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A TAKAGI-SUGENO FUZZY MODEL OF THE INSULIN TO GLUCOSE SYSTEM BASED ON THE MINIMAL MODEL

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Abstract: *In this paper, we introduce a Takagi-Sugeno fuzzy model of the glucose-insulin system, derived from the well-known minimal model, with application in the design of fuzzy control systems for blood glucose concentration. The fuzzy model can be an alternative approach to the classical linearization method applied to the nonlinear minimal model, which is often used in the analytical design of such systems. The main benefit of the fuzzy model is the possibility to use it as the source for finding or tuning control rules of a fuzzy controller. For model validation, simulated time responses for step inputs of insulin and glucose are presented.*

Key words: *glucose-insulin system, glucose-insulin minimal model, Takagi-Sugeno fuzzy models, simulation of glucose-insulin models.*

1. Introduction

A Takagi-Sugeno (T-S) fuzzy model consists in a set of fuzzy *if-then* rules, which represent local linear input-output relations of a nonlinear system [8]. The overall dynamics of the nonlinear system is expressed, or approximated, by a fuzzy blending of these linear "sub-models". The local operating conditions are defined by setting premise variables, which usually are time functions in the nonlinear process.

It should be pointed out that premise variables are varying in wide domains, so that a "nominal" operating point is invalid for most of the time, or it is very difficult to even set one. In this case, some would suggest defining multiple operating points over the range of premise variables and then apply linearization techniques around each of those points. But the idea is similar to the main feature of a fuzzy model and it gives the reasons for searching and testing a T-S fuzzy model, as an alternative to the more classical linearization methods.

Based on the aforementioned arguments, we searched for a T-S fuzzy model to approximate the well-known insulin to glucose minimal model. This describes the dynamics of blood glucose concentration under the effect of an exogenous insulin infusion rate, for a type 1 diabetic. In this case, without a careful diet, the glucose concentration can rise well above the basal value (81 mg/dL), which often considered as a kind of nominal value. On the other hand, the minimal model is quite simple but still nonlinear.

Not least, an automatic control system for blood glucose concentration is an aim of numerous studies and experiments. A good part of them focused on using fuzzy

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controllers, and many say that having a fuzzy model of the process would ease the design of a fuzzy control system.

2. The Insulin-to-Glucose Minimal Model and Its Usual Linearization

The minimal model describes the effect of exogenous insulin infusion rate and glucose disturbance over the blood glucose concentration, in type 1 diabetes mellitus (T1DM) patients. While several variations of it or similar compartment-based models are introduced in literature (see examples in [1] and [7]), the initial model is a 3-order nonlinear model, described in [2-4] or [6] as follows:

$$\begin{aligned} \dot{G}(t) &= -[p_1 + X(t)]G(t) + p_1 G_b + ad(t), \\ \dot{X}(t) &= -p_2 X(t) + p_3 I(t), \\ \dot{I}(t) &= -nI(t) + bu(t), \end{aligned} \tag{1}$$

where $G(t)$ [mg/dL] is the blood glucose concentration, $I(t)$ [mU/dL] is the blood insulin concentration, $X(t)$ [1/min] is a variable proportional to the absorption of insulin in the insulin-excitable tissues ([5]), $d(t)$ [mg/min] is a disturbance input of glucose and $u(t)$ [mU/min] is the exogenous insulin deliver rate. The rest of the notations are constant parameters described in [2], [5], [9]. The values in the simulations done for this paper are those from [2] and they are listed in Table 1.

In some conventional controller design methods, a linear model is obtained by linearizing the term $X(t)G(t)$ in (1) with the Taylor series expansion about an operating point, which leads to:

$$\begin{aligned} \dot{G}(t) = &-p_1 G(t) - X_a G(t) - G_a X(t) \\ &+ X_a G_a + p_1 G_b + ad(t). \end{aligned} \tag{2}$$

The operating point is defined by X_a and G_a, which are mentioned as nominal values in [2], or average values in [3]. Further, the constant term $X_a G_a + p_1 G_b$ is further neglected [2], leading to the linear Equation:

$$\dot{G}(t) = -(p_1 + X_a)G(t) - G_a X(t) + ad(t). \tag{3}$$

This together with the second and the third equations in (1) forms the linear model of the insulin to glucose system, which is used in conventional controller design methods.

One drawback of this linear model is the parameter X_a, for which we couldn't find any argued value in the literature. In fact, many authors seem to ignore it completely, while others mention some arguments that seem a little unpractical. In the simulations done for this article, we simply adopted the value from [3]. In [2], it is mentioned that the linear model is equivalent to the nonlinear one at a specific time moment if $X_a = X(t)$ and $G_a = G(t)$, meaning that X_a is considered variable. Although this is mathematically correct, it suggests that a control system should be adapted with every iteration in the control algorithm based on the value of $X(t)$. Even more, since it is not a measurable function, we can only have an estimated value of it.

3. Deriving a Takagi-Sugeno Fuzzy Model of the Insulin-to-Glucose System

In order to derive the T-S fuzzy model from the minimal model, we will adopt the blood glucose concentration $G(t)$ as the premise variable to define local operating conditions. Next, its range $[G_{\min}, G_{\max}]$, and a set of *pattern values* (points) within the range are chosen: v_i, $i = \overline{1,R}$. Please notice that there isn't any analytical method for choosing the range and the patterns values, only arguments based on experience and process's specific features.

The patterns values may have different distributions over the range based on the

importance of certain local conditions. In the case of blood glucose control, having more pattern values close to the basal value G_b is motivated by the fact that the glucose concentration should be kept low, but not lower than the basal value.

For each of the pattern values of $G(t)$, the nonlinear term $X(t)G(t)$ in (1) may be approximated with the linear one $X(t)v_i$. In fuzzy logic, this substitution is valid when the measured value of blood glucose concentration is *close* to the pattern value. However, notice that the substitution $G(t) = v_i$ is only done in the nonlinear part of the initial model. Further, the constant term p_1G_b is neglected for the same reasons as in [2], so the first equation in (1) finally becomes:

$$\dot{G}(t) = -p_1 G(t) - v_i X(t) + ad(t). \tag{4}$$

This, together with the second and the third equations in (1), forms a linear state-space model:

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}_i\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t), \end{cases} \tag{5}$$

where $\mathbf{x}(t) = \left[G(t)\ \ X(t)\ \ I(t)\right]^{\mathrm{T}}$ is the state vector, $\mathbf{u}(t) = \left[d(t)\ \ u(t)\right]^T$ is the input vector and $\mathbf{y}(t) = [G(t)]$ is the output vector. The rest of matrices are:

$$\mathbf{A}_i = \begin{bmatrix} -p_1 & -v_i & 0 \\ 0 & -p_2 & p_3 \\ 0 & 0 & -n \end{bmatrix},$$

$$\mathbf{B} = \begin{bmatrix} a & 0 \\ 0 & 0 \\ 0 & b \end{bmatrix},\ \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.$$

Notice that the linear model in (5) in totally valid if $G(t) = v_i$. When $G(t)$ *is close to* v_i the linear model is partly valid, in a fuzzy logic meaning. Also, notice that, we have R linear models, one for each pattern value v_i, which leads to expression of the T-S fuzzy model as a set of R rules:

Rule i:

if $G(t)$ *is* close *to* v_i

$$\textit{then} \begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}_i\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t). \end{cases} \tag{6}$$

The validity of each model at any moment is determined by the truth value of the premise sentence "$G(t)$ *is close to* v_i". Although intuitive to describe proximity by symmetric triangular functions, we considered that the closeness of any value $G(t)$ to the pattern value v_i is:

$$\mu_i(z(t)) = \begin{cases} \dfrac{G(t) - v_{i-1}}{v_i - v_{i-1}},\ G(t) \in [v_{i-1}; v_i] \\ \dfrac{v_{i+1} - G(t)}{v_{i+1} - v_i},\ G(t) \in (v_i; v_{i+1}] \\ 0 \qquad ,\ \textit{else} \end{cases} \tag{7}$$

with its shape depicted in Figure 1.

Given a measured value of blood glucose concentration, $G(t)$, the final output of the T-S fuzzy model is:

$$\begin{cases} \dot{\mathbf{x}}(t) = \sum_{i=1}^{R} h_i(t)\{\mathbf{A}_i\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)\}, \\ \mathbf{y}(t) = \sum_{i=1}^{R} h_i(t)\mathbf{C}\mathbf{x}(t), \end{cases} \tag{7}$$

with $h_i(t)$ being the normalized firing strength if the i-th rule in the fuzzy model, which in this case is:

$$h_i(t) = \frac{\mu_i(t)}{\sum_{i=1}^{R} \mu_i(t)}. \tag{8}$$

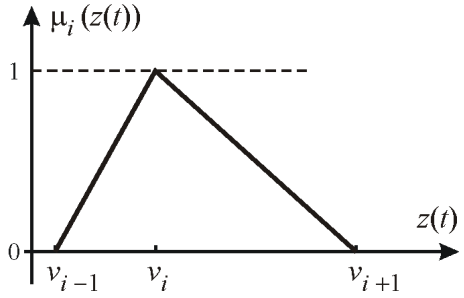

Fig. 1. *The function describing the closeness of the premise variable to one of the patter values, relative to the previous and the next*

4. As a Model Validation

For model validation, several Matlab programs and Simulink models were used to simulate the time responses of: a) original minimal model; b) linearized model; c) linearized model without the term $X_aG_a + p_1G_b$ and d) proposed T-S fuzzy model. The values of all parameters in these simulations are listed in Table 1.

The first round of simulations investigate the decrease of glucose concentration from the initial value of 300 [mg/dL] when an exogenous insulin step input is considered, $u(t) = 1$, and in the absence of any glucose disturbance input, $d(t) = 0$. The results depicted in Figure 2 show that the shape of the time response of the fuzzy T-S model (marked with "4") is much closer to the time response of the original model than those of the linearized models.

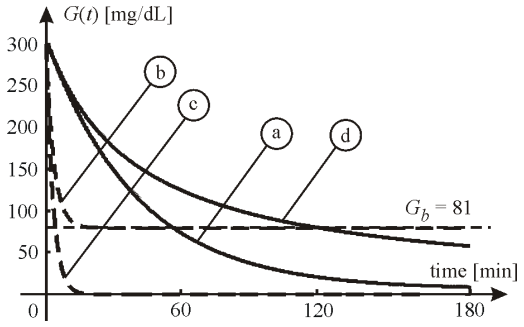

Fig. 2. *Time responses of the simulated models for the step input in exogenous insulin*

Next, the time responses to a step input of glucose disturbance, $d(t) = 1$, without exogenous insulin, $u(t) = 0$, are simulated. This time, the initial glucose concentration is 81 [mg/dL]. The results are depicted in Figure 3. In this case, the response of the fuzzy model is identical to the original one. But the responses of the linearized models are incorrect as the glucose concentration in blood should keep rising when a constant doze is continuously added.

The simulation results suggest that a control system which is designed based on the linearized model and conventional design methods may not be efficient when disturbance is present.

The values of all parameter used in simulations Table 1

parameters of the minimal model	$G_b = 81$ [mg/dL], $p_1 = 0$ [1/min], $p_2 = 0.025$ [1/min], $p_3 = 0.000013$ [dL/(mU×min^2], $n = 5/54$ [1/min],
conversion factors	$a = b = 1/120$ [1/dL]
average values	$G_a = 81$ [mg/dL], $X_a = 0.0054$ [1/min]
premise variable limits and pattern values	$G_{\min} = 0$ [mg/dL], $G_{\max} = 350$ [mg/dL], $v_i \in \{0,80,90,100,120,130,170,220,280,350\}$, $R = 10$
initial conditions	$G_0 = 300/81$[mg/dL], $X_0 = 0$ [1/min], $I_0 = 0$ [mU/dL]
simulation parameters	simulation time: $T = 10000$ [sec] (aprox. 3 hours), sampling time: $T_s = 1$ [sec] ($N_s = 10000$ samples)

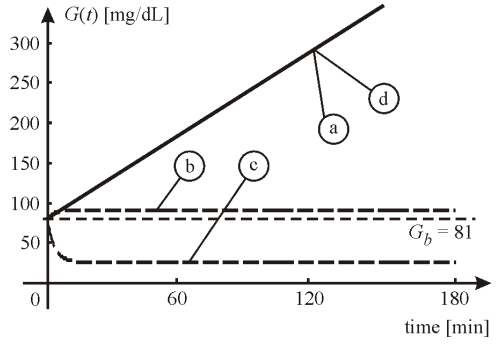

Fig. 3. *Time responses of the simulated models for the step input in glucose disturbance*

For a more precise evaluation of models' validity relative to the original minimal model, we calculated the integral of absolute error (IAE) between the response of the original model $G_{orig}(t)$ and those of the other three, $G(t)$, for all samples, namely:

$$IAE = \frac{1}{N_s}\sum_{t=1}^{T}(G_{orig}[t] - G[t]).$$

The obtained values are listed in Table 2 and they show that the time response of the fuzzy model we derived is much closer to the original, than the linear models used in many papers.

Table 2

The integral of absolute error for the simulated models

	$u(t) = 1$ $d(t) = 0$	$u(t) = 0$ $d(t) = 1$
linear model	~5.9	~40.0
linear model without $X_aG_a + p_1G_b$	~6.6	~48.0
fuzzy model	~4.7	~0.0

5. Conclusions

Although it is often used, the classical linearization of the well-known minimal model of the insulin-to-glucose system seems to lead to linear models that are not reflecting the real dynamics of the system. Indeed, the linear models are very useful for designing conventional control systems for blood glucose concentration through analytical methods. But, nowadays, the fuzzy control is mentioned as being a reliable solution for such application and the design of a fuzzy control system does not necessarily require a linear model of the process. Even more, a fuzzy model of the process offers a different description of the process' dynamics, which is more useful for fuzzy controller design. Hence, deriving a fuzzy model bring important advantages, as in finding and/or fine tuning the control rules.

The fuzzy model we proposed is derived from the well-known minimal model. It is simple, but it imitates the dynamics of the blood glucose concentration better than the linearized models used in many articles. To validate it, we simulated the responses of the original, the linearized and the fuzzy models, to exogenous insulin or glucose disturbance step inputs. The basic idea was to compare the newly introduced fuzzy model and the well-known linearized one, by checking which one is "closer" to the original minimal model. The validation of the fuzzy model is sustained by the idea that if the linearized model is accepted in today's literature, then another model that better imitates the original one should be reliable for the same application.

However, we should mention that the method used here to derive the T-S fuzzy model is strictly related to the original model and it is focused on the subsequent objective of using it in designing a fuzzy control system. Since it is only based on a simple substitution, the method may not be applicable at all for other nonlinear models.

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