmatrix} M_3^D \frac{\dot{x}^2}{c} - M_2^D \frac{\dot{x}^3}{c} & -i\rho_1 N_1^D \\ -M_3^D \frac{\dot{x}^1}{c} + M_1^D \frac{\dot{x}^3}{c} & -i\rho_1 N_2^D \\ M_2^D \frac{\dot{x}^1}{c} - M_1^D \frac{\dot{x}^2}{c} & -i\rho_1 N_3^D \\ \rho_1 N_1^D \frac{\dot{x}^1}{c} + \rho_1 N_2^D \frac{\dot{x}^2}{c} + \rho_1 N_3^D \frac{\dot{x}^3}{c} & 0 \end{bmatrix} \begin{bmatrix} \rho_T \\ \rho_V \end{bmatrix} = \frac{1}{c\rho_V} H^{D\kappa l} \rho_l.
 \end{aligned}$$

In the tensor  $H^{D\kappa l} : \kappa = 1, \dots, 4; l = 1, 2$  and  $\rho_l = \begin{bmatrix} \rho_T \\ \rho_V \end{bmatrix}$ .

This  $\rho_l$  can be expressed by the help of the eigenfunctions of the state function  $\psi$  in the extended Dirac equation presented in subsection 2.4. Since  $\rho_V$  appears in the scalar potential, while  $\rho_V$  in the vector potential and can serve as coefficient of the three components of the vector components, the latter should take the form of a dreibein. This indicates that  $\rho_l$ , and accordingly the eigenfunctions, by the expressions of whom it is demonstrated, should be subject of a special algebra. That algebra must be represented by matrices that reflect the 3 + 1 character of the  $\rho_l$  four-vector.

### 3 The algebra of the isotopic field-charge model applied to QED

#### 3.1 The *tau* algebra for the extended Dirac equation

The  $\psi$  state function in the Dirac equation depends on the  $x_\nu$  space-time coordinates, the  $r$  and  $s$  parameters of  $\rho$  and  $\sigma$ , and in the extended equation on the  $\Delta$  parameter of the isotopic field-charge spin rotation matrix  $\tau$ . (We were more consequent by denoting the latter parameter by  $t$ . We had two reasons not to do

so. Once,  $t$  denotes time in physics. Secondly, we denoted the isotopic field-charge spin by  $\Delta$  in the previous papers.)

Let's denote the eigenfunctions of  $\sigma$  by  $\varepsilon$  and  $\zeta$ , those of  $\rho$  by  $\xi$  and  $\eta$ , and the eigenfunctions of  $\tau$  by  $\vartheta$  and  $\chi$ . Then the eigenfunctions of the  $\rho\sigma$  product can be expressed in the following forms:

$$\varepsilon\xi = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}; \quad \zeta\xi = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}; \quad \varepsilon\eta = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}; \quad \zeta\eta = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

and can be denoted, according to the  $\pm\frac{1}{2}$  values of  $r$  and  $s$ , respectively, by

$$\varphi_+^{(\rho)}, \quad \varphi_-^{(\rho)}, \quad \varphi_+^{(\sigma)}, \quad \varphi_-^{(\sigma)}.$$

Let us represent the eigenfunctions of  $\tau$  in the following form. (Certainly, we had some limited freedom to choose the values in the eigenvectors. This choice makes handling the  $\tau$  transformation matrices convenient. Recall, that in the following case, like in many other cases, the representation of the transformation group coincides with a representation of the respective Lie algebra.)

$$\chi = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}; \quad \vartheta = \begin{bmatrix} 0 \\ 0 \\ 0 \\ i \end{bmatrix}.$$

The respecting eigenfunctions belonging to the  $\pm\frac{1}{2}$  values of  $\Delta$  are

$$\varphi_+^{(\tau)}, \quad \varphi_-^{(\tau)}.$$

Similar to the  $\sigma$  algebra in Dirac's QED, we introduced [14] the  $\tau$  matrices with the following algebraic properties:

$$\begin{aligned} \tau_3\chi &= \chi & \tau_3\vartheta &= \vartheta \\ \tau_2\chi &= \vartheta & \tau_3\vartheta &= \chi \\ & \text{requiring } \tau_i\tau_j &= i\tau_k \text{ } i, j, k \text{ cyclic indices} \\ \tau_1\chi &= -i\vartheta & \tau_1\vartheta &= i\chi. \end{aligned}$$

A set of the following matrices meets the required properties:

$$\tau_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad \tau_2 = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \end{bmatrix}, \quad \tau_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

These  $\tau$  matrices satisfy the following algebra:

$$\tau_1^2 = \tau_2^2 = \tau_3^2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \mathbf{E}_L, \quad \tau_i\tau_j + \tau_j\tau_i = 0, \quad [\tau_i, \tau_j] = 2i\tau_k.$$

The representation of the transformation group, we are looking for in order to extend the invariance group belonging to the extended form of the Dirac equation, is determined by the algebra of these matrices. They transform the two states of the IFCS's among each other by a rotation in the abstract IFCS field. The  $\tau$  matrices are representations of operators that compose the three components of the isotopic field charge spin,  $\Delta_1$ ,  $\Delta_2$ ,  $\Delta_3$  (as derived in [9], sections 4.2.2 and 4.2.3) which follow the same (non-Abelian) commutation rules like do the  $\tau$ . These operators represent the charges of the isotopic field charge spin field. The algebra of the  $\tau$  matrices is discussed in detail in [14].

Similar to the definition of  $\gamma_4$  in the Dirac algebra, let us define artificially  $\tau_4$  :

$$\tau_4^2 = \tau_3 \text{ where } \tau_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \end{bmatrix}.$$

$\tau_i$  ( $i = 1, 2, 3$ ) belong to the set of dyadic shift matrices [15].  $\tau_i^2$  are so called oblique projectors. The asymmetric role of the fourth components in the  $\tau$  algebra confirms the similitude to the asymmetric role of the nucleotides  $U$  and  $T$  of RNA and DNA, respectively, in the algebra of the molecular genetic code. A detailed description of such oblique projection operators which are connected with matrix representations of the genetic coding system in forms of the Rademacher and Hadamard matrices is given among others in [16]. Petoukhov proposes multidimensional vector spaces, whose subspaces are under a selective control (or coding) by means of a set of matrix operators on the basis of genetic projectors. Similitude between the mathematical description of physical fields of interactions and the system of coding molecular genetic information suggests demonstrating that there are fundamental logical principles behind these apparently different ontological phenomena.

Referring to the mathematical discussion presented in [14], we define the following set of  $\delta$  matrices:

$$\delta_1 = \tau_1^2, \quad \delta_2 = \tau_2^2, \quad \delta_3 = \tau_3^2, \quad \delta_4 = \tau_4^2.$$

This choice has two logical reasons.

The  $\gamma$  algebra was created by the multiplication of  $\sigma$  and  $\rho$  matrices. The  $\rho$  matrices were obtained by changing the second and third rows and columns in the  $\sigma$  matrices. In case of the  $\tau$  matrices, the second and third rows and columns coincide, respectively. Therefore, if we considered the roles of the  $\tau$  matrices as the analogues of the  $\sigma$  matrices in the new algebra, the analogue roles corresponding to the  $\rho$  matrices must coincide with the  $\tau$  matrices. Since the squares of all the three  $\tau$  matrices coincide, the analogue set is complete with this choice.

This set has another important advantage that justifies our choice. According to the independent freedom on  $\sigma$  and  $\rho$  dependence, the original Dirac equation had four solutions. The introduction of the isotopic field-charges should not quadruple, only double the solutions of the Dirac equation. For this reason the  $\tau$

algebra should have been chosen in such way that involved only one two-valued property (one new parameter, instead of two). Thus the number of the solutions of the extended Dirac equation will be eight, as we expected.

The  $\psi = \psi(x_\nu, r, s, \Delta)$  state function separates now according to the eigenfunctions of the space-time dependent differential operators, so that the eigenfunctions of the  $\boldsymbol{\sigma}$ ,  $\boldsymbol{\rho}$  and  $\boldsymbol{\tau}$  operators take discrete eigenvalues  $\pm\frac{1}{2}$  each. The differential operators act only on  $\psi(x_\nu)$ , the  $\boldsymbol{\sigma}$  operators on the eigenfunctions of  $\sigma$ , the  $\boldsymbol{\rho}$  operators on the eigenfunctions of  $\rho$ , and the  $\boldsymbol{\tau}$  operators on the eigenfunctions of  $\Delta$ . The state function can be dissociated to the following eight products according to the eigenfunctions of the matrix operators:

$$\begin{aligned} \psi = & \psi_1(x_\nu)\varphi_+^{(\rho)}\varphi_+^{(\sigma)}\varphi_+^{(\tau)} + \psi'_1(x_\nu)\varphi_+^{(\rho)}\varphi_+^{(\sigma)}\varphi_-^{(\tau)} + \psi_2(x_\nu)\varphi_-^{(\rho)}\varphi_+^{(\sigma)}\varphi_+^{(\tau)} + \\ & + \psi'_2(x_\nu)\varphi_-^{(\rho)}\varphi_+^{(\sigma)}\varphi_-^{(\tau)} + \psi_3(x_\nu)\varphi_+^{(\rho)}\varphi_-^{(\sigma)}\varphi_+^{(\tau)} + \psi'_3(x_\nu)\varphi_+^{(\rho)}\varphi_-^{(\sigma)}\varphi_-^{(\tau)} + \\ & + \psi_4(x_\nu)\varphi_-^{(\rho)}\varphi_-^{(\sigma)}\varphi_+^{(\tau)} + \psi'_4(x_\nu)\varphi_-^{(\rho)}\varphi_-^{(\sigma)}\varphi_-^{(\tau)}. \end{aligned}$$

This involves also that the solutions of the extended Dirac equation's  $\psi$  state function can be separated into eight space-time dependent functions. Opposite to the four solutions of the original Dirac equation, this extended one has eight solutions, in accordance with the additional bivariant opposite positions of the IFCS.

### 3.2 Invariance of the extended Dirac equation

Let us introduce the following generalised momentum notations:

$$p'_4 = -i\left(p_4 + \frac{\rho_T}{c}A_4 + \frac{\rho_V}{c}D_4\right); \quad \mathbf{p}' = \mathbf{p} + \frac{\rho_T}{c}\mathbf{A} + \frac{\rho_V}{c}\mathbf{D}.$$

Now the above extended Dirac equation can be written as:

$$\left[-ip'_4 - \gamma_5(\boldsymbol{\sigma}, \mathbf{p}') + \gamma_4 m_V c\right] \cdot \left[ip'_4 - \gamma_5(\boldsymbol{\sigma}, \mathbf{p}') + \gamma_4 m_T c\right] \psi = 0$$

We consider that  $\gamma_4 = \rho_3$ ,  $\gamma_5 = -\rho_1$ , and apply the tau algebra for the isotopic field-charges. The isotopic field charges - both the electric and the gravitational - are rotated by the same transformation matrices in the *isotopic field-charge spin* field. Let's construct, on the analogy of the construction of  $\gamma_i$  by the help of  $\sigma_2$  the upper case tau matrices  $T_i = \tau_2 \tau_i$ . So  $T_i = \tau_2^2 = \delta_2 = \mathbf{E}_L$ . Multiply the expression in the first square bracket with  $T_2 \vartheta$  from right, and the expression in the second square bracket with  $T_2 \chi$  from right, and both expressions in the brackets with  $\rho_3$  from the left. The  $\mathbf{T}$  operators do not affect either the derivative operators or the spin operator, and vice versa. So, they commute within the individual terms. According to a similar observation, and can be dropped behind the square brackets.

$$\left[-\rho_3 T_2 \vartheta p'_4 + i\rho_2(\boldsymbol{\sigma}, T_2 \vartheta \mathbf{p}') + T_2 \vartheta m_V c\right] \cdot \left[\rho_3 T_2 \vartheta p'_4 + i\rho_2(\boldsymbol{\sigma}, T_2 \vartheta \mathbf{p}') + T_2 \chi m_T c\right] \psi = 0$$

and consequently, replacing again the  $\gamma$  matrices for  $\rho$  and  $\sigma$ , the equation takes the form:

$$[i\gamma_4\delta_4p'_4 + i(\boldsymbol{\gamma}, \boldsymbol{\delta}\mathbf{p}') + m_Vc] \cdot [i\gamma_4\delta_4p'_4 + i(\boldsymbol{\gamma}, \boldsymbol{\delta}\mathbf{p}') + m_Tc] \boldsymbol{\psi}'\vartheta\chi = 0$$

where we applied the following notations:

$$T_2p'_1 = \delta_1p'_1 = \tau_1^2p'_1; T_2p'_2 = \delta_2p'_2 = \tau_2^2p'_2; T_2p'_3 = \delta_3p'_3 = \tau_3^2p'_3; \begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{pmatrix} = \begin{pmatrix} \tau_1^2 \\ \tau_2^2 \\ \tau_3^2 \end{pmatrix} = \boldsymbol{\delta}.$$

The same extended Dirac equation can be written with four-components:

$$[i\gamma_\mu(\boldsymbol{\delta}\mathbf{p}')_\mu + m_Vc] \cdot [i\gamma_\nu(\boldsymbol{\delta}\mathbf{p}')_\nu + m_Vc] \boldsymbol{\psi}'\vartheta\chi = 0$$

Executing the matrix multiplication, we can write:

$$[i\Gamma_\mu p'_\mu + m_Vc] \cdot [i\Gamma_\nu p'_\nu + m_Tc] \boldsymbol{\psi} = 0 \quad \text{where } \Gamma_\mu = \gamma_\mu \cdot \delta_\mu.$$

This simplified form provides the extended form of the source of the Dirac equation in the presence of isotopic field-charges with the application of the *tau* algebra. All the rest can be calculated following my previous papers [9, 11, 12, 13].

The latter form demonstrates the covariant character of the equation.

It was a long lasting paradigm in physics that equations of the physical interactions must be subject of invariance under the Lorentz transformation, and this is not only a necessary, but also a sufficient condition. Now, we see this is not the case. Nevertheless, no physical principle stated that Lorentz invariance is a sufficient condition demanded for the equations.

There are other invariances that may appear together and combined with Lorentz's.

How does this observation present itself in this case? As mentioned above in section 2.4, the appearance of the opposite isotopic field-charges in the  $p'_\mu$  generalised momentum and the mass terms destroys the Lorentz invariance of the equation. In addition, there appears a new invariance. It is represented in the equation by the  $\delta(\tau)$  operators that rotates the isotopic field-charges in an additional abstract field. The invariance of the extended Dirac equation is resulted in the convolution of the Lorentz- and this new (IFCS) invariance. In other words, the extended Dirac equation is invariant under the combined transformation [11, 12].

The combined (Lorentz IFCS) invariance of the equation allows to reinterpret our earlier imagination on the invariances of our physical equations. This requires certain intendment change in the approach to the world-picture, especially in its relativistic extension, what can be summarised in the following. The new invariance described algebraically in this (and introduced in the cited previous) papers is interpreted in the presence of a kinetic gauge field. The Lorentz invariance is a combined symmetry in itself. The  $SO^+(3, 1)$  group of the proper Lorentz transformation can be characterised by six independent subgroups. These six independent

subgroups can be separated to and characterised by three  $[4 \times 4]$  rotation matrices  $[R(\varphi)]$  in the space-time, and three  $[4 \times 4]$  velocity boosts  $[\Lambda(\dot{x})]$  into a given direction in the configuration space ( $= R \otimes \Lambda$ ). Since the IFCS invariance  $[\Lambda(\dot{x})]$  is interpreted also in a kinetic field, it seems reasonable to substitute  $R \otimes (\Lambda \otimes \Delta)$  for the transformation. Although this clustering is only formal, we must mention that the reason is to associate the velocity dependent transformations ( $\Lambda$  and  $\Delta$ ) with each other, and formally separate them from the space-time rotation  $R$ .

## 4 Concluding remarks

### Does the quaternion formalism bring any new information in the extended Dirac equation?

The algebra of the  $\sigma$ ,  $\rho$ , and accordingly the  $\gamma$  matrices represented a vector algebra, they could be transformed in a quaternion form, and discussed parallel. The combined covariance of the extended Dirac equation introduced the  $\Gamma$  matrices. Although the  $\gamma$  and the  $\delta$  matrices were subjects of a definite algebra separately, their product ( $\Gamma$ ) matrices do not show up any sign of a regular vector or quaternion algebra. This does not cause a surprise. We saw in section 1: while  $\sigma$  and  $\rho$  represent a complete algebra, the  $\gamma$  matrices, constructed by their combination, decline from that (cf., their commutators). One could have expected further declination when the upper case  $\Gamma$  matrices were constructed from the lower case  $\gamma$  matrices multiplied by elements chosen from the  $\tau$  algebra. So there is no reason to speak about a vector- or quaternion character concerning the properties of the  $\Gamma$  matrices. They mix the four components of the vectors in the Dirac equation in a special way that expresses the combined invariance transformation. The question about a Finsler extension - at least in the present stage of the development of this contexture - is not actual. Nevertheless, the extension of the relativity theory in this direction and applied to the QED was not useless. It provided us with additional information that will be useful in future studies.

Let us see how! Works in this field could be extended with investigations into the third approach of Dirac to the theory of the electron [8], where he introduced a curvilinear co-ordinate system. There, he defined an auxiliary co-ordinate system  $y^\Lambda$ , „which is kept fixed during the variation process and use the functions  $y^\Lambda(x)$  to describe the  $x$  co-ordinate system in terms of the  $y$  co-ordinate system.” He defined the  $y$  system ”so that the metric for the  $x$  system” be  $g_{\mu\nu} = y_{\Lambda,\mu} y_{,\nu}^\Lambda$ . This metric, which then appeared in the Hamiltonian of the electromagnetic interaction, was „at least, according to me“ the first step to a Finsler extension of the theory. My sincere hope is that one will combine the extension of the original [5] (1928) Dirac equation presented in this (and the cited) paper(s) with the conclusions of that third (1962) attempt of Dirac [8] to create a more precise ”approximation” (the expression borrowed from Dirac) to the relativistic quantum theory of the electron. This paper made a step towards it. That expected next step may make the Finsler extension of the relativistic quantum theory of the electron complete.

Nevertheless, one can find a few attempts in the literature to apply Dirac's 1962 approach [8] to an extended relativistic QED. However, (a) none of them referred to Dirac [8] and (b) none of them tried to combine the Finslerian approach with the fully relativistic (called by us  $\Delta[\dot{x}]$ ) transformation. I'd like to mention only a few recent papers from among them (e.g., [17, 18]). H. Brandt published several papers on Finslerian extensions of quantum field theory (e.g., [19] specified to scalar quantum fields) - with the mentioned deficiencies. G. Munteanu - partly together with N. Aldea - made also attempts to such extensions (e.g., [20, 21]). The latter one was presented at the 10<sup>th</sup> FERT conference, 2014, parallel with this paper. I assume that their results could be combined with those presented in this paper, because they could extend each other. They apply Finsler metric and tangent bundles, while this paper considers strong relativistic generalisation and effects of isotopic field charges. Their treatment considers curved *space*, but I am missing in it curvature of *fields* (although, perhaps they are considered, but not specified). Both their and this papers define themselves as initial attempts towards a unified field theory, but none of them are complete without the other. A possible combining point should be found in section 3 of their paper [21], where they refer to Born's reciprocity principle which is based on a symmetry between the space-time coordinates and momentum-energy coordinates, and where they "pass from the complex space-time coordinates ... to the complex momentum-energy coordinates" without defining the transformation group of this symmetry (i.e., of the transition). This transformation is specified in [22]. I hope that combined efforts could lead to promising results.

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