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η-RICCI SOLITONS IN $(LCS)_n$-MANIFOLD

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Abstract

The object of the present paper is to bring out curvature conditions for which η -Ricci solitons in $(LCS)_n$-manifolds are sometimes shrinking or expanding and some other time remain steady. Finally, the existence of shrinking and expanding η-Ricci solitons in such manifolds are ensured by examples.

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1 Introduction

The notion of Lorentzian concircular structure manifolds (briefly $(LCS)_n$ - manifolds) has been initiated by Shaikh [24]. Thereafter, a lot of study has been carried out. For details we refer [25], [26], [27], [28] and the references therein. Recently, in tune with Yano and Sawaki [33], the present authors [20] have introduced and studied generalized quasi-conformal curvature tensor $\mathcal{W}$, in the context of $N(k,\mu)$ -manifold. The generalized quasi-conformal curvature tensor is defined for n-dimensional manifold as

$$\begin{aligned}\mathcal{W}(X,Y)Z &= \frac{n-1}{n}\left[\{1+(n-1)a-b\}-\{1+(n-1)(a+b)\}c\right]C(X,Y)Z\\ &\quad+[1-b+(n-1)a]E(X,Y)Z+(n-1)(b-a)P(X,Y)Z\\ &\quad+\frac{n-1}{n}(c-1)\{1+(n-1)(a+b)\}\hat{C}(X,Y)Z \qquad (1)\end{aligned}$$

for all X, Y & $Z \in \chi(M)$, the set of all vector field of the manifold M , where the scalers triples (a,b,c) being real constants and the symbols C, $\hat{C}$, E, P stand for Conformal, Conharmonic, Concircular and Projective curvature tensor

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respectively. The beauty of such curvature tensor lies in the fact that it has the flavour of Riemann curvature tensor R if $(a, b,c) \equiv (0,0,0)$, Conformal curvature tensor C [12] if $(a, b,c) \equiv (-\frac{1}{n-2}, -\frac{1}{n-2}, 1)$, Conharmonic curvature tensor $\hat{C}$ [15] if $(a, b,c) \equiv (-\frac{1}{n-2}, -\frac{1}{n-2}, 0)$, Concircular curvature tensor E ([2], p. 84) if $(a, b,c) \equiv (0,0,1)$, Projective curvature tensor P([2], p. 84) if $(a, b,c) \equiv (-\frac{1}{n-1}, 0, 0)$ and m-Projective curvature tensor H [21], if $(a, b,c) \equiv (-\frac{1}{2n-2}, -\frac{1}{2n-2}, 0)$. The equation (1) can also be written as

$$\begin{aligned} \mathcal{W}(X,Y)Z &= R(X,Y)Z + a[S(Y,Z)X - S(X,Z)Y] \\ &\quad + b[g(Y,Z)QX - g(X,Z)QY] \\ &\quad - \frac{cr}{n}\left(\frac{1}{n-1} + a + b\right)[g(Y,Z)X - g(X,Z)Y] \end{aligned} \tag{2}$$

where, S, Q, r being Ricci tensor, Ricci operator and scalar curvature respectively.

The study of the Ricci solitons in contact geometry has begun with the work of Ramesh Sharma ([23], [13]). Ricci solitons in contact metric manifolds are also extensively studied by Mukut Mani Tripathi [32], Cornelia Livia Bejan and Mircea Crasmareanu ([7], [6]) and the references therein. Ricci solitons are defined as triples (g, V, λ), where (M, g) is a Riemannian manifold and V is a vector field (the potential vector field) so that the following equation is satisfied

$$\frac{1}{2}\pounds_V g + S + \lambda g = 0 \tag{3}$$

where $\pounds$ denotes the Lie derivative, S is the Ricci tensor and λ a constant on M. A Ricci soliton is said to be shrinking, steady or expanding according to λ negative, zero and positive respectively. A Ricci soliton with V zero is reduced to Einstein equation.

During the last two decades, the geometry of Ricci solitons has been the focus of attention of many mathematicians ([11], [5]). It has become more important after Grigory Perelman applied Ricci solitons to solve the long standing Poincaré conjecture posed in 1904.

η -Ricci solitons (M, g, λ, μ) is the generalization of Ricci solitons (M, g, λ) which is defined as

$$L_\xi g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0, \tag{4}$$

where L_ξ is the Lie derivative operator along the vector field ξ, S is the Ricci curvature tensor field of the metric g, λ and μ are real constants.

Our paper is structured as follows. Section 2 is concerned with $(LCS)_n$-manifolds and some known results. η-Ricci solitons in $(LCS)_n$-manifold satisfying $\mathcal{W}(\xi, X)\cdot S = 0$ has been studied in section 3. It is observed that Ricci soliton of such manifold is expanding, steady or shrinking according to $\alpha^2 \gtreqless \rho$ for each of $\tilde{C}(\xi, X)\cdot S = 0$, $P(\xi, X)\cdot S = 0$ and $H(\xi, X)\cdot S = 0$ provided $\mu \neq -\alpha$.

In section 4, η-Ricci solitons in $(LCS)_n$-manifolds admitting $(\xi \wedge_S X)\cdot\mathcal{W} = 0$ have been investigated. It is also determined that Ricci soliton of such manifold is either expanding, steady or shrinking according to $\alpha^2 \gtreqless \rho$ or $\mu + 3\alpha \lesseqgtr 0$ for each of $(\xi \wedge_S X)\cdot\tilde{C} = 0$, $(\xi \wedge_S X)\cdot P = 0$ and $(\xi \wedge_S X)\cdot H = 0$. Finally, the existence of shrinking and expanding η-Ricci solitons in such manifolds are ensured by examples.

2 $(LCS)_n$-manifolds and some known results

An $n$dimensionally Lorentzian manifold M is a smooth connected paracompact Hausdorff manifold with a Lorentzian metric g, that is, M admits a smooth symmetric tensor field g of type $(0,2)$ such that for each point $p \in M$, the tensor $g_p : T_pM \times T_pM \to R$ is a non-degenerate inner product of signature $(-,+,...,+)$, where T_pM denotes the tangent vector space of M at p and R is the real number space. A non-zero vector $v \in T_pM$ is said to be timelike (resp., non-spacelike, null, spacelike) if it satisfies $g_p(U,U) < 0$ (resp, ≤ 0, $= 0$, > 0)[18]. The category into which a given vector falls is called its causal character.

Let M^n be a Lorentzian manifold admitting a unit timelike concircular vector field ξ, called the characteristic vecotor field of the manifold. Then we have

$$g(\xi,\xi) = -1. \tag{5}$$

Since ξ is a unit concircular vector field, there exists a non-zero 1-form η such that for

$$g(X,\xi) = \eta(X) \tag{6}$$

the equation of the following form holds

$$(\nabla_X \eta)(Y) = \alpha\{g(X,Y) + \eta(X)\eta(Y)\} \qquad (\alpha \neq 0) \tag{7}$$

η for all vector fields X, Y where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g and α is a non-zero scalar function satisfying

$$\nabla_X \alpha = (X\alpha) = \alpha(X) = \rho\eta(X), \tag{8}$$

ρ being a certain scalar function. If we put

$$\phi X = \frac{1}{\alpha}\nabla_X \xi, \tag{9}$$

then from (7) and (9), we have

$$\phi X = X + \eta(X)\xi, \tag{10}$$

from which it follows that ϕ is a symmetric $(1,1)$ tensor. Thus the Lorentzian manifold M^ntogether with the unit timelike concircular vector field ξ, its associated 1-form η and (1,1) tensor field ϕ is said to be a Lorentzian concircular structure manifold (briefly $(LCS)_n$-manifold) [5]. In an $(LCS)_n$-manifold, the following relations hold [24]:

$$\begin{aligned} \eta(\xi) &= -1, \quad \phi \circ \xi = 0, & (11)\\ \eta(\phi X) &= 0, \qquad g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), & (12)\\ \eta(R(X,Y)Z) &= (\rho - \alpha^2)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)], & (13)\\ R(X,Y)\xi &= (\rho - \alpha^2)[\eta(Y)X - \eta(X)Y], & (14) \end{aligned}$$

for any vector fields X, Y, Z.

Let (M, ϕ, ξ, η, g) be a $(LCS)_n$ manifold satisfying (4). Writing $L_\xi g$ in terms of the Levi-Civita connection ∇, we obtain from (4) that,

$$2S(X,Y) = -g(\nabla_X \xi, Y) - g(X, \nabla_Y \xi) - 2\lambda g(X,Y) - 2\mu\eta(X)\eta(Y), \tag{15}$$

for any $X, Y \in \chi(M)$. As a consequence of (9) and (10), the above equation becomes

$$S(X,Y) = -(\lambda + \alpha)g(X,Y) - (\mu + \alpha)\eta(X)\eta(Y). \tag{16}$$

Thus, for $(LCS)_n$-manifold with η-Ricci solution the generalized quasi-conformal curvature tensor W takes the form

$$\begin{aligned} \mathcal{W}(X,Y)Z &= R(X,Y)Z - \left[(\lambda+\alpha)(a+b) + \frac{cr}{n}\left(\frac{1}{n-1} + a + b\right)\right] \\ &\quad [g(Y,Z)X - g(X,Z)Y] - a(\mu+\alpha)\eta(Z)\{\eta(Y)X - \eta(X)Y\} \\ &\quad -b(\mu+\alpha)\{g(Y,Z)\eta(X) - g(X,Z)\eta Y\}\xi. \end{aligned} \tag{17}$$

In particular, replacing $Y = \xi$ in (16), we have

$$S(X,\xi) = [\mu - \lambda]\eta(X). \tag{18}$$

From (13) and (18) one can easily bring out

$$[\mu - \lambda] = (n-1)(\rho - \alpha^2). \tag{19}$$

Also, from (16) we have

$$r = \mu - n\lambda - 2n\alpha. \tag{20}$$

3 η-Ricci solitons in $(LCS)_n$-manifold satisfying $\mathcal{W}(\xi, X) \cdot S = 0$

In this section we consider an $(LCS)_n$-manifold satisfying $W(\xi, X)\cdot S = 0$. Hence, we have

$$S(\mathcal{W}(\xi,X)Y, Z) + S(Y, \mathcal{W}(\xi,X)Z) = 0, \tag{21}$$

for any $X, Y, Z \in \chi(M)$.

In view of the expression (14), (16) and (17) we get

$$-(\mu+\alpha)\left\{\alpha^2-\rho+\lambda(a+b)+\frac{cr}{n}\left(\frac{1}{n-1}+a+b\right)\right\}$$
$$[2\eta(X)\eta(Y)\eta(Z)+g(X,Z)\eta(Y)+g(X,Y)\eta(Z)]=0, \tag{22}$$

which yields for $Y=\xi$

$$(\mu+\alpha)\Big[\alpha^2-\rho+\lambda(a+b)+\frac{cr}{n}\left(\frac{1}{n-1}+a+b\right)\Big]\times$$
$$\times\Big[\eta(X)\eta(Z)+g(X,Z)\Big]=0$$
$$\Rightarrow\quad (\mu+\alpha)\Big[\alpha^2-\rho+\lambda(a+b)$$
$$+\frac{c\{\mu-n\lambda-(n-1)\alpha\}}{n}\left(\frac{1}{n-1}+a+b\right)\Big]g(\phi X,\phi Y)=0. \tag{23}$$

for any X, $Y\in\chi(M)$. This leads to the following

Theorem 1. *Let $M^n(\phi,\xi,\eta,g)$ be an $(LCS)_n$-manifold bearing an η-Ricci soliton satisfying $\mathcal{W}(\xi,X)\cdot S=0$. Then*

Curvature condition	The values of λ & μ
$R(\xi,X)\cdot S=0$	$\mu=-\alpha$
$C(\xi,X)\cdot S=0$	$\mu=-\alpha$ or $\lambda=(n-1)(\alpha^2-\rho)+\frac{(n-1)\alpha-\mu}{2(n-1)}$
$\tilde{C}(\xi,X)\cdot S=0$	$\mu=-\alpha$ or $\lambda=(n-1)(\alpha^2-\rho)$
$E(\xi,X)\cdot S=0$	$\mu=-\alpha$ or $\lambda=(n-1)(\alpha^2-\rho)+\frac{\mu-(n-1)\alpha}{n}$
$P(\xi,X)\cdot S=0$	$\mu=-\alpha$ or $\lambda=(n-1)(\alpha^2-\rho)$
$H(\xi,X)\cdot S=0$	$\mu=-\alpha$ or $\lambda=(n-1)(\alpha^2-\rho)$

Theorem 2. *Let $M^n(\phi,\xi,\eta,g)$ be an $(LCS)_n$-manifold bearing an η-Ricci soliton. Then the η-Ricci soliton of such manifold is expanding, steady or shrinking according to $\alpha^2 \gtreqless \rho$ for each of $\tilde{C}(\xi,X)\cdot S=0$, $P(\xi,X)\cdot S=0$ and $H(\xi,X)\cdot S=0$ provided $\mu\neq-\alpha$.*

Theorem 3. *Let $M^n(\phi,\xi,\eta,g)$ be an $(LCS)_n$-manifold bearing an η-Ricci soliton. Then the η-Ricci soliton of such a manifold is expanding, steady or shrinking according as $(n-1)(\alpha^2-\rho)+\frac{(n-1)\alpha-\mu}{2(n-1)}\gtreqless 0$ for each of $C(\xi,X)\cdot S=0$ and $E(\xi,X)\cdot S=0$ provided $\mu\neq-\alpha$.*

4 $(LCS)_n$-manifold satisfying $((\xi,\wedge_S X)\cdot\mathcal{W})=0$

Let $M^n(\phi,\xi,\eta,g)(n>1)$, be an $(LCS)_n$-manifold satisfying the condition

$$((\xi,\wedge_S X)\cdot\mathcal{W})(Y,Z)U=0, \tag{24}$$

which is equivalent to

$$\begin{aligned}&S(X,\mathcal{W}(Y,Z)U)\xi - S(\xi,\mathcal{W}(Y,Z)U)X - S(X,Y)\mathcal{W}(\xi,Z)U\\&+S(\xi,Y)\mathcal{W}(X,Z)U - S(X,Z)\mathcal{W}(Y,\xi)U + S(\xi,Z)\mathcal{W}(Y,X)U\\&-S(X,U)\mathcal{W}(Y,Z)\xi + S(\xi,U)\mathcal{W}(Y,Z)X = 0.\end{aligned} \tag{25}$$

Taking the inner product with ξ, we have

$$\begin{aligned}0 \;=\;& -S(X,\mathcal{W}(Y,Z)U) - S(\xi,\mathcal{W}(Y,Z)U)\eta(X) - S(X,Y)\eta(\mathcal{W}(\xi,Z)U)\\&+S(\xi,Y)\eta(\mathcal{W}(X,Z)U) - S(X,Z)\eta(\mathcal{W}(Y,\xi)U) + S(\xi,Z)\eta(\mathcal{W}(Y,X)U)\\&+S(X,U)\eta(\mathcal{W}(Y,Z)\xi) + S(\xi,U)\eta(\mathcal{W}(Y,Z)X).\end{aligned} \tag{26}$$

In view of (16) and (18), we get

$$\begin{aligned}&\left\{\rho - \alpha^2 - \lambda(a+b) - \frac{cr}{n}\left(\frac{1}{n-1} + a + b\right)\right\}[(\lambda+\alpha)\{2g(X,Y)g(Z,U)\\&+2\eta(X)\eta(Y)g(Z,U) + 2\eta(U)\eta(Y)g(X,Z) - 2g(X,U)\eta(Z)\eta(Y)\}\\&+(\mu+\alpha)\{2\eta(X)\eta(Y)\eta(Z)\eta(U) + 2\eta(Z)\eta(Y)g(X,U) + g(X,Z)\eta(Y)\eta(U)\\&-g(X,Y)\eta(Z)\eta(U)\}] = 0,\end{aligned} \tag{27}$$

which yields for $U = \xi$ that

$$\begin{aligned}&\left\{\rho - \alpha^2 - \lambda(a+b) - \frac{cr}{n}\left(\frac{1}{n-1} + a + b\right)\right\}\\&[2(\lambda+\alpha) + (\mu+\alpha)][g(X,Y)\eta(Z) - g(X,Z)\eta(Y)] = 0,\end{aligned} \tag{28}$$

which leads to

$$\lambda = -\frac{(\mu+3\alpha)}{2} \quad \text{or,}\ \lambda = \frac{n(n-1)(\alpha^2-\rho) + c\{\mu-(n-1)\alpha\}[1+(n-1)(a+b)]}{(c-a-b)[1+(n-1)(a+b)]}. \tag{29}$$

Theorem 4. *Let $M^n(\phi,\xi,\eta,g)$ be an $(LCS)_n$-manifold bearing an η-Ricci soliton and $(\xi \wedge_S X)\cdot \mathcal{W} = 0$. Then*

Curvature condition	The values of λ & μ
$(\xi\wedge_S X)\cdot R = 0$	$\lambda = -\frac{(\mu+3\alpha)}{2}$
$(\xi \wedge_S X)\cdot C = 0$	$\lambda = -\frac{(\mu+3\alpha)}{2}$ or $\lambda = (n-1)(\alpha^2-\rho) + \frac{(n-1)\alpha-\mu}{2(n-1)}$
$(\xi \wedge_S X)\cdot \tilde{C} = 0$	$\lambda = -\frac{(\mu+3\alpha)}{2}$ or $\lambda = (n-1)(\alpha^2-\rho)$
$(\xi \wedge_S X)\cdot E = 0$	$\lambda = -\frac{(\mu+3\alpha)}{2}$ or $\lambda = (n-1)(\alpha^2-\rho) + \frac{\mu-(n-1)\alpha}{n}$
$(\xi \wedge_S X)\cdot P = 0$	$\lambda = -\frac{(\mu+3\alpha)}{2}$ or $\lambda = (n-1)(\alpha^2-\rho)$
$(\xi \wedge_S X)\cdot H = 0$	$\lambda = -\frac{(\mu+3\alpha)}{2}$ or $\lambda = (n-1)(\alpha^2-\rho)$

Theorem 5. *Let $M^n(\phi,\xi,\eta,g)$ be an $(LCS)_n$-manifold bearing an η-Ricci soliton. The η-Ricci soliton of such a manifold is either expanding, steady or shrinking according to $\alpha^2 \gtreqless \rho$ or $\mu + 3\alpha \lesseqgtr 0$ for each of $(\xi \wedge_S X)\cdot\tilde{C} = 0$, $(\xi \wedge_S X)\cdot P = 0$ and $(\xi \wedge_S X)\cdot H = 0$.*

Theorem 6. *Let $M^n(\phi, \xi, \eta, g)$ be an $(LCS)_n$-manifold bearing an η-Ricci soliton. The η-Ricci soliton of such a manifold is either expanding, steady or shrinking according to the Ricci soliton of such manifold is expanding, shrinking or steady according as $(n-1)(\alpha^2-\rho)+\frac{(n-1)\alpha-\mu}{2(n-1)} \gtreqless 0$ or $\mu+3\alpha \lesseqgtr 0$ for each of $(\xi \wedge_S X)\cdot C = 0$ and $(\xi \wedge_S X)\cdot E(\xi, X)\ S = 0$.*

5 Existence of expanding and shrinking η-Ricci soliton

Example 1. *Let us consider a 4dimensional manifold $M = \{(x^1, x^2, x^3, x^4) \in R^4 : x^4 \neq 0$, where (x^1, x^2, x^3, x^4) being standard coordinates in R^4. Let $\{e_1, e_2, e_3, e_4\}$ be a linearly independent global frame on M given by*

$$e_1 = \cosh \frac{\partial}{\partial x^1}, \quad e_2 = \cosh x^4 \frac{\partial}{\partial x^2}, \quad e_3 = \cosh x^4 \frac{\partial}{\partial x^3}, \quad e_4 = \frac{\partial}{\partial x^4}.$$

Let g be the Lorentzian metric defined by $g(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^1}) = g(\frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^2}) = g(\frac{\partial}{\partial x^3}, \frac{\partial}{\partial x^3}) =$ $\sec h^2 x^4$, $g(\frac{\partial}{\partial x^4}, \frac{\partial}{\partial x^4}) = -1$ and $g(\frac{\partial}{\partial x^i}\ , \frac{\partial}{\partial x^j}) = 0$ for $i \neq j =$ 1, 2, 3, 4. Let η be the 1-form defined by $\eta(U) = g(U,\ e_4)$ for any $U \in \chi(M)$. Let ϕ be a tensor field of type $(1,1)$, defined by $\phi e_1 = e_1$, $\phi e_2 = e_2$, $\phi e_3 = e_3$ $\phi e_4 = 0$. Then using the linearity of ϕ and g we have $\eta(e_4) = -1$, $\phi^2 U = U + \eta(U)\ e_4$ and $g(\phi U, \phi W) = g(U, W) + \eta(U)\eta(W)$ for any $U, W \in \chi(M)$. Thus for $e_4 = \xi$, (ϕ, ξ, η, g) defines a Lorentzian paracontact structure on M.

Let ∇ be the Levi-Civita connection with respect to the Lorentzian metric g and R be the curvature tensor of g. Then we have

$$[e_1, e_2] = -\cosh x^4 e_2, \quad [e_1, e_3] = -\cosh x^4 e_3, \quad [e_1, e_4] = -\tanh x^4 e_1,$$

$$[e_2, e_4] = -\tanh x^4 e_2, \quad [e_3, e_4] = -\tanh x^4 e_3.$$

Taking $e_4 = \xi$ and using Koszul formula for the Lorentzian metric g, we can easily calculate

$$\nabla_{e_1} e_4 = -\tanh x^4 e_1, \qquad \nabla_{e_2} e_4 = -\tanh x^4 e_2, \qquad \nabla_{e_3} e_4 = -\tanh x^4 e_3,$$

$$\nabla_{e_1} e_1 = -\tanh x^4 e_4, \qquad \nabla_{e_2} e_1 = \cosh x^4 e_2, \qquad \nabla_{e_3} e_1 = \cosh x^4 e_3,$$

$$\nabla_{e_2} e_2 = -\tanh x^4 e_4 - \cosh x^4 e_1, \qquad \nabla_{e_3} e_3 = -\tanh x^4 e_1 - \cosh x^4 e_1,$$

$$\nabla_{e_1} e_3 = 0, \qquad \nabla_{e_4} e_1 = 0, \qquad \nabla_{e_1} e_2 = 0, \qquad \nabla_{e_3} e_2 = 0,$$

$$\nabla_{e_4} e_2 = 0, \qquad \nabla_{e_4} e_3 = 0, \qquad \nabla_{e_4} e_4 = 0, \qquad \nabla_{e_2} e_3 = 0.$$

From the above, it can be easily seen that (ϕ, ξ, η, g) is an $(LCS)_4$ structure on M. Consequently $M^4(\phi, \xi, \eta, g)$ is an $(LCS)_4$-manifold with $\alpha = -\tanh x^4 e_4 \neq 0$ and $\rho = \sec h^2 x^4$.

Using the above relations, we can easily calculate the non-vanishing components of the curvature tensor as follows :

$$\begin{aligned}
R(e_2,e_3)e_2 &= (\cosh^2 x^4 - \tanh^2 x^4)e_3, \quad R(e_2,e_3)e_3 = (\tanh^2 x^4 - \cosh^2 x^4)e_2,\\
R(e_1,e_3)e_1 &= (\cosh^2 x^4 - \tanh^2 x^4)e_1, \quad R(e_1,e_3)e_3 = (\tanh^2 x^4 - \cosh^2 x^4)e_1,\\
R(e_3,e_4)e_4 &= (\sec h^2 x^4 - \tanh^2 x^4)e_3, \quad R(e_3,e_4)e_3 = (\sec h^2 x^4 - \tanh^2 x^4)e_4,\\
R(e_1,e_2)e_2 &= (\tanh^2 x^4 - \cosh^2 x^4)e_1, \quad R(e_1,e_2)e_1 = (\cosh^2 x^4 - \tanh^2 x^4)e_2,\\
R(e_1,e_4)e_1 &= (\sec h^2 x^4 - \tanh^2 x^4)e_4, \quad R(e_1,e_4)e_4 = (\sec h^2 x^4 - \tanh^2 x^4)e_1,\\
R(e_2,e_4)e_2 &= (\sec h^2 x^4 - \tanh^2 x^4)e_4, \quad R(e_2,e_4)e_4 = (\sec h^2 x^4 - \tanh^2 x^4)e_2,
\end{aligned}$$

The non-vanishing components of the Ricci tensor in $(LCS)_4$ manifold under consideration satisfying η- Ricci solution are

$$S(e_1,e_1) = S(e_2,e_2) = S(e_3,e_3) = \tanh x^4 - \lambda,$$
$$S(e_4,e_4) = \lambda - \mu.$$

Using the above relations, we can easily calculate the non-vanishing components as follows

$$\begin{aligned}
(\mathcal{W}(e_4,e_i)\cdot S)(e_4,e_i) &= -(\lambda - \tan hx^4)\Big[\sec h^2x^4 - \tan h^2x^4 - a(\lambda-\mu)\\
&\quad -b(\mu - \tan hx^4) - \frac{c}{4}(\mu - 4\lambda + 8\tan hx^4)\left(\frac{1}{3}+a+b\right)\Big]\\
&\quad -(\mu-\lambda)\Big[\sec h^2x^4 - \tan h^2x^4 - a(\lambda - \tan hx^4)\\
&\quad +b(\mu - \tan hx^4) - \frac{c}{4}(\mu - 4\lambda + 8\tan hx^4)\left(\frac{1}{3}+a+b\right)\Big]
\end{aligned}$$

for $i = 1,2,3$ and the components which can be obtained from these by the symmetric properties. Using the above relation we can easily bring out the following :

Theorem 7. *There exists an $(LCS)_4$-manifold bearing an η-Ricci soliton where the Ricci soliton is expanding or shrinking according to* $\sinh^2 \gtreqless 1$ *for each of* $\tilde{C}(\xi,X)\cdot S = 0$, $P(\xi,X)\cdot S = 0$ *and* $H(\xi,X)\cdot S = 0$ *provided* $\mu \neq \tanh x^4$.

Theorem 8. *There exists an $(LCS)_4$-manifold bearing an η-Ricci soliton where the Ricci soliton is expanding or shrinking according to* $3(\tanh^2 x^4 - \mathrm{sech}^2 x^4) \gtreqless \frac{3\tanh x^4 + \sec h^2 x^4}{6}$ *for each of* $C(\xi,X)\cdot S = 0$ *and* $E(\xi,X)\cdot S = 0$ *provided* $\mu \neq \tanh x^4$.

Example 2. *Let us consider a 4dimensional manifold* $M = \{(x^1,x^2,x^3,x^4) \in R^4 : x^4 \neq 0$, *where* (x^1,x^2,x^3,x^4) *being standard coordinates in* R^4. *Let* $\{e_1,e_2,e_3,e_4\}$ *be a linearly independent global frame on* M *given by*

$$e_1 = x^1x^4\,\frac{\partial}{\partial x^1}, \qquad e_2 = x^4\frac{\partial}{\partial x^2}, \qquad e_3 = x^4\frac{\partial}{\partial x^3}, \quad e_4 = (x^4)^3\frac{\partial}{\partial x^4}.$$

Let g be the Lorentzian metric defined by $g(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^1}) = \left(\frac{1}{x^1 x^4}\right)^2$, $g(\frac{\partial}{\partial x^2} \frac{\partial}{\partial x^2}) =$ $= g(\frac{\partial}{\partial x^3}, \frac{\partial}{\partial x^3}) = \left(\frac{1}{x^4}\right)^2$, $g(\frac{\partial}{\partial x^4}, \frac{\partial}{\partial x^4}) = -\left(\frac{1}{x^4}\right)^6$ *and* $g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = 0$ *for* $i \neq j = 1, 2, 3, 4$. *Let η be the 1-form defined by* $\eta(U) = g(U, e_4)$ *for any* $U \in \chi(M)$. *Let ϕ be the (1, 1) tensor field defined by* $\phi e_1 = e_1$, $\phi e_2 = e_2$, $\phi e_3 = e_3$ $\phi e_4 = 0$. *Then using the linearity of ϕ and g we have* $\eta(e_4) = -1$, $\phi^2 U = U + \eta(U)\, e_4$ *and* $g(\phi U, \phi W) = g(U, W) + \eta(U)\eta(W)$ *for any* $U, W \in \chi(M)$. *Thus for* $e_4 = \xi$, (ϕ, ξ, η, g) *defines a Lorentzian paracontact structure on M.*

Let ∇ be the Levi-Civita connection with respect to the Lorentzian metric g and R be the curvature tensor of g. Then we have

$$[e_1, e_2] = -x^4 e_2, \;\; [e_1, e_4] = -\left(x^4\right)^2 e_1, \;\; [e_2, e_4] = -\left(x^4\right)^2 e_2, \;\; [e_3, e_4] = -\left(x^4\right)^2 e_3.$$

Taking $e_4 = \xi$ and using Koszul formula for the Lorentzian metric g, we can easily calculate

$$\nabla_{e_1} e_4 = -\left(x^4\right)^2 e_1, \qquad \nabla_{e_2} e_4 = -\left(x^4\right)^2 e_2, \qquad \nabla_{e_3} e_4 = -\left(x^4\right)^2 e_3,$$
$$\nabla_{e_1} e_1 = -\left(x^4\right)^2 e_4, \qquad \nabla_{e_2} e_1 = x^4 e_2, \qquad \nabla_{e_3} e_3 = -\left(x^4\right)^2 e_4,$$
$$\nabla_{e_2} e_2 = -\left(x^4\right)^2 e_4 - \; x^4 e_1, \qquad \nabla_{e_4} e_1 = 0, \qquad \nabla_{e_3} e_2 = 0,$$
$$\nabla_{e_1} e_3 = 0, \qquad \nabla_{e_1} e_2 = 0, \qquad \nabla_{e_3} e_1 = 0,$$
$$\nabla_{e_4} e_2 = 0, \qquad \nabla_{e_4} e_3 = 0, \qquad \nabla_{e_4} e_4 = 0.$$

From the above it can be easily seen that (ϕ, ξ, η, g) is an $(LCS)_4$ structure on M. Consequently $M^4(\phi, \xi, \eta, g)$ is an $(LCS)_4$-manifold with $\alpha = -\left(x^4\right)^2 \neq 0$ and $\rho = 2\left(x^4\right)^2$. Using the above relations, we can easily calculate the non-vanishing components of the curvature tensor as follows :

$$R(e_2, e_3)e_2 = -\left(x^4\right)^4 e_3, \qquad R(e_2, e_3)e_3 = \left(x^4\right)^4 e_2, \qquad R(e_1, e_4)e_1 = \left(x^4\right)^4 e_4,$$
$$R(e_1, e_3)e_1 = -\left(x^4\right)^4 e_3, \qquad R(e_1, e_4)e_4 = \left(x^4\right)^4 e_1, \qquad R(e_2, e_4)e_2 = \left(x^4\right)^4 e_4,$$
$$R(e_1, e_3)e_3 = \left(x^4\right)^4 e_1, \quad R(e_3, e_4)e_4 = \left(x^4\right)^4 e_3, \quad R(e_3, e_4)e_3 = \left(x^4\right)^4 e_4,$$

$$\begin{aligned} R(e_1, e_2)e_2 &= \left(x^4\right)^4 e_1 - \left(x^4\right)^2 e_1, \quad R(e_2, e_4)e_4 = \left(x^4\right)^4 e_4, \\ R(e_1, e_2)e_1 &= -\left(x^4\right)^4 e_2 + \left(x^4\right)^2 e_2 \end{aligned}$$

and the components which can be obtained from these by the symmetric properties.

The non-vanishing components of the Ricci tensor in $(LCS)_4$ manifold under consideration satisfying η- Ricci solution are

$$S(e_1, e_1) = S(e_2, e_2) = S(e_3, e_3) = \left(x^4\right)^2 - \lambda$$
$$S(e_4, e_4) = -(\mu - \lambda)$$

As a consequence of the above, one can easily bring out the non-vanishing components as follows

$$\begin{aligned}((e_4, \wedge_S e_i)\cdot \mathcal{W})(e_i, e_2)e_4 &= Le_2\\ ((e_4, \wedge_S e_i)\cdot \mathcal{W})(e_i, e_3)e_4 &= Le_3\\ ((e_4, \wedge_S e_i)\cdot \mathcal{W})(e_i, e_4)e_4 &= Le_4\end{aligned}$$

for $i = 1, 2, 3$, where

$$L = [2\left(x^4\right)^2 - \left(x^4\right)^4 - \lambda(a+b) - \frac{c}{n}\left(\mu - n\lambda + 2n\left(x^4\right)^2\right)\left(\frac{1}{n-1} + a + b\right)][2\lambda - 3\left(x^4\right)^2 + \mu)].$$

This motivates

Theorem 9. *There exists an $(LCS)_4$-manifold bearing an η-Ricci soliton where the Ricci soliton is expanding or shrinking according to $\left(x^4\right)^2 \gtreqless 2$ or $\mu \lessgtr 3\left(x^4\right)^2$ for each of $(\xi \wedge_S X)\cdot\tilde{C} = 0$, $(\xi \wedge_S X)\cdot P = 0$ and $(\xi \wedge_S X)\cdot H = 0$.*

Theorem 10. *There exists an $(LCS)_4$-manifold bearing an η-Ricci soliton where the Ricci soliton is expanding or shrinking according to $3(\left(x^4\right)^4 - 2\left(x^4\right)^2) \gtreqless \frac{3\left(x^4\right)^2+\mu}{6} 0$ or $\mu \lessgtr 3\left(x^4\right)^2$ for each of $(\xi \wedge_S X)\cdot C = 0$ and $(\xi \wedge_S X)\cdot E(\xi, X)\ S = 0$.*

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