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PLANAR TENSEGRITY FORM-FINDING BY THE FORCE DENSITY METHOD

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Abstract: *A numerical form-finding algorithm capable of solving the initial equilibrium problem of free-standing planar tensgrities is presented herein. The only information needed at the start of the form-finding procedure is the topology of the structure and the member types (cables and struts). The eigenvalue decomposition of the force density matrix and the singular value decomposition of the equilibrium matrix are performed iteratively during the search of the nodal coordinates and the feasible set of force density coefficients.*

Key words: *Tensegrity, Form-finding, Force density method, Singular value decomposition, Eigenvalue decomposition*

1. Introduction

Tensegrity structures, first proposed for engineering applications by R.B. Fuller [6], have been in a constant development over the past 3 decades due to an ever-increasing interest in lightweight structural applications for engineering purposes. They belong to the subclass of prestressable pin-jointed frameworks, more exactly to the family of statically and kinematically indeterminate structures which can be rendered stable only under the effect of pre-stress [9]. Furthermore, the research of Connelly showed that the presence of the aforementioned pre-tensioning force depends on the geometry of the structure [3, 5]. In this context the exact definition of the initial geometry of a tensegrity structure is of capital importance. Thus, the design process of these structures must start by finding an initial geometric configuration which allows the existence of a set of pre-tensioning forces (which can stiffen the infinitesimal mechanisms of the proposed system) even in the lack of external loading.

This procedure of finding the equilibrium configuration of tensegrity structures is known as form-finding. The importance of this step is emphasized by the existence of several different methods for the initial geometry problem. In the early stages of tensegrity study the researchers tried to solve the problem of the equilibrium geometry through analytical methods, but soon they found out that these are only suitable for highly symmetric structures [4, 7, 10]. Later, the development of pioneering numerical methods such as dynamic relaxation [1, 14], force density method [11, 17], non-linear programming [15] or the reduced coordinates method [19] made a huge impact on the research of prestressable lightweight structures, such as the tensegrity system. A full review of the existing form-finding methods for tensegrity structures can be found in [20].

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In this paper a slightly modified formulation of the force density method, one similar to that proposed by Estrada et al [8] will be used for solving the prestressability problem of planar tensegrity structures.

2. Equilibrium Equations of the Force Density Method

The force density method was first proposed by Schek and Linkwitz [11, 17] for the analysis of purely tensioned cable networks, but later this had been modified for the analysis of tensegrity structures. This method capitalizes of an ingenious mathematical trick by transforming the non-linear equations of equilibrium into a series of linear equations. For example, the equilibrium equation in the *x* direction of a generalized node *j* can be formulated as

$$\sum_k \frac{n_{jk}}{l_{jk}}\left(x_j - x_k\right) = f_{jx} \,. \tag{1}$$

where node *j* is connected to node *k*, n_{jk} is the value of the axial force in the element *jk* and f_{jx} represents the projected value of the external loading onto the *x* direction. Although this equation seems to be linear the lengths of elements l_{jk} in the denominator are also functions of the nodal coordinates, thus making it non-linear. Schek's solution to the problem was to linearize these equations of equilibrium by introducing the value of force density for each element.

$$q_{jk} = \frac{n_{jk}}{l_{jk}} \,. \tag{2}$$

Although the original method could be of great importance for many researchers this paper will emphasize only on the modified version of the method, which is adapted for the analysis of free-standing tensegrities.

2.1. Basic Assumptions

The following assumptions are made throughout this article:

- The members of the presented structures are pin-jointed
- The topology of structures in terms of nodal connectivity is known
- No external load is applied onto the analysed structures and the self-weight of the elements is neglected in the form-finding procedure
- The structure does not need any fixed nodes to be rendered stable

2.2. Self-Equilibrium Equations for Free-Standing Tensegrity Structures

For a *D*-dimensional structure with *nB* members and *nN* free nodes the topology of the structure can be defined by its connectivity matrix $\mathbf{C}_s$ ($\in$ $^{nB \times nN}$) as presented in [11, 12,

17]. Let us imagine a member *k* that connects the nodes *i* and *j*, then the elements of the *k*th row of the incidence matrix $\mathbf{C}_s$ will be defined by the following rule:

$$C_{s(k,m)} = \begin{cases} +1 & \text{for } m = i \\ -1 & \text{for } m = j \\ 0 & \textit{otherwise} \end{cases}. \tag{3}$$

The equilibrium equations in each direction of a generalised pin-jointed assembly could be formulated as

$$\mathbf{C}_s{}^T \mathbf{Q} \mathbf{C}_s \mathbf{x} = \mathbf{f}_x \tag{4.1}$$

$$\mathbf{C}_s{}^T \mathbf{Q} \mathbf{C}_s \mathbf{y} = \mathbf{f}_\mathbf{y} \tag{4.2}$$

where $\mathbf{f}_\mathbf{x}$ and $\mathbf{f}_\mathbf{y}$ are the vectors of the external loads in the *x*- and *y*-directions. **Q** is a square matrix which has on its diagonals the components of the vector of force densities **q** as suggested in [17], defined by the following rule

$$\mathbf{q} = \left(q_1 \;\; q_2 \;\; q_3 \ldots . q_{nB}\right)^T \tag{5}$$

each component of this vector represents the force density coefficient of a member defined in Eq. (2). As a general rule the values of force density coefficients are positive for tensioned elements (cables) and negative for compressed members (struts).

In order to simplify the equilibrium equations one could make the following notation

$$\mathbf{D} = \mathbf{C}_s{}^T \mathbf{Q} \mathbf{C}_s \tag{6}$$

where **D** is the square symmetric matrix of force densities [11, 17], which essentially is the analogue of the stress matrix presented in Connelly's Energy Method [2].

Adapting the equations of equilibrium of a generalised pin-jointed framework to the case of free-standing tensegrity structures and taking into account Eq. (6) the equilibrium of the system can be defined as

$$\mathbf{D}\mathbf{x} = 0\,. \tag{7.1}$$

$$\mathbf{D}\mathbf{y} = 0\,. \tag{7.2}$$

Reorganizing Eq. (7.1-7.2) the equilibrium condition of free-standing planar tensegrity structures can be defined as

$$\mathbf{D}\left[\mathbf{x}\,\mathbf{y}\right] = 0\,. \tag{8}$$

On the other hand by substituting Eq. (6) into Eq. (7.1, 7.2) the self-equilibrium equations of the free-standing tensegrity structures are

$$\mathbf{Aq} = 0\,. \tag{9}$$

where $\mathbf{A}$ ($\in$ $^{DnN\ x\ nB}$) is the equilibrium matrix presented in [13], defined as

$$\mathbf{A} = \begin{bmatrix} \mathbf{C}_s{}^T \mathrm{diag}(\mathbf{C}_s\mathbf{x}) \\ \mathbf{C}_s{}^T \mathrm{diag}(\mathbf{C}_s\mathbf{y}) \end{bmatrix}. \tag{10}$$

Equation (8) presents the relationship between the force densities and the nodal coordinates, while Eq. (9) shows the relationship between the projected lengths of members in each direction and their force density coefficients. Both equations representing now linear homogeneous systems of self-equilibrium.

3. Prestressability Conditions

As noted earlier the tensegrity structures are part of the family of statically and kinematically indeterminate pin-jointed assemblies. A classification of pin-jointed structures according to Pellegrino [16] is presented in Table 1.

Classification of pin jointed structures Table 1

Category	*s and m values*	*Type of Structure*
I	s=0; m=0	Statically and kinematically determinate
II	s=0; m >0	Statically determinate and kinematically indeterminate
III	s>0; m=0	Statically indeterminate and kinematically determinate
IV	s>0; m>0	Statically and kinematically indeterminate

This type of structure must fulfil two necessary, but not sufficient conditions in a *D*-dimensional space to be in a stable equilibrium configuration. The first one refers to the existence of the states of self-stress, which essentially translates to the rank deficiency condition of the equilibrium matrix [13]. The number of inextensional mechanisms *m* and the corresponding states of self-stress *s* can be defined using the rank of the equilibrium matrix r_A, as follows

$$m = D \cdot nN - r_A \geq 1\,. \tag{11}$$

$$s = nB - r_A \geq 1\,. \tag{12}$$

The second condition refers to the rank of the force density matrix **D** :

$$rank(\mathbf{D}) < nN - D\,. \tag{13}$$

The later condition is in concordance with the rank deficiency condition of the prestress stable state defined by Connelly [5].

4. The Form-finding Procedure

Contrary to the majority of the known form-finding methods the one presented in this article needs only a handful of initial parameters, such as the topology of the structure (defined through the connectivity matrix $\mathbf{C}_s$) and the member types, in order to solve the initial equilibrium problem. This means that any initial assumptions regarding the nodal coordinates, member length ratios or symmetric properties are not needed, making the form-finding procedure considerably easier.

The prototype of the force density vector $\mathbf{q}^0$ can be defined by assigning trivial values, such as +1 and -1 for the cables, respective the struts:

$$\mathbf{q}^0 = \left(\underbrace{+1\ +1\ +1 \ldots +1}_{\text{cables}} \quad \underbrace{-1\ -1 - 1 \ldots . -1}_{\text{struts}} \right)^T . \tag{14}$$

First, the force density matrix **D** will be calculated starting from $\mathbf{q}^0$ and $\mathbf{C}_s$ by Eq. (6). Subsequently the nodal coordinates will be derived from the force density matrix by eigenvalue decomposition. These assumed nodal coordinates then will be substituted in Eq. (10) defining the equilibrium matrix **A**. Finally, the equilibrium conditions defined in Eq. (8, 9) will be verified. If these are not fulfilled the force density matrix **D** will be updated through a least squares type algorithm by choosing the updated vector of force densities **q** by the singular value decomposition of the equilibrium matrix. The process is iteratively searching for a set of nodal coordinates [**x y**] and force density coefficients **q** until the rank deficiency conditions presented in Eq. (12, 13) are fulfilled, thus forcing the equilibrium conditions presented in Eq. (8, 9) to become true.

4.1. Eigenvalue Decomposition of the Force Density Matrix

The square symmetric force density matrix **D** can be factorized using eigenvalue decomposition as follows [18]:

$$\mathbf{D} = \Phi \Lambda \Phi^T . \tag{15}$$

where Φ ($\in$ $^{nN \times nN}$) is the orthogonal matrix whose columns are the eigenvector basis $\mathbf{u}_i$ of **D**. While Λ ($\in$ $^{nN \times nN}$) is the diagonal matrix of the corresponding eigenvalues λ_i. These will be in an increasing order.

The number of zero eigenvalues (noted with *e*) of **D** is equal to the dimension of its null space *nD* [18]. It is clear that the null space of **D** contains the potential nodal coordinates

of the tensegrity structure. There are two cases to be considered. **The first one** is $e < nD$. In this case the first nD eigenvectors of Φ are taken as potential nodal coordinates

$$[\mathbf{x}\ \mathbf{y}] \in [¤_1\ \ ¤_2 \ldots ¤_{nD}]. \tag{16}$$

The algorithm then determines the force density vector **q**, which will be repeatedly approximated from Eq. (9), thus providing the least squares type of solution to the equilibrium problem.

An eigenvector $¤_i$ will be removed from the nD eigenvector bases of Φ if its projected length ($\mathbf{C}_s ¤_i$) is either equal to 0 or forces a zero-length member.

The remaining eigenvectors will constitute the linearly independent basis of nodal coordinates for the tensegrity structure

The second case is when $e > nD$, i.e. the rank deficiency is larger than needed and the force density matrix **D** is not positive definite during the iteration. An additional verification of the positive semi-definiteness of the tangent stiffness matrix is necessary as per Connelly's super-stability conditions defined in [3].

4.2. Selecting a Feasible Set of Force Densities

Once a potential set of coordinates **[x y]** is found from Eq. (8) the equilibrium matrix **A** will be determined through substitution in Eq. (10). The solution of the linear homogeneous system presented in Eq. (9) can be found by the singular value decomposition (SVD) of the equilibrium matrix **A** [16]

$$\mathbf{A} = \mathbf{U}\mathbf{S}\mathbf{v}\mathbf{W}^T. \tag{17}$$

where $\mathbf{U}\ (\in {}^{\text{DnN x DnN}}) = [\mathbf{u}_1\ \ \mathbf{u}_2 \ldots \mathbf{u}_{DnN}]$ and $\mathbf{W}\ (\in {}^{\text{DnN x DnN}}) = [\mathbf{w}_1\ \ \mathbf{w}_2 \ldots \mathbf{w}_{nB}]$ are the left and right orthonormal bases of singular values. **Sv** respectively is the diagonal matrix of nonnegative singular values of **A** in a decreasing order [18].

The iterative form-finding procedure will stop when both the rank deficiency condition of Eq. (12) and the equilibrium condition of Eq. (8) are met. Accordingly, there are two cases to be considered with respect to the number of self-stress states s during the form-finding process. **Case 1** $s = 0$, there is no self-stress state which could balance the infinitesimal mechanisms of the structure. The form-finding procedure will use a least squares type of approach to select a possible self-stress state from the right orthonormal bases of singular values which matches in sign with the prototype of force densities $\mathbf{q}^0$.

Case 2 $s \geq 1$, it is known that the left and right orthonormal bases of singular values contain the m inextensional mechanisms and the vectors **q** of force densities which indeed satisfy the linear homogeneous equations of equilibrium [16]. In this context the matrices **U** and **W** can be grouped as follows:

$$\mathbf{U} = [\mathbf{u}_1\ \mathbf{u}_2 \ldots \mathbf{u}_{\mathrm{rA}} \,|\, \mathbf{m}_1\ \mathbf{m}_2 \ldots \mathbf{m}_{\mathrm{m}}]. \tag{18.1}$$

$$\mathbf{W} = [\mathbf{w}_1\ \mathbf{w}_2 \ldots \mathbf{w}_{\mathrm{rA}} \,|\, \mathbf{q}_1\ \mathbf{q}_2 \ldots \mathbf{q}_{\mathrm{s}}]. \tag{18.2}$$

For evaluating the accuracy of results the Euclidean norm of the unbalanced internal forces in each direction $£_x = \mathbf{Dx}$ and $£_y = \mathbf{Dy}$ is used

$$£_T = \sqrt{(£_x)^T \cdot £_x + (£_y)^T \cdot £_y} \,. \tag{19}$$

5. Numerical Example-A Planar Hexagonal Tensegrity

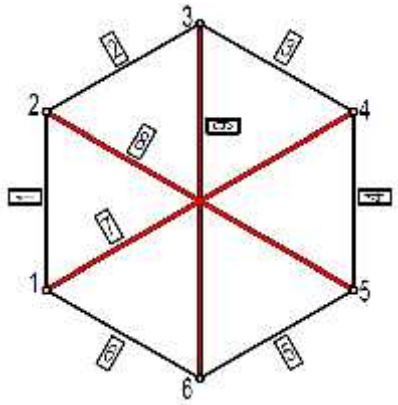

Fig. 1. *Planar hexagonal tensegrity*

The initial geometry of the planar hexagonal tensegrity comprising of three struts and six cables (Fig. 1) is studied herein for verification purposes. The known information at the beginning of the form-finding procedure is the incidence matrix and the prototype of force densities

$$\mathbf{q^0} = \left(q_1 - q_6 = +1, q_7 - q_9 = -1\right)^T . \tag{20}$$

The results of the presented form-finding procedure concur with those found by Tibert and Pellegrino [20], respectively by Estrada et al [8]. The form-finding procedure converges in ten iterations with a total design error of $£_T = 6.3906 \cdot 10^{-16}$ (Fig. 2). The solution of the initial equilibrium problem normalised with respect to cable 1 is a force density vector with $q = +1$ for cables and $q = -0.5$ for struts.

6. Conclusions

The presented form-finding method solves the initial equilibrium problem with minimal information regarding the properties of the analysed structures (the incidence matrix and element types).

Trivial values (+1, -1) may be used for the prototype of force densities to find a feasible range of nodal coordinates and force densities which fulfil the rank deficiency conditions forced upon the force density, respectively the equilibrium matrix.

The presented algorithm is very well suited for the design of both free-form and regular tensegrities.

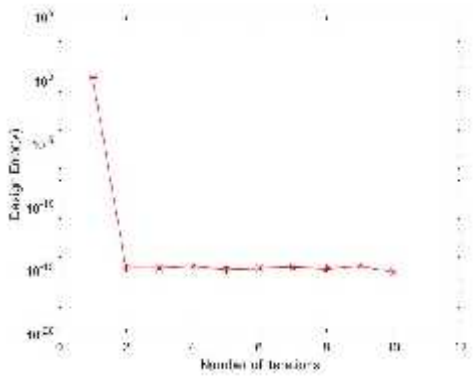

Fig. 2. *Convergence of the iterative algorithm for the hexagonal tensegrity*

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