

## EXISTENCE AND UNIQUENESS OF THE WEAK SOLUTION OF ELECTRO-VISCOELASTIC CONTACT PROBLEM

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### Abstract

In this work, the material is assumed to be electro-viscoelastic and the foundation is assumed to be electrically conductive and the friction is modeled with Tresca's law. For each problem we present the mathematical model, its variational formulation, and state an existence and uniqueness result.

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## 1 Introduction

Antiplane shear deformations are one the simplest examples of deformations that solids can undergo : in the antiplane shear of a cylindrical body, the displacement is parallel to the generators on the cylinder and is dependent of the axial coordinate [7, 8, 9]. Piezoelectric materials for which the mechanical properties are elastic are called electro-elastic materials see for more details [2, 3, 4, 5, 6] and those for which the mechanical properties are elastic are called electro-elastic materials. The Mathematics and Mechanics of the art on Contact Mechanics can be found in [1].

The present paper is devoted to functional analysis of electro-viscoelastic antiplane contact problem with friction. The process is static and the friction is modeled with the well known Tresca's law in which the friction bound is given. The behavior of the material is described with a linear electro-viscoelastic constitutive law.

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Our paper is structured as follows. In Section 2 we describe the model of the frictional contact process between electro-viscoelastic body and a conductive deformable foundation. In Section 3 we derive the variational formulation. It consists of a variational inequality for the displacement field coupled with a dependent variational equation for the electric potential. We state our main result, the existence of a unique weak solution to the model in theorems 1 and 2. The proof of Theorem 2 is provided in the end of section 3.2, where it is based on arguments of evolutionary inequalities.

## 2 The Mathematical Model and its Well-Posedness

The physical setting is as follows. We consider a piezoelectric body  $\mathcal{B}$  identified with a region in  $\mathbb{R}^3$  it occupies in a fixed and undistorted reference configuration. We assume that  $\mathcal{B}$  is a cylinder with generators parallel to the  $x_3$ -axes with a cross-section which is a regular region  $\Omega$  in the  $x_1, x_2$ -plane,  $Ox_1x_2x_3$  being a Cartesian coordinate system. The cylinder is assumed to be sufficiently long so that the end effects in the axial direction are negligible. Thus,  $\mathcal{B} = \Omega \times (-\infty, +\infty)$ . The cylinder is acted upon by body forces of density  $\mathbf{f}_0$  and has volume free electric charges of density  $q_0$ . It is also constrained mechanically and electrically on the boundary. To describe the boundary conditions, we denote by  $\partial\Omega = \Gamma$  the boundary of  $\Omega$  and we assume a partition of  $\Gamma$  into three open disjoint parts  $\Gamma_1, \Gamma_2$  and  $\Gamma_3$ , on the one hand, and a partition of  $\Gamma_1 \cup \Gamma_2$  into two open parts  $\Gamma_a$  and  $\Gamma_b$ , on the other hand. We assume that the one-dimensional measure of  $\Gamma_1$  and  $\Gamma_a$ , denoted  $\text{meas}\Gamma_1$  and  $\text{meas}\Gamma_a$ , are positive. The cylinder is clamped on  $\Gamma_1 \times (-\infty, +\infty)$  and therefore the displacement field vanishes there. Surface tractions of density  $\mathbf{f}_2$  act on  $\Gamma_2 \times (-\infty, +\infty)$ . We also assume that the electrical potential vanishes on  $\Gamma_a \times (-\infty, +\infty)$  and a surface electrical charge of density  $q_2$  is prescribed on  $\Gamma_b \times (-\infty, +\infty)$ . The cylinder is in contact over  $\Gamma_3 \times (-\infty, +\infty)$  with a conductive obstacle, the so called foundation. The contact is frictional and is modeled with Tresca's law. We assume that

$$\mathbf{f}_0 = (0, 0, f_0) \quad \text{with} \quad f_0 = f_0(x_1, x_2) : \Omega \rightarrow \mathbb{R}, \quad (1)$$

$$\mathbf{f}_2 = (0, 0, f_2) \quad \text{with} \quad f_2 = f_2(x_1, x_2) : \Gamma_2 \rightarrow \mathbb{R}, \quad (2)$$

$$q_0 = q_0(x_1, x_2) : \Omega \rightarrow \mathbb{R}, \quad (3)$$

$$q_2 = q_2(x_1, x_2) : \Gamma_b \rightarrow \mathbb{R}. \quad (4)$$

The forces (1), (2) and the electric charges (3), (4) would be expected to give rise to deformations and to electric charges of the piezoelectric cylinder corresponding to a displacement  $\mathbf{u}$  and to an electric potential field  $\varphi$  which are independent on  $x_3$  and have the form

$$\mathbf{u} = (0, 0, u) \quad \text{with} \quad u = u(x_1, x_2) : \Omega \rightarrow \mathbb{R}, \quad (5)$$

$$\varphi = \varphi(x_1, x_2) : \Omega \rightarrow \mathbb{R}. \quad (6)$$

Below in this paper the indices  $i$  and  $j$  denote components of vectors and tensors and run from 1 to 3, summation over two repeated indices is implied, and the index that follows a comma represents the partial derivative with respect to the corresponding spatial variable. We use  $\mathcal{S}^3$  for the linear space of second order symmetric tensors on  $\mathbb{R}^3$  or, equivalently, the space of symmetric matrices of order 3, and “ $\cdot$ ”,  $\|\cdot\|$  will represent the inner products and the Euclidean norms on  $\mathbb{R}^3$  and  $\mathcal{S}^3$ ; we have :

$$\mathbf{u} \cdot \mathbf{v} = u_i v_i, \quad \|\mathbf{v}\| = (\mathbf{v} \cdot \mathbf{v})^{1/2} \quad \text{for all } \mathbf{u} = (u_i), \mathbf{v} = (v_i) \in \mathbb{R}^3$$

and

$$\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \sigma_{ij} \tau_{ij}, \quad \|\boldsymbol{\tau}\| = (\boldsymbol{\tau} \cdot \boldsymbol{\tau})^{1/2} \quad \text{for all } \boldsymbol{\sigma} = (\sigma_{ij}), \boldsymbol{\tau} = (\tau_{ij}) \in \mathcal{S}^3.$$

The infinitesimal strain tensor is denoted  $\boldsymbol{\varepsilon}(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u}))$  and the stress field by  $\boldsymbol{\sigma} = (\sigma_{ij})$ . We also denote by  $\mathbf{E}(\varphi) = (E_i(\varphi))$  the electric field and by  $\mathbf{D} = (D_i)$  the electric displacement field. Here and below, in order to simplify the notation, we do not indicate the dependence of various functions on  $x_1, x_2, x_3$  or  $t$  and we recall that

$$\varepsilon_{ij}(\mathbf{u}) = \frac{1}{2} (u_{i,j} + u_{j,i}), \quad E_i(\varphi) = -\varphi_{,i}.$$

The material's is modeled by the following electro-elastic constitutive law

$$\boldsymbol{\sigma} = 2\theta\boldsymbol{\varepsilon}(\dot{\mathbf{u}}) + \zeta \operatorname{tr} \boldsymbol{\varepsilon}(\dot{\mathbf{u}}) \mathbf{I} + 2\mu\boldsymbol{\varepsilon}(\mathbf{u}) + \lambda \operatorname{tr} \boldsymbol{\varepsilon}(\mathbf{u}) \mathbf{I} - \mathcal{E}^* \mathbf{E}(\varphi), \quad (7)$$

$$\mathbf{D} = \mathcal{E} \boldsymbol{\varepsilon}(\mathbf{u}) + \alpha \mathbf{E}(\varphi), \quad (8)$$

where  $\zeta$  and  $\theta$  are viscosity coefficients,  $\lambda$  and  $\mu$  are the Lamé coefficients,  $\operatorname{tr} \boldsymbol{\varepsilon}(\mathbf{u}) = \varepsilon_{ii}(\mathbf{u})$ ,  $\mathbf{I}$  is the unit tensor in  $\mathbb{R}^3$ ,  $\beta$  is the electric permittivity constant,  $\mathcal{E}$  represents the third-order piezoelectric tensor and  $\mathcal{E}^*$  is its transpose.

In the antiplane context (5), (6), using the constitutive equations (7), (8) it follows that the stress field and the electric displacement field are given by

$$\boldsymbol{\sigma} = \begin{pmatrix} 0 & 0 & \sigma_{13} \\ 0 & 0 & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & 0 \end{pmatrix}, \quad (9)$$

$$\mathbf{D} = \begin{pmatrix} eu_{,1} - \alpha\varphi_{,1} \\ eu_{,2} - \alpha\varphi_{,2} \\ 0 \end{pmatrix} \quad (10)$$

where

$$\sigma_{13} = \sigma_{31} = \mu \partial_{x_1} u$$

and

$$\sigma_{23} = \sigma_{32} = \mu \partial_{x_2} u.$$

We assume that

$$\mathcal{E} \boldsymbol{\varepsilon} = \begin{pmatrix} e(\varepsilon_{13} + \varepsilon_{31}) \\ e(\varepsilon_{23} + \varepsilon_{32}) \\ e\varepsilon_{33} \end{pmatrix} \quad \forall \boldsymbol{\varepsilon} = (\varepsilon_{ij}) \in \mathcal{S}^3, \quad (11)$$

where  $e$  is a piezoelectric coefficient. We also assume that the coefficients  $\mu$ ,  $\alpha$  and  $e$  depend on the spatial variables  $x_1$ ,  $x_2$ , but are independent on the spatial variable  $x_3$ . Since  $\mathcal{E}\boldsymbol{\varepsilon} \cdot \mathbf{v} = \boldsymbol{\varepsilon} \cdot \mathcal{E}^*\mathbf{v}$  for all  $\boldsymbol{\varepsilon} \in \mathcal{S}^3$ ,  $\mathbf{v} \in \mathbb{R}^3$ , it follows from (11) that

$$\mathcal{E}^*\mathbf{v} = \begin{pmatrix} 0 & 0 & ev_1 \\ 0 & 0 & ev_2 \\ ev_1 & ev_2 & ev_3 \end{pmatrix} \quad \forall \mathbf{v} = (v_i) \in \mathbb{R}^3. \quad (12)$$

We assume that the process is mechanically quasistatic and electrically static and therefore is governed by the equilibrium equations

$$\text{Div } \boldsymbol{\sigma} + \mathbf{f}_0 = \mathbf{0}, \quad D_{i,i} - q_0 = 0 \quad \text{in } \mathcal{B},$$

where  $\text{Div } \boldsymbol{\sigma} = (\sigma_{ij,j})$  represents the divergence of the tensor field  $\boldsymbol{\sigma}$ . Taking into account (1), (3), (5), (6), (9) and (10), the equilibrium equations above reduce to the following scalar equations

$$\text{div}(\theta \nabla \dot{u}) + \text{div}(\mu \nabla u) + \text{div}(e \nabla \varphi) + f_0 = 0, \quad \text{in } \Omega, \quad (13)$$

$$\text{div}(e \nabla u) - \text{div}(\alpha \nabla \varphi) = q_0, \quad \text{in } \Omega. \quad (14)$$

Here and below we use the notation

$$\text{div } \boldsymbol{\tau} = \tau_{1,1} + \tau_{1,2} \quad \text{in } \boldsymbol{\tau} = (\tau_1(x_1, x_2), \tau_2(x_1, x_2))$$

and

$$\nabla v = (v_{,1}, v_{,2}), \quad \partial_\nu v = v_{,1} \nu_1 + v_{,2} \nu_2 \quad \text{for } v = v(x_1, x_2).$$

We now describe the boundary conditions. During the process the cylinder is clamped on  $\Gamma_1 \times (-\infty, +\infty)$  and the electric potential vanishes on  $\Gamma_1 \times (-\infty, +\infty)$ ; thus, (5) and (6) imply that

$$u = 0 \quad \text{on } \Gamma_1, \quad (15)$$

$$\varphi = 0 \quad \text{on } \Gamma_a. \quad (16)$$

Let  $\boldsymbol{\nu}$  denote the unit normal on  $\Gamma \times (-\infty, +\infty)$ . We have

$$\boldsymbol{\nu} = (\nu_1, \nu_2, 0) \quad \text{with } \nu_i = \nu_i(x_1, x_2) : \Gamma \rightarrow \mathbb{R}, \quad i = 1, 2. \quad (17)$$

For a vector  $\mathbf{v}$  we denote by  $v_\nu$  and  $\mathbf{v}_\tau$  its normal and tangential components on the boundary, defined by

$$v_\nu = \mathbf{v} \cdot \boldsymbol{\nu}, \quad \mathbf{v}_\tau = \mathbf{v} - v_\nu \boldsymbol{\nu}. \quad (18)$$

respectively. In (18) and everywhere in this paper “ $\cdot$ ” represents the inner product on the space  $\mathbb{R}^d$  ( $d = 2, 3$ ). Moreover, for a given stress field  $\boldsymbol{\sigma}$  we denote by  $\sigma_\nu$  and  $\boldsymbol{\sigma}_\tau$  the normal and the tangential components on the boundary, that is

$$\sigma_\nu = (\boldsymbol{\sigma} \boldsymbol{\nu}) \cdot \boldsymbol{\nu}, \quad \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma} \boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}. \quad (19)$$

From (9), (10) and (17) we deduce that the Cauchy stress vector and the normal component of the electric displacement field are given by

$$\boldsymbol{\sigma}\boldsymbol{\nu} = (0, 0, \theta\partial\dot{u} + \mu\partial_\nu u + e\partial_\nu\varphi), \quad \mathbf{D} \cdot \boldsymbol{\nu} = e\partial_\nu u - \alpha\partial_\nu\varphi. \quad (20)$$

Taking into account (2), (4) and (20), the traction condition on  $\Gamma_2 \times (-\infty, \infty)$  and the electric conditions on  $\Gamma_b \times (-\infty, \infty)$  are

$$\theta\partial\dot{u} + \mu\partial_\nu u + e\partial_\nu\varphi = f_2 \quad \text{on } \Gamma_2, \quad (21)$$

$$e\partial_\nu u - \alpha\partial_\nu\varphi = q_2 \quad \text{on } \Gamma_b. \quad (22)$$

We now describe the frictional contact condition and the electric conditions on  $\Gamma_3 \times (-\infty, +\infty)$ . First, from (5) and (17) we infer that the normal displacement vanishes,  $u_\nu = 0$ , which shows that the contact is bilateral, that is, the contact is kept during the whole process. Using now (5) and (17)–(19) we conclude that

$$\mathbf{u}_\tau = (0, 0, u), \quad \boldsymbol{\sigma}_\tau = (0, 0, \sigma_\tau) \quad (23)$$

where

$$\sigma_\tau = (0, 0, \theta\partial\dot{u} + \mu\partial_\nu u + e\partial_\nu\varphi).$$

We assume that the friction is invariant with respect to the  $x_3$  axis and is modeled with Tresca's friction law, that is

$$\begin{cases} |\boldsymbol{\sigma}_\tau| \leq g, \\ |\boldsymbol{\sigma}_\tau| = -g \frac{\mathbf{u}}{|\mathbf{u}|} \quad \text{if } \mathbf{u} \neq 0 \quad \text{on } \Gamma_3. \end{cases} \quad (24)$$

Here  $g : \Gamma_3 \rightarrow \mathbb{R}_+$  is a given function, the friction bound. Using now (23) it is straightforward to see that the friction law (24) implies

$$\begin{cases} |\theta\partial\dot{u} + \mu\partial_\nu u + e\partial_\nu\varphi| \leq g, \\ |\theta\partial\dot{u} + \mu\partial_\nu u + e\partial_\nu\varphi| = -g \frac{u}{|u|} \quad \text{if } u \neq 0 \quad \text{on } \Gamma_3. \end{cases} \quad (25)$$

Finally, we collect the above equations and conditions to obtain the following mathematical model which describes the antiplane shear of an electro-elastic cylinder in frictional contact with a conductive foundation.

### Problem P.

Find the displacement field  $u : \Omega \rightarrow \mathbb{R}$  and the electric potential  $\varphi : \Omega \rightarrow \mathbb{R}$  such that

$$\operatorname{div}(\theta\nabla\dot{u}) + \operatorname{div}(\mu\nabla u) + \operatorname{div}(e\nabla\varphi) + f_0 = 0, \quad \text{in } \Omega, \quad (26)$$

$$\operatorname{div}(e\nabla u) - \operatorname{div}(\alpha\nabla\varphi) = q_0 \quad \text{in } \Omega, \quad (27)$$

$$u = 0 \quad \text{on } \Gamma_1, \quad (28)$$

$$\theta\partial\dot{u} + \mu\partial_\nu u + e\partial_\nu\varphi = f_2 \quad \text{on } \Gamma_2, \quad (29)$$

$$\begin{cases} |\theta \partial_i u + \mu \partial_\nu u + e \partial_\nu \varphi| \leq g, \\ |\theta \partial_i u + \mu \partial_\nu u + e \partial_\nu \varphi| = -g \frac{u}{|u|} \text{ if } u \neq 0 \text{ on } \Gamma_3, \end{cases} \quad (30)$$

$$\varphi = 0 \text{ in } \Gamma_a. \quad (31)$$

$$e \partial_\nu u - \alpha \partial_\nu \varphi = q_2 \text{ on } \Gamma_b, \quad (32)$$

Note that once the displacement field  $u$  and the electric potential  $\varphi$  which solve Problem **P** are known, then the stress tensor  $\boldsymbol{\sigma}$  and the electric displacement field  $\mathbf{D}$  can be obtained by using the constitutive laws (7) and (8), respectively.

### 3 Variational Formulation and Main Results

#### 3.1 Variational Formulation

We derive now the variational formulation of Problem **P**. To this end we introduce the function spaces

$$V = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_1\}, \quad W = \{\psi \in H^1(\Omega) : \psi = 0 \text{ on } \Gamma_a\}$$

where, here and below, we write  $w$  for the trace  $\gamma w$  of a function  $w \in H^1(\Omega)$  on  $\Gamma$ . Since  $\text{meas } \Gamma_1 > 0$  and  $\text{meas } \Gamma_a > 0$ , it is well known that  $V$  and  $W$  are real Hilbert spaces with the inner products

$$(u, v)_V = \int_{\Omega} \nabla u \cdot \nabla v \, dx \quad \forall u, v \in V, \quad (\varphi, \psi)_W = \int_{\Omega} \nabla \varphi \cdot \nabla \psi \, dx \quad \forall \varphi, \psi \in W.$$

Moreover, the associated norms

$$\|v\|_V = \|\nabla v\|_{L^2(\Omega)} \quad \forall v \in V, \quad \|\psi\|_W = \|\nabla \psi\|_{L^2(\Omega)} \quad \forall \psi \in W \quad (33)$$

are equivalent on  $V$  and  $W$ , respectively, with the usual norm  $\|\cdot\|_{H^1(\Omega)}$ . By Sobolev's trace theorem we deduce that there exist two positive constants  $c_V > 0$  and  $c_W > 0$  such that

$$\|v\|_{L^2(\Gamma_3)} \leq c_V \|v\|_V \quad \forall v \in V, \quad \|\psi\|_{L^2(\Gamma_3)} \leq c_W \|\psi\|_W \quad \forall \psi \in W. \quad (34)$$

Let  $X = V \times W$  be a real Hilbert space with inner product  $(\cdot, \cdot)$  and the norm  $\|\cdot\|_X$ .

In the study of Problem **P** we assume that the viscosity coefficient and the electric permittivity coefficient satisfy

$$\theta \in L^\infty(\Omega) \text{ and there exists } \theta^* > 0 \text{ such that } \theta(\mathbf{x}) \geq \theta^* \text{ a.e. } \mathbf{x} \in \Omega, \quad (35)$$

$$\beta \in L^\infty(\Omega) \text{ and there exists } \beta^* > 0 \text{ such that } \beta(\mathbf{x}) \geq \beta^* \text{ a.e. } \mathbf{x} \in \Omega. \quad (36)$$

The forces, tractions, volume and surface free charge densities have the regularity

$$f_0 \in L^2(\Omega), \quad f_2 \in L^2(\Gamma_2), \quad (37)$$

$$q_0 \in L^2(\Omega), \text{ and } q_2 \in L^2(\Gamma_b) \text{ such that } q_2 = 0 \text{ a.e. } \mathbf{x} \in \Gamma_b. \quad (38)$$

Finally, we assume that the electric conductivity coefficient and the friction bound function  $g$  satisfies the following properties

$$g \in L^\infty(\Gamma_3) \text{ and } g(\mathbf{x}) \geq 0 \text{ a.e. } \mathbf{x} \in \Gamma_3. \quad (39)$$

We define now the functional  $j : V \rightarrow \mathbb{R}_+$  by the formula

$$j(v) = \int_{\Gamma_3} g|v| da \quad \forall v \in V. \quad (40)$$

We also define the mappings  $f \in V$  and  $q \in W$ , respectively, by

$$(f, v)_V = \int_{\Omega} f_0 v dx + \int_{\Gamma_2} f_2 v da, \quad (41)$$

$$(q, \psi)_W = \int_{\Gamma_b} q_2 \psi da - \int_{\Omega} q_0 \psi dx, \quad (42)$$

for all  $v \in V$  and  $\psi \in W$ . The definitions of  $f$  and  $q$  are based on Riesz's representation theorem.

Next, we define the bilinear forms  $a_\theta : V \times V \rightarrow \mathbb{R}$ ,  $a_\mu : V \times V \rightarrow \mathbb{R}$ ,  $a_e : V \times W \rightarrow \mathbb{R}$ ,  $a_e : W \times V \rightarrow \mathbb{R}$ , and  $a_\alpha : W \times W \rightarrow \mathbb{R}$ , by equalities

$$a_\theta(u, v) = \int_{\Omega} \theta \nabla u \cdot \nabla v dx, \quad (43)$$

$$a_\mu(u, v) = \int_{\Omega} \mu \nabla u \cdot \nabla v dx, \quad (44)$$

$$a_e(u, \varphi) = \int_{\Omega} e \nabla u \cdot \nabla \varphi dx = a_e(\varphi, u), \quad (45)$$

$$a_\alpha(\varphi, \psi) = \int_{\Omega} \alpha \nabla \varphi \cdot \nabla \psi dx, \quad (46)$$

for all  $u, v \in V$ ,  $\varphi, \psi \in W$ . Assumptions (40)–(42) imply that the integrals above are well defined and, using (33) and (34), it follows that the forms  $a_\theta$ ,  $a_\mu$ ,  $a_e$  and  $a_e$  are continuous; moreover, the forms  $a_\mu$  and  $a_\alpha$  are symmetric and, in addition, the forms  $a_\theta$  and  $a_\alpha$  are  $W$ -elliptic, since

$$a_\alpha(\psi, \psi) \geq \alpha^* \|\psi\|_V^2 \quad \forall \psi \in W \quad (47)$$

and

$$a_\theta(v, v) \geq \theta^* \|v\|_W^2 \quad \forall v \in V. \quad (48)$$

The variational formulation of Problem **P** is based on the following result.

### 3.2 Main Results

**Lemma 1.** *If  $(u, \varphi)$  is a smooth solution to Problem **P**, then  $(u, \varphi) \in X$  and*

$$a_\theta(u, v - u) + a_\mu(u, v - u) + a_e(\varphi, v - u) + j(v) - j(u) \geq (f, v - u)_V \quad \forall v \in V, \quad (49)$$

$$a_\alpha(\varphi, \psi) - a_e(u, \psi) = (q, \psi)_W \quad \forall \psi \in W. \quad (50)$$

### Proof of Lemma 3.1

Let  $(u, \varphi)$  denote a smooth solution to Problem **P**, we have  $u \in V$  and  $\varphi \in W$  and, from (26), (28) and (29), we get

$$\int_{\Omega} \theta \nabla u \cdot \nabla(v - u) dx + \int_{\Omega} \mu \nabla u \cdot \nabla(v - u) dx + \int_{\Omega} e \nabla \varphi \cdot \nabla(v - u) dx = \quad (51)$$

$$\int_{\Omega} f_0(v - u) dx + \int_{\Gamma} (\mu \partial_\nu u + e \partial_\nu \varphi)(v - u) da \quad \forall v \in V, \quad (52)$$

and from (27) and (32) we have

$$\int_{\Omega} \alpha \nabla \varphi \cdot \nabla \psi dx - \int_{\Omega} e \nabla u \cdot \nabla \psi dx = \int_{\Gamma} (\mu \partial_\nu u - e \partial_\nu \varphi) \psi da - \int_{\Omega} q_0 \psi dx \quad (53)$$

$\forall \psi \in W.$

#### • Proof of (49)

From the friction law (30) we can write

$$-(\theta \partial_\nu \dot{u} + \mu \partial_\nu u + e \partial_\nu \varphi)u = -g|u| \quad \text{on } \Gamma_3, \quad (54)$$

it's very easy to see that for all  $x$  and  $y \in \mathbb{R}$ ,  $xy \geq -|x||y|$ , then equation (54) takes the form

$$(\theta \partial_\nu \dot{u} + \mu \partial_\nu u + e \partial_\nu \varphi)(v - u) \geq g|u| - g|v| \quad \text{on } \Gamma_3. \quad (55)$$

By integration on  $\Gamma$ , we get

$$\int_{\Gamma} (\theta \partial_\nu \dot{u} + \mu \partial_\nu u + e \partial_\nu \varphi)(v - u) da \geq \int_{\Gamma} (g|u| - g|v|) da \quad \text{on } \Gamma_3. \quad (56)$$

The left term in inequality (56) can be written as follows

$$\int_{\Gamma} (\theta \partial_\nu \dot{u} + \mu \partial_\nu u + e \partial_\nu \varphi)(v - u) da = \int_{\Gamma_1} (\theta \partial_\nu \dot{u} + \mu \partial_\nu u + e \partial_\nu \varphi)(v - u) da +$$

$$\int_{\Gamma_2} (\theta \partial_\nu \dot{u} + \mu \partial_\nu u + e \partial_\nu \varphi)(v - u) da + \int_{\Gamma_3} (\theta \partial_\nu \dot{u} + \mu \partial_\nu u + e \partial_\nu \varphi)(v - u) da \quad \forall v \in V.$$



Using (28), (29) in the last equation, then for all  $v \in V$  we have:

$$\int_{\Gamma} (\theta \partial_{\nu} \dot{u} + \mu \partial_{\nu} u + e \partial_{\nu} \varphi)(v - u) da = \int_{\Gamma_3} (\theta \partial_{\nu} \dot{u} + \mu \partial_{\nu} u + e \partial_{\nu} \varphi)(v - u) da \quad \forall v \in V. \quad (57)$$

We combine (56) and (57)

$$\int_{\Gamma} (\theta \partial_{\nu} \dot{u} + \mu \partial_{\nu} u + e \partial_{\nu} \varphi)(v - u) da \geq \int_{\Gamma_3} g|u| da - \int_{\Gamma_3} g|v| da \quad \text{on } \Gamma_3 \quad \forall v \in V. \quad (58)$$

Now, we use (29), (40), (41), (44), (45) and (58) in (52), then

$$a_{\theta}(u, v - u) + a_{\mu}(u, v - u) + a_e(\varphi, v - u) + j(v) - j(u) \geq (f, v - u)_V \quad \forall v \in V. \quad (59)$$

### • Proof of (50)

On the other hand, the integral on  $\Gamma$  in the left of inequality (53) can be written as

$$\int_{\Gamma} (\mu \partial_{\nu} u - e \partial_{\nu} \varphi) \psi da = \int_{\Gamma_a} (\mu \partial_{\nu} u - e \partial_{\nu} \varphi) \psi da + \int_{\Gamma_b} (\mu \partial_{\nu} u - e \partial_{\nu} \varphi) \psi da. \quad (60)$$

Using (31) and (32), then we get

$$\int_{\Gamma} (\mu \partial_{\nu} u - e \partial_{\nu} \varphi) \psi da = \int_{\Gamma_b} q_2 \psi da. \quad (61)$$

Now, we combine equality (61) with (53), we infer

$$\int_{\Omega} \alpha \nabla \varphi \cdot \nabla \psi dx - \int_{\Omega} e \nabla u \cdot \nabla \psi dx = \int_{\Gamma_b} q_2 \psi da - \int_{\Omega} q_0 \psi dx \quad \forall \psi \in W. \quad (62)$$

Keeping in mind (42), (45) and (46), we find the second equality in Lemma 1, i.e.,

$$a_{\alpha}(\varphi, \psi) - a_e(u, \psi) = (q, \psi)_W \quad \forall \psi \in W, \quad (63)$$

which conclude the proof.

Now, the use of Lemma 3.1 gives the following variational Problem:

### Problem PV.

Find a displacement field  $u : \Omega \rightarrow \mathbb{R}$  and an electric potential field  $\varphi : \Omega \rightarrow \mathbb{R}$  such that

$$a_{\theta}(u, v - u) + a_{\mu}(u, v - u) + a_e(\varphi, v - u) + j(v) - j(u) \geq (f, v - u)_V, \quad \forall v \in V, \quad (64)$$

$$a_{\alpha}(\varphi, \psi) - a_e(u, \psi) = (q, \psi)_W, \quad \forall \psi \in W. \quad (65)$$

Now, let us define the bilinear form  $a(\cdot, \cdot) : X \times X \rightarrow \mathbb{R}$ , the functional  $J(\cdot) : X \rightarrow \mathbb{R}$  and the element  $F$  as follows:

$$\begin{aligned} a(x, y) &= a_\theta(u, v - u) + a_\mu(u, v - u) + a_\alpha(\varphi, \psi) + a_e(\varphi, v - u), \\ \forall x &= (u, \varphi) \in X, \forall y = (v, \psi) \in X, \end{aligned} \quad (66)$$

$$J(x) = j(u), \quad \forall x = (u, \varphi) \in X, \quad (67)$$

and

$$F = (f, q) \in X. \quad (68)$$

Using expressions (66), (67) and (68) in (64) and (65), we obtain the new Problem:

### Problem PVG.

Find a displacement field  $x \in X$  such that

$$a(x, y - x) + J(y) - J(x) \geq (F, y - x)_X, \quad \forall y \in X. \quad (69)$$

Our main equivalent result between Problems **PV** and **PVG** is the following:

**Theorem 1.** *The Problems **PV** and **PVG** are equivalent.*

## Proof of Theorem 3.2

We start with the Proof of Theorem 1 which will be carried out in several steps.

### • Proof that PV implies PVG:

In the first step we will suppose that  $(u, \varphi)$  is solution of Problem **PV**. We change in (65) the element  $\psi$  by  $(\psi - \varphi)$  and we add the resulting equation to both sides of inequality (64), hence we have

$$\begin{aligned} a_\theta(u, v - u) + a_\mu(u, v - u) + a_e(\varphi, v - u) + a_\alpha(\varphi, \psi - \varphi) - a_e(u, \psi - \varphi) + \\ + j(v) - j(u) \geq (f, v - u)_V + (q, \psi - \varphi)_W. \end{aligned}$$

Using now notations (66), (67) and (68), then for all  $\psi \in W$  and for all  $y \in X$ , we get

$$a(x, y - x) + J(y) - J(x) \geq (F, y - x)_X, \quad (70)$$

which conclude the proof of the first setp.

• **Proof that PV implies PVG:**

In the second step we will suppose that  $x = (u, \varphi)$  is the solution of Problem **PVG**. We change the bilinear form  $a(\cdot, \cdot)$  by (66),  $(F, y - x)_X$  by (68) and the functional  $J(\cdot)$  by (67); then for all  $(v, \psi) \in X$  we obtain

$$\begin{aligned} a_\theta(u, v - u) + a_\mu(u, v - u) + a_e(\varphi, v - u) + a_\alpha(\varphi, \psi - \varphi) + j(v) - j(u) \\ \geq (f, v - u)_V + (q, \psi - \varphi)_W. \end{aligned} \quad (71)$$

We test in the last inequality (71) with  $\psi = \varphi$ , then we obtain (64). Next, we take  $v = u$  and  $\psi - \varphi = \varphi \pm \psi - \varphi$  in (71), it follows that for all  $\psi \in W$

$$a_\alpha(\varphi, \pm\psi) - a_e(u, \pm\psi) \geq (q, \pm\psi)_W \quad \forall \psi \in W, \quad (72)$$

which conclude the proof of the second setp. Then, the Problems **PV** and **PVG** are equivalent.

Our main existence and uniqueness result, which we state now and prove in the next section, is the following :

**Theorem 2.** *Assume that (36)–(42) hold. Then the variational problem **PVG** possesses a unique solution  $x = (u, \varphi)$  satisfies*

$$a(x, y - x) + J(y) - J(x) \geq (F, y - x)_X, \quad \forall y \in X. \quad (73)$$

We note that an element  $x = (u, \varphi)$  which solves Problem **PV** is called a weak solution of the antiplane contact Problem **PV**. We conclude by Theorem 1 that the element  $x = (u, \varphi)$  also solves Problem **PVG**, then the element  $x$  is called a weak solution of the antiplane contact Problem **PVG**. Hence, the antiplane contact Problem **P** has a unique weak solution, provided that (36)–(42) hold.

## Proof of Theorem 2

The proof of Theorem 2 is based on an abstract result for **evolutionary variational inequalities** that we present in what follows.

Let  $X$  be a real Hilbert space with the inner product  $(\cdot, \cdot)_X$  and the associated norm  $\|\cdot\|_X$  and consider the problem of finding a displacement field  $x \in X$  such that

$$a(x, y - x) + J(y) - J(x) \geq (F, y - x)_X, \quad \forall y \in X. \quad (74)$$

In the study of Problem (73) we assume that

$$a : X \times X \rightarrow \mathbb{R} \text{ is a bilinear symmetric form and} \quad (75)$$

$$(a) \text{ there exists } M > 0 \text{ such that } |a(u, v)| \leq M \|u\|_X \|v\|_X, \text{ for all } u, v \in X \quad (76)$$

$$(b) \text{ there exists } m > 0 \text{ such that } a(v, v) \geq m \|v\|_X^2, \text{ for all } v \in V. \quad (77)$$

$b : X \times X \rightarrow \mathbb{R}$  is a bilinear symmetric form and there exists  $M' > 0$  (78)

such that  $|b(u, v)| \leq M' \cdot \|u\|_X \|v\|_X$ , for all  $u, v \in X$ . (79)

$j : X \rightarrow \mathbb{R}$  is a convex lower semicontinuous functional. (80)

We start with the Proof of Theorem 2 which will be carried out in several steps. To this end, in the rest of this section we assume that (36)–(42) hold. In the first step we will use (80), then we obtain that the bilinear form (66) satisfies

$$|a(x, y)| \leq (\|\mu\|_{L^\infty(\Omega)} + \|\alpha\|_{L^\infty(\Omega)} + 2\|e\|_{L^\infty(\Omega)}) \|x\|_X \cdot \|y\|_Y, \quad \forall x, y \in X, \quad (81)$$

i.e., then the bilinear form  $a(\cdot, \cdot)$  is continuous. It follows that  $a(\cdot, \cdot)$  is elliptic because

$$a(x, y) \geq \mu^* \cdot \|u\|_V^2 + \alpha^* \cdot \|\varphi\|_W^2, \quad \forall x \in X, \quad (82)$$

consequently, from the last inequality we get

$$a(x, y) \geq \min(\mu^*, \alpha^*) \cdot \|x\|_X^2, \quad \forall x \in X. \quad (83)$$

Now, using the hypothesis (39), then we have

$$J(x) = j(u) \leq c\|u\|_{L^2(\Gamma_3)} \leq c\|u\|_V \leq c\|x\|_X, \quad \forall x \in X, \quad (84)$$

where  $c > 0$  dependent on function  $g$ . It follows that the functional  $J$  defined in (67) is a seminorm continuous on space  $X$ , then  $J$  is convex and is a convex lower semicontinuous functional. Now we have all the ingredients to prove Theorem 1. Using Theorem 2, it follows that Problem **PVG** has a unique solution  $x = (u, \varphi) \in X$ . Now, we coupled Theorems 1 and 2, it follows that Problem **PV** has a unique solution  $x = (u, \varphi)_X$ . This solution can be interpreted as weak solution of the antiplane contact Problem **P**.

## References

- [1] Chau, O., *Dynamic contact problems for viscoelastic materials*; Proceedings of Fourteenth International Symposium on Mathematical theory of networks and systems, (MTNS), Perpignan, (2000).
- [2] Han, W., Sofonea, M., *Quasistatic contact problems in viscoelasticity and viscoplasticity*; Studies in Advanced Mathematics, Americal Mathematical Society, Providence, RI–International Press, Somerville, **30** (2002), MA.
- [3] Hoarau-Mantel, T.-V., Matei, A., *Analysis of a viscoelastic antiplane contact problem with slip-dependent friction*; Int. J. Appl. Math. Comput. Sci. **12** (2002), 51–58.
- [4] Matei, A., Motreanu, V. V., Sofonea, M., *A quasistatic antiplane contact problem with slip dependent friction*; Advances in Nonlinear Variational Inequalities, **4** (2007), 1-21.

- [5] Shillor, M., Sofonea, M., Telega, J. J., *Models and analysis of quasistatic contact*; Lect. Notes Phys, Springer, Berlin Heidelberg, **655** (2004), .
- [6] Sofonea, M., Essoufi, El H., *Quasistatic frictional contact of a viscoelastic piezoelectric body*; Adv. Math. Sci. Appl. **14** (2004), 613–631.
- [7] Sofonea, M., Dalah, M., *Antiplane frictional contact of electro-Viscoelastic cylinders*; Electronic Journal of Differential Equations. no. 161, (2007), 1–14., 2007.
- [8] Sofonea, M., Dalah, M., Ayadi, A., *Analysis of an antiplane electro-elastic contact problem*; Adv. Math. Sci. Appl., **17** (2007), 385–400.
- [9] Dalah, M., *Analysis of electro-viscoelastic antiplane contact problem with total slip rate dependent friction*; Electronic Journal of Differential Equations. no. 118, (2009), 1–15, .

