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SUFFICIENT CONDITIONS FOR UNIVALENCE OF A NEW INTEGRAL OPERATOR

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Abstract

In this paper we introduce a new integral operator $K_{\alpha,\beta}$, for analytic functions f and g defined in the open unit disk $\mathcal{U}$, α and β complex numbers, and we derive sufficient conditions for the univalence of this integral operator.

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1 Introduction

Let $\mathcal{A}$ be the class of analytic functions f in the unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$, normalized by $f(0) = f'(0) - 1 = 0$ and $\mathcal{S}$ be the subclass of univalent functions in the class $\mathcal{A}$.

We denote by $\mathcal{P}$ the class of functions p which are analytic in $\mathcal{U}$, and satisfy $p(0) = 1$ and $Re\, p(z) > 0$, for all $z \in \mathcal{U}$.

We define an integral operator

$$K_{\alpha,\beta}(z) = \int_0^z \left(\frac{f(u)}{u}\right)^{\alpha} \left(g'(u)\right)^{\beta} du, \tag{1}$$

for $\alpha,\ \beta \in \mathbb{C}$ and the functions $f,\ g \in \mathcal{A}$.

For $\beta = 0$, α a complex number and $f \in \mathcal{A}$, from (1) we have the integral operator Kim-Merkes [2],

$$G_{\alpha}(z) = \int_0^z \left(\frac{f(u)}{u}\right)^{\alpha} du. \tag{2}$$

From (1), for $\alpha = 0$, β a complex number and $g \in \mathcal{A}$, we obtain the integral operator Pfaltzgraff [4],

$$H_{\beta}(z) = \int_0^z \left(g'(u)\right)^{\beta} du. \tag{3}$$

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2 Preliminary results

To discuss our problems for univalence of integral operator $K_{\alpha,\beta}$, we need the following lemmas.

Lemma 1. *([1]). If the function f is analytic in $\mathcal{U}$ and*

$$\left(1-|z|^2\right)\left|\frac{zf''(z)}{f'(z)}\right| \leq 1, \tag{4}$$

for all $z \in \mathcal{U}$, then the function f is univalent in $\mathcal{U}$.

Lemma 2. *(generalized Schwarz lemma, [3]). Let f be the function regular in the disk*
$\mathcal{U}_R = \{z \in \mathbb{C} : |z| < R\}$ with $|f(z)| < M$, M fixed. If $z = 0$ is a zero of order at least m for f, then

$$|f(z)| \leq \frac{M}{R^m}|z|^m,\ (z \in \mathcal{U}_R)\,. \tag{5}$$

Moreover, the equality occurs in (5) for $z \neq 0$ if and only if

$$f(z) = e^{i\theta}\frac{M}{R^m}z^m,$$

where θ is constant.

3 Main results

Theorem 1. *Let α, β be complex numbers, M, L positive real numbers and the functions $f \in \mathcal{A}$, $g \in \mathcal{A}$.*

If

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq M, \quad (z \in \mathcal{U}), \tag{6}$$

$$\left|\frac{zg''(z)}{g'(z)}\right| \leq L, \quad (z \in \mathcal{U}) \tag{7}$$

and

$$|\alpha|M + |\beta|L \leq \frac{3\sqrt{3}}{2}, \tag{8}$$

then the integral operator $K_{\alpha,\beta}$ is in the class $\mathcal{S}$.

Proof. The function $K_{\alpha,\beta}(z)$ is analytic in $\mathcal{U}$ and satisfies $K_{\alpha,\beta}(0) = K'_{\alpha,\beta}(0) - 1$. We have

$$\frac{zK''_{\alpha,\beta}(z)}{K'_{\alpha,\beta}(z)} = \alpha\left(\frac{zf'(z)}{f(z)} - 1\right) + \beta\frac{zg''(z)}{g'(z)}, \tag{9}$$

for all $z \in \mathcal{U}$.

From (9) we obtain

$$\left(1-|z|^2\right)\left|\frac{zK''_{\alpha,\beta}(z)}{K'_{\alpha,\beta}(z)}\right| \le \left(1-|z|^2\right)\left[|\alpha|\left|\frac{zf'(z)}{f(z)} - 1\right| + |\beta|\left|\frac{zg''(z)}{g'(z)}\right|\right], \tag{10}$$

for all $z \in \mathcal{U}$. By Lemma 2, from (6) and (7) we get

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \le M|z|, \quad (z \in \mathcal{U}), \tag{11}$$

$$\left|\frac{zg''(z)}{g'(z)}\right| \le L|z|, \quad (z \in \mathcal{U}) \tag{12}$$

and by (10), we obtain

$$\left(1-|z|^2\right)\left|\frac{zK''_{\alpha,\beta}(z)}{K'_{\alpha,\beta}(z)}\right| \le \left(1-|z|^2\right)|z|\left(|\alpha|M + |\beta|L\right), \tag{13}$$

for all $z \in \mathcal{U}$. Since

$$\max_{|z|\le 1}\left[\left(1-|z|^2\right)|z|\right] = \frac{2}{3\sqrt{3}},$$

hence, and from (8), (13), we have

$$\left(1-|z|^2\right)\left|\frac{zK''_{\alpha,\beta}(z)}{K'_{\alpha,\beta}(z)}\right| \le 1 \quad (z \in \mathcal{U}). \tag{14}$$

From (14) and Lemma 1, we obtain that the integral operator $K_{\alpha,\beta}$ defined by (1) is in the class $\mathcal{S}$. □

Theorem 2. *Let α, β be complex numbers, L positive real number and the functions $f \in \mathcal{S}$, $g \in \mathcal{A}$.*

If

$$\left|\frac{zg''(z)}{g'(z)}\right| \le L, \quad (z \in \mathcal{U}), \tag{15}$$

and

$$12\sqrt{3}|\alpha| + 2|\beta|L \le 3\sqrt{3}, \tag{16}$$

then the integral operator $K_{\alpha,\beta}$ is in the class $\mathcal{S}$.

Proof. From (9) we have

$$\left(1-|z|^2\right)\left|\frac{zK''_{\alpha,\beta}(z)}{K'_{\alpha,\beta}(z)}\right| \leq$$

$$\leq \left(1-|z|^2\right)\left[|\alpha|\left(\left|\frac{zf'(z)}{f(z)}\right|+1\right)+|\beta|\left|\frac{zg''(z)}{g'(z)}\right|\right], \tag{17}$$

for all $z \in \mathcal{U}$. Using [6], for $f \in \mathcal{S}$ we get

$$\left|\frac{zf'(z)}{f(z)}\right| \leq \frac{1+|z|}{1-|z|}, \quad (z \in \mathcal{U}). \tag{18}$$

By (15) and Lemma 2 we obtain

$$\left|\frac{zg''(z)}{g'(z)}\right| \leq L|z|, \quad (z \in \mathcal{U}). \tag{19}$$

From (18), (19) and (17) we get

$$\left(1-|z|^2\right)\left|\frac{zK''_{\alpha,\beta}(z)}{K'_{\alpha,\beta}(z)}\right| \leq \left(1-|z|^2\right)\frac{2}{1-|z|}|\alpha| + \left(1-|z|^2\right)|z||\beta|L, \tag{20}$$

for all $z \in \mathcal{U}$. Because

$$\max_{|z|\leq 1}\left[\left(1-|z|^2\right)|z|\right] \leq \frac{2}{3\sqrt{3}},$$

hence, and from (20), we have

$$\left(1-|z|^2\right)\left|\frac{zK''_{\alpha,\beta}(z)}{K'_{\alpha,\beta}(z)}\right| \leq 4|\alpha| + \frac{2}{3\sqrt{3}}|z||\beta|L, \quad (z \in \mathcal{U})$$

and by (16), we obtain

$$\left(1-|z|^2\right)\left|\frac{zK''_{\alpha,\beta}(z)}{K'_{\alpha,\beta}(z)}\right| \leq 1, \quad (z \in \mathcal{U}). \tag{21}$$

From (21) and Lemma 1, it follows that the integral operator $K_{\alpha,\beta}$ belongs to the class $\mathcal{S}$. □

Theorem 3. *Let α, β be complex numbers, M positive real number and the functions $f \in \mathcal{A}$, $g \in \mathcal{A}$ and $g' \in \mathcal{P}$.*

If

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq M, \quad (z \in \mathcal{U}) \tag{22}$$

and

$$2|\alpha|M + 6\sqrt{3}|\beta| \leq 3\sqrt{3}, \tag{23}$$

then the integral operator $K_{\alpha,\beta}$ is in the class $\mathcal{S}$.

Proof. By (22) and Lemma 2 we obtain

$$\left|\frac{zf'(z)}{f(z)}-1\right|\le M|z|,\quad (z\in\mathcal{U}). \tag{24}$$

Using [7], for $g'\in\mathcal{P}$, we have

$$\left|\frac{zg''(z)}{g'(z)}\right|\le\frac{2|z|}{1-|z|^2},\quad (z\in\mathcal{U}). \tag{25}$$

From (24), (25) and (10) we obtain

$$\left(1-|z|^2\right)\left|\frac{zK''_{\alpha,\beta}(z)}{K'_{\alpha,\beta}(z)}\right|\le\left(1-|z|^2\right)|z|M|\alpha|+2|z||\beta|,\quad (z\in\mathcal{U})$$

and because

$$\max_{|z|\le 1}\left[\left(1-|z|^2\right)|z|\right]=\frac{2}{3\sqrt{3}},$$

we get

$$\left(1-|z|^2\right)\left|\frac{zK''_{\alpha,\beta}(z)}{K'_{\alpha,\beta}(z)}\right|\le\frac{2}{3\sqrt{3}}|\alpha|M+2|\beta|,\quad (z\in\mathcal{U}). \tag{26}$$

From (23) and (26), we get

$$\left(1-|z|^2\right)\left|\frac{zK''_{\alpha,\beta}(z)}{K'_{\alpha,\beta}(z)}\right|\le 1,\quad (z\in\mathcal{U})$$

and hence, by Lemma 1 we obtain that the integral operator $K_{\alpha,\beta}$ is in the class $\mathcal{S}$. □

Theorem 4. *Let α, β be complex numbers and the functions $f\in\mathcal{S}$, $g\in\mathcal{A}$ and $g'\in\mathcal{P}$.*

If

$$2|\alpha|+|\beta|\le\frac{1}{2}, \tag{27}$$

then the integral operator $K_{\alpha,\beta}$ is in the class $\mathcal{S}$.

Proof. Using [6], for $f\in\mathcal{S}$, we have

$$\left|\frac{zf'(z)}{f(z)}\right|\le\frac{1+|z|}{1-|z|},\quad (z\in\mathcal{U}) \tag{28}$$

and using [7], for $g'\in\mathcal{P}$, we have

$$\left|\frac{zg''(z)}{g'(z)}\right|\le\frac{2|z|}{1-|z|^2},\quad (z\in\mathcal{U}). \tag{29}$$

From (17), (28) and (29) we obtain

$$\left(1-|z|^2\right)\left|\frac{zK''_{\alpha,\beta}(z)}{K'_{\alpha,\beta}(z)}\right| \leq 4|\alpha|+2|\beta|, \quad (z\in\mathcal{U})$$

and by (27) we get

$$\left(1-|z|^2\right)\left|\frac{zK''_{\alpha,\beta}(z)}{K'_{\alpha,\beta}(z)}\right| \leq 1, \quad (z\in\mathcal{U}). \tag{30}$$

From (30) and Lemma 1 it follows that the integral operator $K_{\alpha,\beta}$ is in the class $\mathcal{S}$. □

4 Corollaries

Corollary 1. *Let α be a complex number, M positive real number and the function $f\in\mathcal{A}$.*

If

$$\left|\frac{zf'(z)}{f(z)}-1\right| \leq M, \quad (z\in\mathcal{U}), \tag{31}$$

and

$$|\alpha| \leq \frac{3\sqrt{3}}{2M}, \tag{32}$$

then the integral operator G_α defined by (2), is in the class $\mathcal{S}$.

Proof. Follows by taking $\beta=0$ in Theorem 1. □

Corollary 2. *Let β be a complex number, L positive real number and the function $g\in\mathcal{A}$.*

If

$$\left|\frac{zg''(z)}{g'(z)}\right| \leq L, \quad (z\in\mathcal{U}), \tag{33}$$

and

$$|\beta| \leq \frac{3\sqrt{3}}{2L}, \tag{34}$$

then the integral operator H_β, defined by (3), belongs to the class $\mathcal{S}$.

Proof. Follows from Theorem 1 by taking $\alpha=0$. □

Corollary 3. *Let α be a complex number and the function $f \in \mathcal{S}$.*
If

$$|\alpha| \leq \frac{1}{4}, \tag{35}$$

then the integral operator $G_\alpha \in \mathcal{S}$.

Proof. Follows by taking $\beta = 0$ in Theorem 2. □

Corollary 4. *Let β be a complex number and the function $g \in \mathcal{A}$, $g' \in \mathcal{P}$.*
If

$$|\beta| \leq \frac{1}{2}, \tag{36}$$

then the integral operator $H_\beta \in \mathcal{S}$.

Proof. Follows from Theorem 3 by taking $\alpha = 0$. □

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