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A STUDY OF THE FINITE ELEMENT MODELING OF AN END LOADED CANTILEVER BEAM

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Abstract: *The beam theory and the elasticity solutions for the deflection and stresses in an end-loaded cantilever are intensely used to verify and to evaluate the performances of different finite elements. The finite element modeling of the displacement boundary conditions is difficult and in the most part of cases is not solved correctly, which leads to errors. In this paper a new solution is given for the modeling of the clamped edge, which is equivalent with the hypotheses used at the analytical models. The stresses and the displacements given by this finite element model are converging to the theoretical solutions of the idealized cantilever beam.*

Key words: *Finite element method, Beam, Model errors*

1. Introduction

The capability of the finite elements to approximate complex displacement and stress fields is often shown by comparison to problems for which there are known solutions. One of these problems is the end loaded cantilever beam [1]. Although this problem is simple, the finite element modelling of the loading and boundary conditions which were at the basis of the idealized solutions is difficult. Not respecting these conditions is a source of errors and the use of such a model is not appropriate for testing finite elements capabilities.

In the paper [2] there are given five solutions for the modeling of the cantilever beam. The introduction of the vertical load as a parabolic varying shear traction over the depth of the free end solves correctly the problem of loading. The problem of modeling the clamped edge is more difficult. In the paper [2] the correct solution for this problem is solved by introducing essential boundary conditions at the fixed end to match the displacements given by the elasticity solution.

In this paper we use the parabolic shear loading on the free end and zero mechanical work conditions of the clamped edge tensions with the corresponding displacements. As a result on the clamped edge the mean displacements are fixed and the stresses and displacements on the whole cantilever are converging to the theoretical solutions for the idealized cantilever.

2. Problem Definition

Figure 1 shows a cantilever beam of depth D, length L and unit thickness, which is fully fixed to a support at *x=0* and carries a load P at the free end. The stress field is

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given by the beam theory [1], [3]:

$$\pm_x = \frac{P(L-x)y}{I}, \quad (1)$$

$$\pm_y = 0, \quad (2)$$

$$²_{xy} = -\frac{3P}{2D}\left(1-\frac{4y^2}{D^2}\right), \quad (3)$$

and the tip displacement is:

$$v = \frac{PL^3}{3EI} + \frac{6PL}{5GD}. \quad (4)$$

where E is Young's modulus, G is the shear modulus, and I is the second moment of area of the cross-section.

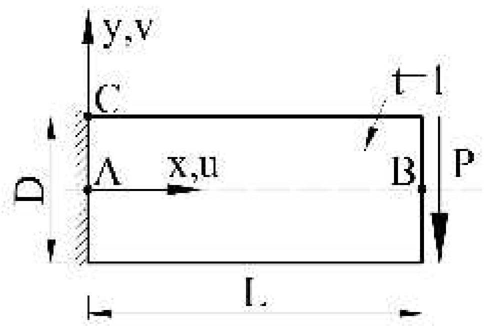

Fig. 1. *Coordinate system for the cantilever problem*

3. Analysis of the Cantilever Beam

For the numerical calculations we used the following data: *L=24, D=12, E=160, ¬=1/4, P=40.*

Applying equations (1),...(4) results:

$$\pm_{x,\max} = 40,$$

$$²_{xy,\max} = 5,$$

$$v_{\max} = 9.5.$$

At a first finite element model (i) we used parabolic load distribution on the section of the free end and full restrictions of the displacements of the nodes of the clamped section. We used a mesh of 6x12 of pairs of six node triangular elements. The displacement interpolation functions of these elements are parabolic, which means that the strains and the stresses are linear on each element. The parabolic distributed loads were concentrated to the nodes using the relation:

$$\begin{Bmatrix} F_i \\ F_{i+1} \\ F_{i+2} \end{Bmatrix} = \frac{h}{30}\begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix}\begin{Bmatrix} q_i \\ q_{i+1} \\ q_{i+2} \end{Bmatrix} \quad (5)$$

where

F_i, F_{i+1}, F_{i+2}, are the resulting nodal forces acting in the nodes on the side of an element;

h is the length of the side of the element;

q_i, q_{i+1}, q_{i+2}, are the intensities of the distributed load in the nodes on the side of the element.

As it can be observed form Figure 2 and 3, in the clamped section the stresses are perturbed. Because of the full restraint provided to the left-hand end, stress concentrations occur in the top and bottom lef-hand corners, and the resulting shear stress distribution exhibits singularities at the top and bottom corners. The differences between the finite element solution and the theortical solution are considerable in the area *x<D/2*. These perturbations depend on the number of divisions of the finite element mesh. The tip displacement was $v_{\max}^{FE(i)} = 9.41$. The error of this displacement is due to the finite element approximation and to the clamped end constraints, which are not allowing the shear deplanation and the developement of transversal contraction on this edge.

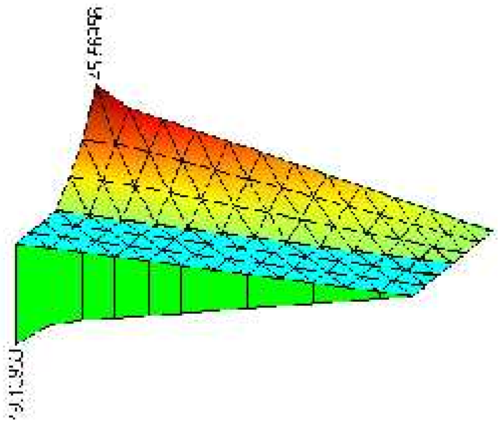

Fig. 2. *Normal stresses σ_x for the model "i" of the cantilever beam*

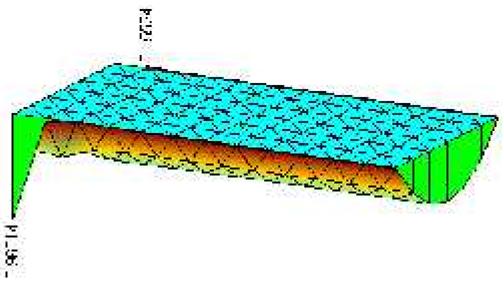

Fig. 3. *Shear stresses τ_{xy} for the model "i" of the cantilever beam*

As it can be observed from these results, it is not correct to model the clamped edge by constraining all the displacements of the nodes of the fixed end.

The next model is based on the observation from [1] that the theoretical solution "represents an exact solution only if the shearing forces on the ends are distributed according to the same parabolic law as the shearing stress τ_{xy} and the intensity of the normal forces at the built end is proportional to *y*." This conditions were enforced using mechanical work relations and added to the equilibrium equations with Lagrange multipliers.

The equations are:

$$\begin{bmatrix} \mathbf{K} & \mathbf{V}_x & \mathbf{V}_y & \mathbf{V}_\theta \\ \mathbf{V}_x^T & 0 & 0 & 0 \\ \mathbf{V}_y^T & 0 & 0 & 0 \\ \mathbf{V}_\theta^T & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \mathbf{a} \\ \lambda_x \\ \lambda_y \\ \lambda_\theta \end{Bmatrix} = \begin{Bmatrix} \mathbf{F} \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad (6)$$

where

K is the stiffness matrix of the free cantilever;

$\mathbf{V}_x$, $\mathbf{V}_y$ and $\mathbf{V}_\theta$, are load vectors with nodal loads resulting from a constant distributed x load, a parabolic varying shear load and respectively a linear varying x load acting on the clamped edge;

a is the vector of the nodal displacements;

λ_x, λ_y, λ_θ are the Lagrange multipliers;

F is the load vector.

As a result, on the clamped edge only the mean displacements are fixed, but the developement of shear and transversal contraction displacements are not restricted. The developement of these displacements can better be observed on a short cantilever (L=6), where the effect of bending is smaller.

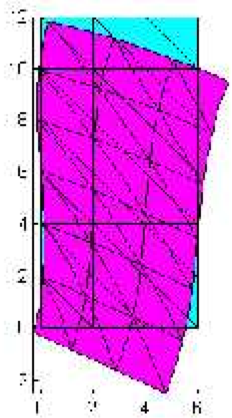

Fig. 4. *Deformed shape of a short cantilever (L=6)*

The normal and shear stresses in the beam are presented in Figures 5 and 6. The tip displacement was $v_{\max}^{FE(ii)} = 9.4996$.

Despite of the relatively coarse finite element mesh, it can be observed a good agreement with the theoretical results.

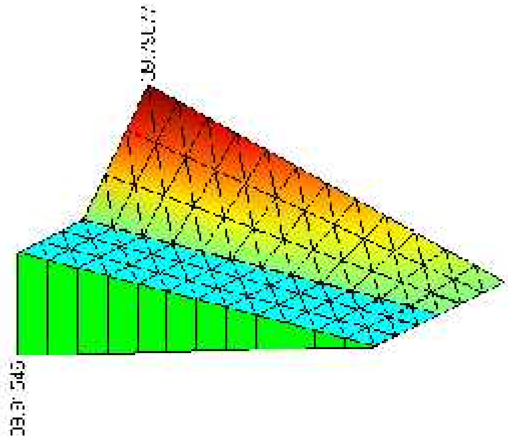

Fig. 5. *Normal stresses* $\pm_x$ *for the model "ii" of the cantilever beam*

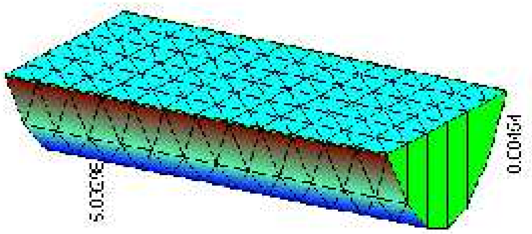

Fig. 6. *Shear stresses* $^{2}{}_{xy}$ *for the model "ii" of the cantilever beam*

4. Conclusions

This paper has examined the effect of boundary conditions on the correct solution for a cantilever beam problem. Replication of the solutions of beam theory and elasticity theory require the implementation of precise displacement and stress conditions at the built in end together with the application of the vertical load as parabolic varying shear traction over the depth of the free end.

There are many examples in the literature where this has been done incorrectly. This paper illustrates the effect of the complete constraints on the displacements of the nodes of the clamped end and introduces a new model where the zero displacements of the clamped end are introduced in a variational manner. This approach allowes the free development of some shear and transversal contraction displacements. As a result, the displacements and the stresses of this finite element model are converging to those obtained by the colsed form solutions of the idealized cantilever beam.

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