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A STUDY OF NONHOLONOMIC FINSLER FRAME ON GAUGE TRANSFORMATIONS

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Abstract

The notion of nonholonomic Finsler frame on space time was introduced by P. R. Holland in 1982, when he studied electromagnetism. Then R. G. Beil studied on gauge transformation viewed as a nonholonomic frame on the tangent bundle of a four dimensional manifold. In this paper we are determined the nonholonomic Finsler frame for Finsler space with special (α, β)-metric $L = (\alpha + \beta + \frac{\beta^2}{\alpha} + \frac{\beta^3}{\alpha^2})^2$ and we discussed that Finsler nonholonomic frames acts as a gauge transformation of Finsler metric tensors.

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1 Introduction

In 1982, P.R.Holland [9], studied that when the classical theory of a charged particles moving in an external electromagnetic field is geometrically reformed as the existence of nonholonomic frames in a Finsler space time. In 2000, P.L.Antonelli and I.Bucataru [1, 2] motivated by quantum mechanics and developed nonholonomic geometry of Randers space using Holland frame and determine two Finsler connections, one of which is Crystallographic connection. In fact R.S.Ingardan [10], was first to point out that the Lorentz force law, in this case, could be written as geodesic equation on a Finsler space called Randers space. Next, R.G.Beil viewed a gauge transformation as a nonholonomic frame on the tangent bundle of a four dimensional base manifold [6].

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The geometry that follows from these considerations gives a more unified approach to gravitation and gauge symmetries. In the above mentioned papers, the common Finsler idea used by physicists R.G.Beil and P.R.Holland is the existence of the nonholonomic frame on the vertical subbundle VTM of the tangent bundle on an n-dimensional manifold M. In [1, 2], P.L.Antonelli and I.Bucataru found such a nonholonomic Finsler frame for two important classes of Finsler spaces that are Randers space and Kropina space. In this paper we evaluated nonholonomic Finsler frame for special Finsler (α, β)-metric and we discussed that nonholonomic Finsler frames are the gauge transformations of Finsler metric tensors.

2 Preliminaries

A Finsler manifold is a manifold M, where each tangent space is equipped with a Minkowski norm, i.e., a norm not necessarily induced by an inner product, such a norm is called Finsler metric. However in sharp contrast to the Riemannian case, these Finsler metrics are not parameterized by points of M, but by directions in TM. The formal definition of Finsler space as follows;

Definition 1. *[3] A Finsler space is the ordered pair $F^n = (M, F)$, where M is an n-dimensional manifold and F is a Finsler metric defined as a function $F : TM \to [0, \infty)$ with the following properties:*
i. Regular: F is C^∞ on the entire tangent bundle $TM\backslash\{0\}$.
ii. Positive homogeneous: $F(x, \lambda y) = \lambda F(x, y)$, $\forall \lambda \geq 0$.
iii. Strong convexity: The $n \times n$ Hessian matrix

$$g_{ij} = \frac{1}{2}[F^2]_{y^i y^j}$$

is positive definite at every point on $TM\backslash\{0\}$, where $TM\backslash\{0\}$ denotes the tangent vector y is non zero in the tangent bundle TM.

An important class of Finsler metrics is (α, β)-metrics [11]. The first Finsler space with (α, β)-metric were introduced by the Physicist G.Randers in 1940 are called Randers space.

Definition 2. *A Finsler space $F^n = (M, F(x, y))$ is called with (α, β)-metric if there exists a 2-homogeneous function L of two variables such that the Finsler metric $F : TM \to R$ is given by,*

$$F^2(x, y) = L(\alpha(x, y), \beta(x, y)), \tag{1}$$

where $\alpha(x, y) = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric on M, and $\beta(x, y) = b_i(x)y^i$ is a 1-form on M.

Randers metric $L = \alpha + \beta$, Kropina metric $L = \frac{\alpha^2}{\beta}$, Matsumoto metric $L = \alpha + \beta + \frac{\beta}{\alpha}$, etc are the examples of Finsler (α, β)-metrics. If we neglect the function L to be homogeneous of order two with respect to the (α, β) variables, then we have a Lagrange space with (α, β)-metric.

Definition 3. *[8] Let M be the n-dimensional Finsler manifold and U be a open set of tangent bundle TM. Then the mapping*

$$V_i : u \in U \to V_i(u) \in V_u TM, \quad i \in 1, 2, ..., n \tag{2}$$

be a vertical frame over U. If $V_i(u) = V_i^j(u) \frac{\partial}{\partial y^j} |_u$, then $V_i^j(u)$ are the entries of a invertible matrix of all $u \in U$ and its inverse is denoted by $\tilde{V}_j^k(u)$ and so that

$$V_i^j \tilde{V}_j^k = \delta_i^k, \quad \tilde{V}_j^i V_k^j = \delta_k^i. \tag{3}$$

Then V_i^j is called as nonholonomic Finsler frame.

Naturally, every geometric object filed on TM can be expressed in nonholonomic Finsler frames V_j^i. For example, if T_j^i are the components of a $(1, 1)$ Finsler field, then the nonholonomic components are given by:

$$T_\beta^\alpha = V_i^\alpha T_j^i V_\beta^j. \tag{4}$$

In [7], Ioan Bucataru consider a generalized Lagrange space with Beil metric defined as

$$g_{ij}(x, y) = a(x, y) a_{ij}(x) + b(x, y) B_i(x, y) B_j(x, y) \tag{5}$$

and denote $B^2(x, y) = a_{ij}(x) B^i(x, y) B^j(x, y)$, then

$$V_j^i = \sqrt{a}\delta_j^i - \frac{1}{B^2}(\sqrt{a} \pm \sqrt{a + bB^2}) B^i B_j \tag{6}$$

is a nonholonomic Finsler frame. Then the Beil metric defined in the equation (5) and the Riemannian metric $a_{ij}(x)$ are related by

$$g_{ij}(x, y) = V_i^k(x, y) V_j^l(x, y) a_{kl}(x). \tag{7}$$

For a four dimensional manifold M^4 with a Minkowski metric η_{ij}, consider the Lagrangian per unit mass of a charged particle moving in M^4 in an external field A_i:

$$L(x, y) = (\eta_{ij} y^i y^j)^{1/2} + k A_i y^i.$$

Then the metrical coefficients of a Finsler space are defined by

$$g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 L^2}{\partial y^i \partial y^j}. \tag{8}$$

The Minkowski metric η_{ij} and the Finsler metric g_{ij} are related by the formula

$$g_{ij} = V_i^k V_j^l \eta_{kl}, \tag{9}$$

where $V_i^k(x, y)$ is a nonholonomic frame on TM.

3 Nonholonomic Finsler frame for Finsler space with special (α,β)-metric $L = (\alpha + \beta + \frac{\beta^2}{\alpha} + \frac{\beta^3}{\alpha^2})^2 = F^2$:

Let us consider a Finsler space $F^n = (M, F)$ on an $n-$dimensional manifold M. For a Lagrange space with (α,β)-metric $F^2(x,y) = L(\alpha(x,y),\beta(x,y))$, the metric tensor g_{ij} is given by,

$$g_{ij}(x,y) = \rho a_{ij}(x) + \rho_0 b_i(x)b_j(x) + \rho_{-1}(b_i(x)y_j + b_j(x)y_i) + \rho_{-2}y_iy_j, \tag{10}$$

where $\rho_1, \rho_0, \rho_{-1}$ and ρ_{-2} are some Finsler invariants defined by:

$$\rho_1 = \frac{1}{2\alpha}\frac{\partial L}{\partial \alpha}; \quad \rho_0 = \frac{1}{2}\frac{\partial^2 L}{\partial \beta^2}; \quad \rho_{-1} = \frac{1}{2\alpha}\frac{\partial^2 L}{\partial \alpha \partial \beta}; \quad \rho_{-2} = \frac{1}{2\alpha^2}(\frac{\partial^2 L}{\partial \alpha^2} - \frac{1}{\alpha}\frac{\partial L}{\partial \alpha}) \tag{11}$$

where, subscripts -2,-1, 0, 1 gives us the degree of homogeneity of these invariants.

For a Finsler space with (α,β)-metric, that L is homogeneous of degree two with respect to α and β we have,

$$\rho_{-1}\beta + \rho_{-2}\alpha^2 = 0 \tag{12}$$

Consider $g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ the fundamental tensor of the Finsler space (M, F). Taking into account the homogeneity of α and F we have the following formula:

$$\begin{aligned}
&p^i = \frac{1}{\alpha}y^i = a^{ij}\frac{\partial \alpha}{\partial y^j}; \qquad p_i = a_{ij}p^j = \frac{\partial \alpha}{\partial y^i}\\
&l^i = \frac{1}{L}y^i = g_{ij}\frac{\partial L}{\partial y^j}; \qquad l_i = g^{ij}\frac{\partial L}{\partial y^j} = p_i + b_i;\\
&l^i = \frac{1}{L}p^i; \quad l^i l_i = p^i p_i = 1; \quad l^i p_i = \frac{\alpha}{L};\\
&p^i l_i = \frac{L}{\alpha}; \quad b_i p^i = \frac{\beta}{\alpha}; \quad b_i l^i = \frac{\beta}{L}.
\end{aligned} \tag{13}$$

With respect to these notations, the metric tensors (a_{ij}) and (g_{ij}) are related by [12],

$$g_{ij} = \frac{L}{\alpha}a_{ij} + b_ip_j + p_ib_j + b_ib_j - \frac{\beta}{\alpha}p_ip_j = \frac{L}{\alpha}(a_{ij} - p_ip_j) + l_il_j. \tag{14}$$

The metric tensor g_{ij} of a Lagrange space with $(\alpha,\beta)-$ metric can be arranged into the form:

$$g_{ij} = \rho a_{ij} + \frac{1}{\rho_{-2}}(\rho_{-1}b_i + \rho_{-2}y_i)(\rho_{-1}b_j + \rho_{-2}y_j) + \frac{1}{\rho_{-2}}(\rho_0\rho_{-2} - \rho_{-1}^2)b_ib_j. \tag{15}$$

From the above expression (15), we can see that g_{ij} is the result of two Finsler deformations:

$$a_{ij} \to h_{ij} = \rho a_{ij} + \frac{1}{\rho_{-2}}(\rho_{-1}b_i + \rho_{-2}y_i)(\rho_{-1}b_j + \rho_{-2}y_j) \tag{16}$$

$$h_{ij} \to g_{ij} = h_{ij} + \frac{1}{\rho_{-2}}(\rho_0\rho_{-2} - \rho_{-1}^2)b_ib_j. \tag{17}$$

The nonholonomic Finsler frame that corresponds to the first deformation (16) according to the equation (6) and reference [7], given by,

$$X^i_j = \sqrt{\rho}\delta^i_j - \frac{1}{A^2}\left(\sqrt{\rho} \pm \sqrt{\rho + \frac{A^2}{\rho_{-2}}}\right)(\rho_{-1}b_i + \rho_{-2}y_i)(\rho_{-1}b_j + \rho_{-2}y_j), \tag{18}$$

where $A^2 = a_{ij}(\rho_{-1}b_i + \rho_{-2}y_i)(\rho_{-1}b_j + \rho_{-2}y_j) = \rho^2_{-1}b^2 + \beta\rho_{-1}\rho_{-2}$. The metric tensors a_{ij} and h_{ij} are related by:

$$h_{ij} = X^k_i X^l_j a_{kl}. \tag{19}$$

Similarly, by the equation (6) and reference [7], the nonholonomic Finsler frame that corresponds to the second deformation (17) is given by

$$Y^i_j = \delta^i_j - \frac{1}{B^2}(1 \pm \sqrt{1 + \frac{\rho_{-2}B^2}{\rho_0\rho_{-2} - \rho^2_{-1}}})b^i b_j, \tag{20}$$

where $B^2 = h_{ij}b^i b^j = \rho b^2 + \frac{1}{\rho_{-2}}(\rho_{-1}b^2 + \rho_{-2}\beta)^2$. The metric tensor h_{ij} and g_{ij} are related by the formula:

$$g_{mn} = Y^i_m Y^j_n h_{ij}. \tag{21}$$

From (19) and (21) we have that $V^k_m = X^k_i Y^i_m$, with X^k_i given by (18) and Y^i_m given by (20), is a nonholonomic Finsler frame of the Finsler space with $(\alpha, \beta)-$metric.

In this section we consider Finsler space with special (α, β)-metrics with the fundamental function $L = (\alpha + \beta + \frac{\beta^2}{\alpha} + \frac{\beta^3}{\alpha^2})^2 = F^2$, then we determined nonholonomic Finsler frame.

For the above (α, β)-metric, Finsler invariants becomes:

$$\begin{aligned}
\rho &= \frac{(\alpha^5 - 3\alpha\beta^4)(\alpha + \beta) - 2\beta^3(\alpha^3 + \beta^3)}{\alpha^6} \\
\rho_0 &= \frac{3\alpha^4 + 6\alpha^2\beta(2\alpha + 3\beta) + 5\beta^3(4\alpha + 3\beta)}{\alpha^4} \\
\rho_{-1} &= \frac{\alpha^5 - 6\alpha^2\beta^2(\alpha + 2\beta) - 3\beta^4(5\alpha + 4\beta)}{\alpha^6} \\
\rho_{-2} &= \frac{-\alpha^5\beta + 3\alpha\beta^3(2\alpha^2 + 5\beta^2) + 12\beta^4(\alpha^2 + \beta^2)}{\alpha^8} \\
A^2 &= (b^2 - \frac{\beta^2}{\alpha^2})[\frac{\alpha^5 - 12\beta^3(\alpha^3 + \beta^3) - 3\alpha\beta^2(2\alpha^2 + 5\beta^2)}{\alpha^6}]^2.
\end{aligned} \tag{22}$$

These invariants satisfies the relation $\rho_{-1}\beta + \rho_{-2}\alpha^2 = 0$. Then by (18) and (20) the expression for horizontal and vertical nonholonomic Finsler frames with respect to two deformations (16) and (17) we get:

$$
\begin{aligned}
X^i_j &= \sqrt{\frac{(\alpha+\beta)(\alpha^5-3\alpha\beta^4)-2\beta^3(\alpha^3+\beta^3)}{\alpha^6}}\delta^i_j \\
&- \frac{\alpha^2}{(\alpha^2b^2-\beta^2)}\left\{\sqrt{\frac{(\alpha+\beta)(\alpha^5-3\alpha\beta^4)-2\beta^3(\alpha^3+\beta^3)}{\alpha^6}}\delta^i_j\right\} \\
&\pm \frac{\alpha^2}{(\alpha^2b^2-\beta^2)}(b^i-\frac{\beta y^i}{\alpha^2})(b_j-\frac{\beta y_j}{\alpha^2})\sqrt{\frac{T(\alpha,\beta)}{\alpha^6}}, \qquad (23)
\end{aligned}
$$

where

$$
\begin{aligned}
T(\alpha,\beta) &= (\alpha+\beta)(\alpha^5-3\alpha\beta^4)-2\beta^3(\alpha^3+\beta^3) \\
&\quad -(\alpha^2b^2-\beta^3)(\alpha^5-12\beta^3(\alpha^3+\beta^3)-3\alpha\beta^2(2\alpha^2+5\beta^2))
\end{aligned}
$$

and

$$
Y^i_j = \delta^i_j - \frac{1}{B^2}\left[1\pm\sqrt{1+\frac{\alpha^4\beta B^2}{U(\alpha,\beta)}}\right]b^ib_j, \qquad (24)
$$

where

$$
\begin{aligned}
B^2 &= \frac{(\alpha+\beta)(\alpha^5-3\alpha\beta^4)-2\beta^3(\alpha^3+\beta^3)}{\alpha^6}b^2 \\
&\quad -\frac{\alpha^2}{\beta}(b^2-\frac{\beta^2}{\alpha^2})[\alpha^5-12\beta^3(\alpha^2+\beta^2)-3\alpha\beta^2(2\alpha^2+5\beta^2)]. \\
U(\alpha,\beta) &= \alpha^5+3\alpha^4\beta+3\alpha^2\beta^2(4\alpha+6\beta)-3\alpha\beta^2(2\alpha^2+5\beta^2)+5\beta^3(4\alpha+3\beta)
\end{aligned}
$$

Then we obtained the nonholonomic Finsler frame as follows,

$$
\begin{aligned}
V^i_j &= X^i_kY^k_j \\
&= C\delta^i_j - \frac{C}{B^2}(1\pm E)b^ib^j - \frac{C\alpha^2}{\alpha^2b^2-\beta^2}\left\{\delta^i_j+\frac{1\pm E}{B^2}b^ib^j\right\} \qquad (25)\\
&\pm \frac{\alpha^2D}{\alpha^2b^2-\beta^2}\left\{(b^i-\frac{\beta y^i}{\alpha^2})(b_j-\frac{\beta y_j}{\alpha^2})+(1\pm E)(b^i-\frac{\beta y^i}{\alpha^2})(b^2b_j-\frac{\beta b_jb^ky_k}{\alpha^2})\right\},
\end{aligned}
$$

where

$$
\begin{aligned}
C &= \sqrt{\frac{(\alpha+\beta)(\alpha^5-3\alpha\beta^4)-2\beta^3(\alpha^3+\beta^3)}{\alpha^6}} \\
D &= \sqrt{C-\frac{(\alpha^2b^2-\beta^3)[\alpha^5-12\beta^3(\alpha^3+\beta^3)-3\alpha\beta^2(2\alpha^2+5\beta^2)]}{\alpha^6}} \\
E &= \sqrt{1+\frac{\alpha^4\beta B^2}{\alpha^5+3\alpha^4\beta+3\alpha^2\beta^2(4\alpha+6\beta)-3\alpha\beta^2(2\alpha^2+5\beta^2)+5\beta^3(4\alpha+3\beta)}}.
\end{aligned}
$$

Theorem 1. *Consider a Finsler space with $(\alpha,\beta)-$metric given by*

$$
L = \alpha+\beta+\frac{\beta^2}{\alpha}+\frac{\beta^3}{\alpha^2}
$$

for which the condition (12) is true. Then $V^i_j = X^i_k Y^k_j$ *is a nonholonomic Finsler frame given in equation (25), with* X^i_k *and* Y^k_j *are given by (23) and (24) respectively. Then if inverse of* V^i_j *is denoted by* $\tilde{V}^i_j$*, then we get* $V^i_k \tilde{V}^k_j = \delta^i_j$.

4 Gauge transformation of Finsler metrics

Suppose a particle is moving in an non geodesic path then it should be under the influence of some external force. In this case, to explain the path of motion an external force term is added to the equation of motions. Alternative point of view is that motion can be explained by a new metric, which would result from a gauge transformation. In this way, physical force fields can be geometrized, and general relativistic idea of space time curvature determining the path of the particle will also include fields other than gravitation. For this purpose a class of gauge transformations which act on tangent space is considered [6].

Under these kind of transformations, the tangent vector y^i transforms as

$$y^j = Y^j_i \tilde{y^i}. \tag{26}$$

where $i, j = 0, 1, 2, 3$ are indices corresponding the space components, and

$$Y^i_j = \frac{\partial y^i}{\partial \tilde{y^j}}. \tag{27}$$

Then the inverse of Y^i_j given by $\tilde{Y^i_j} = \frac{\partial \tilde{y^i}}{\partial y^j}$. Hence

$$Y^i_k \tilde{Y^k_j} = \delta^i_j. \tag{28}$$

and these transformations Y^i_j are called Y transformations.

Even though the transformation does not act on the base space coordinates, it will seen to produce changes in the base space. Therefore, these transformations also depend on the base coordinates, such as

$$\tilde{Y^i_j} = \tilde{Y^i_j}(x, y). \tag{29}$$

The Y transformation of the metric tensor is given as

$$\tilde{g_{ij}}(x, y) = Y^\alpha_i(x, y) Y^\beta_j(x, y) g_{\alpha\beta}(x, y). \tag{30}$$

Under this transformation, Finsler metric function is invariant, such as

$$\begin{aligned} \tilde{F}^2(x, \tilde{y}) &= \tilde{g_{ij}} \tilde{y}^i \tilde{y}^j \\ &= g_{\alpha\beta}(x, y) Y^\alpha_i Y^\beta_j \tilde{y}^i \tilde{y}^j \\ &= g_{\alpha\beta} y^\alpha y^\beta \\ &= F^2(x, y). \end{aligned} \tag{31}$$

Here y^j is the contravariant vector and the covariant vector associated with it is y_i, where $y_i = g_{ij}y^j$. Covariant vector y_i transforms as

$$\tilde{y_i} = Y_i^{\alpha} y_{\alpha}. \tag{32}$$

Since

$$\frac{\partial \tilde{y^i}}{\partial \tilde{y^j}} = \tilde{g_{ij}} + Y_j^{\beta} \frac{\partial Y_i^{\alpha}}{\partial y^{\beta}} y_{\alpha} \tag{33}$$

$$= Y_i^{\alpha} Y_j^{\beta} g_{\alpha\beta} + Y_j^{\beta} \frac{\partial Y_i^{\alpha}}{\partial y^{\beta}} y_{\alpha}. \tag{34}$$

The Y transformation of the Finslerian metric tensor does not yield a tensor unless

$$\frac{\partial Y_i^{\alpha}}{\partial y^{\beta}} y_{\alpha} = 0. \tag{35}$$

The condition (35) is called as the metric condition [5].

Now we have to study how many of the Finsler metrics can be obtained by this sort of gauge transformation. At this point one can only list those for which a specific Y matrix is known: Randers, Kropina, Beil, Weyl, and metrics where Y gives a conformal transformation. Obviously, nonlinear metrics are not included. What does this gauge transformation mean physically? It can be interpreted as what happens when a non-gravitational field is turned on in a region of space. For example, electromagnetic field. A metric has also been given for the electroweak field $SU(2)U(1)$[9]. The gauge transformation could also be interpreted as a distortion or deformation of the original Lorentz space. In other words, the gauge field twists or distorts the space. The relative effect is, by the way, a torsion rather than a curvature. Although, remarkably, the final outcome is a curved space. The torsion interpretation has been advocated by Holland in [9], who relates the transformation to nonholonomic frames. The nonholonomic frame viewpoint is explained in a very useful new paper by Bucataru [7].

It is obvious that Y transformations, when Y_j^i is a function of x only, that is

$$Y_j^i = Y_j^i(x)$$

satisfy the metric condition. These type of transformations are called K-group or linear transformations. Y transformations can be interpreted as the transformations from an original space where there exists no external field, to a space that also contains external fields which are turned on by some physical potentials contained in Y_j^i [6].

It is now time to get to some specific physics using the above developments. There are several gauge transformations which might give useful results. The Gauge transformation from Minkowski metric η_{ij} to the Finsler metric g_{ij} given by the in the equation (9) as

$$g_{ij} = V_i^k V_j^l \eta_{kl}. \tag{36}$$

Here V_j^i is the nonholonomic frame.

R.G.Beil[3] had studied Y transformations for general Finsler (α, β)-metrics.

$$Y_j^i = \sqrt{a}\delta_j^i - \frac{1}{B^2}\{\sqrt{a} - \sqrt{a + bB^2}\}B^i B^j \tag{37}$$

where B^j is a vector which can be associated to a physical potential, and $B^2 = g_{ij}B^i B^j$. Here B is a constant depending on the physical space that will be geometrized. The inverse transformation is given by the inverse of the matrix (37), such as

$$\tilde{Y_j^i} = \frac{1}{\sqrt{a}}\delta_i^j - \frac{1}{B^2}\{\frac{1}{\sqrt{a}} - \frac{1}{\sqrt{a + bB^2}}\}B^j B^i. \tag{38}$$

Similarly, gauge transformation of metric tensors a_{ij} and h_{ij} of the Finsler spaces with (α, β)-metrics is given in equation (19) by,

$$h_{ij} = X_i^k X_j^l a_{kl}, \tag{39}$$

and gauge transformation of metric tensors h_{ij} and g_{ij} given by (21) as,

$$g_{mn} = Y_m^i Y_n^j h_{ij}, \tag{40}$$

where X_j^i and Y_j^i are nonholonomic Finsler frames acts as gauge transformations with respect to two deformations (16) and (17). Then metric tensors a_{ij} and g_{ij} related by nonholonomic Finsler frames V_j^i is given by,

$$g_{mn} = V_m^i V_n^j a_{ij}. \tag{41}$$

Hence nonholonomic Finsler frames V_j^i are the gauge transformations between Finsler metric tensors.

5 Conclusion

Holland studies a unified formalism which uses a nonholonomic frame (non integrable 1-form) on space-time, a sort of plastic deformation, arising from consideration of a charged particle moving in an external electromagnetic field in the background space-time viewed as a strained medium [9]. Ioan Bucataru constructed nonholonomic Finsler frame for a class of generalized Lagrange spaces, for a class of Lagrange spaces with (α, β)-metric and for Finsler spaces with (α, β)-metric. For a Finsler space with (α, β)-metric, the fundamental tensor field might be taught as the result of two Finsler deformation. Then we can determine a corresponding frame for each of these two Finsler deformations. Consequently, a nonholonomic frame for a Finsler space with (α, β)-metric will appear as a product of two Finsler frames formerly determined. In this paper we are determined the nonholonomic Finsler frame for Finsler space with special (α, β)-metric $L = (\alpha + \beta + \frac{\beta^2}{\alpha} + \frac{\beta^3}{\alpha^2})^2$. And also we discussed that Finsler nonholonomic frames acts as a gauge transformation of Finsler metric tensors.

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