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MODAL ANALYSIS OF ISOTROPIC RECTANGULAR PLATES USING AN ANALYTICAL METHOD

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Abstract: *The paper aims is to determine the modal and total responses of an isotropic rectangular flat plates. As is known, for the rectangular flat plates, the modal analysis method is applied to an infinite number of degrees of freedom. The study has been made considering the simplifying hypothesis according to which the plate material is isotropic. The studied plate considered in the paper is loaded on its surface by a uniformly distributed harmonic force. In applying modal analysis the damping was considered neglected.*

Key words: *plate, vibration, shape, functions, analytical.*

1. Introduction

In the paper the problem of dynamic calculation of flat plate was formulated into the theory limits of engineering plates assumptions (Kirchhoff model).

The material of the studied plate studied is considered to be isotropic.

Considering the principle of D'Alembert the differential equation of plate motion have the form [1], [3], [9], [10]:

$$\nabla^4 w(x,y,t) = \frac{\rho h}{D}\ddot{w}(x,y,t) = \frac{p(x,y,t)}{D} \qquad (1)$$

Where:

∇^4 - double Laplacian operator [1]

$$\nabla^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4} \qquad (2)$$

ρ - material density.

h - constant thickness of the plate.

$$D = \frac{Eh^3}{12(1-\nu^2)} \qquad (3)$$

D - bending strength stiffness of the plate [6].

ν - Poisson ratio.

$p(x,y,t) = p((x,y) \cdot f(t)$ - distributed harmonic disturbing force [1], [2], [5].

$w(x,y,t)$ - transversal displacement rectangular plate points.

2. Objectives

The objective in this work is to determine the plate numerical values of modal and total displacements, the own pulsations the and dynamic multiplier functions using an analytical method.

It was also followed the solving of

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algebraic equations system resulting from imposing the boundary conditions.

The analysis was performed for the vibration modes ij (1,1) and (1,3).

The objective in this work is to determine the numerical values of their movements and pulsation functions multiplier dynamic modal and total plate using an analytical method.

3. Material and Methods

The plate considered in the paper have two opposite sides clamped and the others two simply supported,.

The plate considered is loaded with a system of distributed harmonic forces on the surface having the intensity P_0 and pulsation Ω. Material of rectangular plate is homogeneous and isotropic considered. Median plate surface dimensions considered are:

$a = 4\,[m]$ - plate length.

$b = 2\,[m]$ - plate width.

$h = 0.2\,[m]$ - the plate thickness.

The plate considered have the following boundary conditions:

- for $x = 0$, $x = 4$, the boundary is simply supported.
- for $y = 0$, $y = 2$, the boundary is clamped.

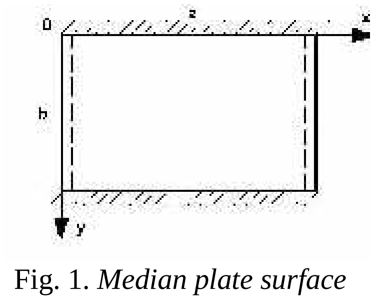

Fig. 1. *Median plate surface*

For determining the own pulsations and vibrations shapes functions it starts from the free vibrations differential equation [1], [2], [6]:

$$\nabla^4 w(x,y,t) = \frac{°h}{D}\ddot{w}(x,y,t) = 0 \qquad (4)$$

Differential equations of free vibration has the following general solutions form [1], [2], [6]

$$w(x,y,t) = \sum_{i=1}^{m}\sum_{j=1}^{n}\Phi_{ij}(x,y\cdot ¦_{ij}(t) \qquad (5)$$

Where:

$\Phi_{ij}(x,y)$ - the shapes functions plate.

$¦_{ij}(t)$ - the time functions.

After replacing the general solution into the differentials equations of free vibration will be obtained the normal modes differential equation of vibration.

The form is [4], [6], [9]:

$$\nabla^4 \Phi_{ij}(x,y) - \frac{°h\mu_{ij}^{\,2}}{D} = 0 \qquad (6)$$

μ_{ij} - the rectangular plate own pulsations

For the normal modes differential equation of vibrations was adopted the following solutions

$$\Phi_{ij}(x,y) = Y_j(y)\cdot\sin\frac{i®x}{a} \qquad (7)$$

Differential equation of free vibration becomes [1], [4], [5]

$$Y_j^4 - 2\frac{i^2®^2}{a^2}Y_j^2 + \left(\frac{i^4®^4}{a^4} - \frac{°h\mu^2}{D}\right)Y_j = 0 \quad (8)$$

Differential Equation characteristic equation (8) has the form:

$$r^4 - 2\frac{i^2®^2}{a^2}r^2 + \left(\frac{i^4®^4}{a^4} - \frac{°h\mu^2}{D}\right)r = 0 \quad (9)$$

The roots of this equation are the following

$$r_{1,2} = \pm\sqrt{\mu\sqrt{\frac{°h\mu^2}{D} + \frac{i^2®^2}{a^2}}} \quad (10)$$

$$r_{3,4} = \pm i\sqrt{\mu\sqrt{\frac{°h\mu^2}{D} - \frac{i^2®^2}{a^2}}} \quad (11)$$

For the determined solutions are adopted the following notations:

$$\sim_i = \sqrt{\mu\sqrt{\frac{°h\mu^2}{D} + \frac{i^2®^2}{a^2}}} \quad (12)$$

$$_i = \sqrt{\mu\sqrt{\frac{°h\mu^2}{D} - \frac{i^2®^2}{a^2}}} \quad (13)$$

General solution of differential equations can be written as [1]:

$$Y_i = A_i ch\sim_i y + B_i sh\sim_i y + C_i \cos \;_i y + D_i \sin \;_i y$$

where:

A_i, B_i, C_i, D_i - unknown integration constants to be determined.

The unknown integration constants of plate equation are determined by imposing the boundary conditions. For clamped sides of plate $y = 0$, $y = b$ with the length b=2, the boundary conditions are written as [5], [6]:

$$y = 0 \rightarrow \begin{cases} Y(0) = 0 \\ Y'(0) = 0 \end{cases} \quad (14)$$

$$y = b \rightarrow \begin{cases} Y(b) = 0 \\ Y'(b) = 0 \end{cases} \quad (15)$$

Following the calculation algorithm adopted are obtained 4 homogeneous algebraic equations having 4 unknowns. The unknowns of these equations are the constants of integration.

The system of equations has the form:

$$\begin{cases} A_i + C_i = 0 \\ \sim_i B_i + \;_i D_i = 0 \\ A_i ch\sim_i b + B_i sh\sim_i b + C_i \cos \;_i b + D_i \sin \;_i b = 0 \\ \sim_i A_i sh\sim_i b + \sim_i B_i ch\sim_i b - \;_i C_i \sin \;_i b + \;_i D_i \cos \;_i b = 0 \end{cases}$$

In order to have non-zero solution the homogeneous system of equations must have the determinant equal to zero.

Corresponding to the own pulsations of their normal modes of vibration considered are determined the parameters pulsations values a_1, a_3.

The expression for calculus of the own pulsations has the form [1], [3]:

$$\mu_{ij} = \frac{a_{ij}}{a^2}\sqrt{\frac{D}{°h}} \quad (16)$$

Dynamic multiplication functions for normal modes of vibration considered was determined using the relationship [1], [9], [10]:

$$\Psi_i = \frac{1}{1 - \frac{\Omega^2}{\mu_i^2}} \quad (17)$$

Using the expressions presented is determined the modal and total displacements points plate for normal modes of vibration considered..

Tthe modal responses into displacement was determined using relations:

$$w_i = \Phi_1 \Psi_i \quad (18)$$

The total plate response in displacements was determined using equation (16) by applying the principle of superposition:

$$w(x, y, t) = w_1(x, y, t) + w_3(x, y, t) \quad (19)$$

The total and modal value loading is also determined for the case studied of dynamic plate.

The relasion is:

$$p_i = P_0 \cdot \Psi_i \qquad (20)$$

4. Results and Discussions

At conducting numerical study analysis were considered the following initial data of the problem presented.

Are taken into consideration the following values:

1. Young's modulus:

$$E = 2.1 \cdot 10^6 \left[\frac{daN}{cm^2}\right] \qquad (21)$$

2. Poisson ratio:

$$« = 0.2 \qquad (22)$$

3. Distributed uniformly disturbing force module on the surface plate:

$$P_0 = 10\left[\frac{KN}{m^2}\right] \qquad (23)$$

4. The disturbing force pulsation:

$$\Omega = 10\left[\frac{rad}{\sec}\right] \qquad (24)$$

5. The value bending strength stiffness plate value determined is:

$$D = 2.181 \cdot 10^4 \left[\frac{daN}{cm^2}\right] \qquad (25)$$

Substituting into (14), are determined their own pulsations values.

The pulsations parameters determined are [1]:

$$a_1 = 54.8$$
$$a_3 = 69.3 \qquad (26)$$

$$\mu_1 = \frac{a_1}{a^2}\sqrt{\frac{D}{°h}} = \frac{54,8}{a^2}\sqrt{\frac{D}{°h}} = 24.65\left[\frac{rad}{\sec}\right] \qquad (27)$$

$$\mu_3 = \frac{a_3}{a^2}\sqrt{\frac{D}{°h}} = \frac{69,3}{a^2}\sqrt{\frac{D}{°h}} = 31.17\left[\frac{rad}{\sec}\right] \qquad (28)$$

The coefficients of the general solution for normal vibration modes considered are:

$$\sim_1 = \sqrt{\mu_1\sqrt{\frac{°h}{D} + \frac{i^2®^2}{a^2}}} = 2.01 \qquad (29)$$

$$_1 = \sqrt{\mu_1\sqrt{\frac{°h}{D} - \frac{i^2®^2}{a^2}}} = 1.675 \qquad (30)$$

$$\sim_3 = \sqrt{\mu_3\sqrt{\frac{°h}{D} + \frac{i^2®^2}{a^2}}} = 3.141 \qquad (31)$$

$$_i = \sqrt{\mu\sqrt{\frac{°h\mu^2}{D} - \frac{i^2®^2}{a^2}}} = -1.1 \qquad (32)$$

The unknown integration constants determined by the homogeneous algebraic system for the mode number 1 of vibration are:

$$\begin{cases} A_1 + C_1 = 0 \\ \sim_1 B_1 + \quad _1 D_1 = 0 \\ A_1 ch\sim_1 b + B_1 sh\sim_1 b + C_1 \cos \quad _1 b + D_1 \sin \quad _1 b = 0 \\ \sim_1 A_1 sh\sim_1 b + \sim_1 B_1 ch\sim_1 b - \quad _1 C_1 \sin \quad _1 b + \quad _1 D_1 \cos \quad _1 b = 0 \end{cases} \quad (33)$$

By performing the algebraic calculations by the system equations are obtained the values

$$ch\sim_1 = \frac{e^{\sim 1} + e^{-\sim 1}}{2} = 27.308 \quad (34)$$

$$sh\sim_1 = \frac{e^{\sim 1} - e^{-\sim 1}}{2} = 27.29 \quad (35)$$

$$\begin{aligned} \cos \quad _1 &= 0.998 \\ \sin \quad _1 &= 0.058 \end{aligned} \quad (36)$$

By following the calculations are obtained the constants of integration values.

$$\begin{aligned} A_1 &= 1; \\ B_1 &= -0.966 \\ C_1 &= -1 \\ D_1 &= 1.16 \end{aligned} \quad (37)$$

Following the same algorithm for vibration mode i = 3, by solving the homogenous algebraic equations system integration constants obtained are:

$$\begin{aligned} A_3 &= 1; \\ B_3 &= 1.01 \\ C_3 &= -1 \\ D_3 &= 2.88 \end{aligned} \quad (38)$$

For the shape vibration functions corresponding to the normal modes of vibrations considered, we obtained the following values:

$$\begin{aligned} \Phi_1 &= 1.507 \\ \Phi_3 &= 2.26 \end{aligned} \quad (39)$$

The corresponding dynamic multiplier dynamic functions [1], [2], [4], for the studied vibration modes are:

$$\Psi_1 = \frac{1}{1 - \frac{\Omega^2}{\mu_1^2}} = 1{,}19 \quad (40)$$

$$\Psi_3 = \frac{1}{1 - \frac{\Omega^2}{\mu_3^2}} = 1{,}11 \quad (41)$$

Using expressions of the modal responses in displacement are obtained the following numerical values:

-for mode number 1:

$$w_1 = \Phi_1 \Psi_1 = 1.804\,[cm] \quad (42)$$

- for mode number 3:

$$w_3 = \Phi_3 \Psi_3 = 2.5\,[cm] \quad (43)$$

The total plate response was determined by applying the principle of superposition [1], [2], [4].

$$w(x,y,t) = w_1(x,y,t) + w_3(x,y,t) = 4.304\,[cm] \quad (44)$$

The determinated loads corresponding to the normal modes of vibration are:

$$p_1 = P_0 \cdot \Psi_1 = 11.9 \left[\frac{KN}{m^2}\right] \quad (45)$$

$$p_3 = P_0 \cdot \Psi_3 = 11.1 \left[\frac{KN}{m^2}\right] \quad (46)$$

Watching the results obtained by the modal analysis (displacements and loadings) for the rectangular plate, can be expressed some conclusions with theoretical and practical aspects.

As a conclusion from the results obtained in paper are:

- for mode 1 of vibration the dynamic multiplication factor is about 7% higher than for mode 3 of vibration;

- the modal responses into displacement values are inversely proportional for mode 1 and are about 27% lower than for mode 3;

- regarding the modal loads is ascertained that for vibration mode number 1, the loading mode is about 7% higher than for the vibration mode 3

5. Conclusions

The paper is intended to be an example of analytical calculation of rectangular plates type structures using the modal analysis method. The modal analysis method allows the determination of total and the modal responses into displacement and the modal and total responses into sectional efforts. The work follows and shows detailed calculation algorithm through a detailed presentation of the work stages. Thus, following the steps algorithm presented in the paper, can be determined the dynamic modal responses depending on the product of the static bending moment and static shear forces static with the dynamic multiplication factor determined [1], [7], [8] [10].

The expressions are

$$M_{ij}(x,y,t) = M_{ij}(x,y)\Psi_{ij} \quad (47)$$

$$T_{ij}(x,y,t) = T_{ij}(x,y)\Psi_{ij} \quad (48)$$

$$M(x,y,t) = \sum_{i=1}^{m}\sum_{j=1}^{n} M_{ij}(x,y)\Psi_{ij} \quad (49)$$

The application of modal analysis using analytical or numerical method for structure strength, is a very important mathematical engineering calculus regarding the dynamic study of different types of structures.

References

1. Bârsan, G: *Dinamica şi stabilitatea construcţiilor,* Editura Didactică şi Pedagogică Bucureşti, 1979.
2. Bârsan, G., M. *Vibraţiile şi stabilitatea plăcilor poligonale.* In *Ph.D. Thesis* Cluj-Napoca, 1971.
3. *Bia, C., Ille, V., Soare, M.: Rezistenţa materilalelor şi teoria elasticităţii, Editura Didactică şi Pedagogică., Bucureşti, 1983.*
4. Borş, I.: *Aplicaţii ale problemei de valori proprii în mecanica construcţiilor – Sisteme finit-dimensionale,* Editura U.T. Pres, Cluj-Napoca, 2005.
5. Fetea, M.: *Calculul analitic si numeric in rezistenta materialelor, Notite curs, aplicatii si lucrari practice* Universitatea din Oradea, Oradea, Romania, 2010.
6. Soare, M.: *Ecuaţii diferenţiale cu aplicaţii în mecanica construcţiilor,* Editura Tehnică, Bucureşti, 1999.
7. Timoshenko, St., P: *Strenght of Materials, Part I Elementary Theory and Problems, 3-rd Edition, 1962, Part II Advaced Theory and Problems,* 3-rd Edition1962, D. Van Nostrand Company, Inc., Princeton, New Jersey.
8. Warburton, G., B: *The Vibration of Rectangular Plates,* Proc. Inst. Mech. Eng., ser. A, vol. 168, no. 12, 1954, papp. 371-384.
9. Warburton, G., B: *The dynamical behaviour of structures,* Pergamon Press, London, 1976.
10. Zienkiewicz C.: *The Finite Element Method in Engineering*. Science, Ed. Graw-Hill, 1971.