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BIANCHI IDENTITIES IN THE THEORY OF THE HOMOGENEOUS LIFT TO THE 2-OSCULATOR BUNDLE OF A FINSLER METRIC

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Abstract

In this article we present a study of the subspaces of the manifold Osc^2M, the total space of the 2-osculator bundle of a real manifold M. We obtain the induced connections of the canonical N-linear metric connection determined by the homogeneous prolongation of a Finsler metric to the manifold Osc^2M. We present the Bianchi identities of the associated 2-osculator submanifold.

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1 Introduction

The Sasaki N-prolongation $\mathbb{G}$ to the 2-osculator bundle without the null section $\widetilde{Osc^2M} = Osc^2M \setminus \{0\}$ of a Finslerian metric g_{ab} on the real manifold M given by

$$\mathbb{G} = g_{ab}\left(x, y^{(1)}\right) dx^a \otimes dx^b + g_{ab}\left(x, y^{(1)}\right) \delta y^{(1)a} \otimes \delta y^{(1)b} + g_{ab}\left(x, y^{(1)}\right) \delta y^{(2)a} \otimes \delta y^{(2)b} \quad (*)$$

is a Riemannian structure on $\widetilde{Osc^2M}$, which depends only on the metric g_{ab}.

The tensor $\mathbb{G}$ is not invariant with respect to the homothetis on the fibres of $\widetilde{Osc^2M}$, because $\mathbb{G}$ is not homogeneous with respect to the variable $y^{(1)a}$.

In this paper, we use a new kind of prolongation $\mathring{\mathbb{G}}$ to $\widetilde{Osc^2M}$, ([8]), which depends only on the metric g_{ab}. Thus, $\mathring{\mathbb{G}}$ determines on the manifold $\widetilde{Osc^2M}$ a Riemannian structure which is 0-homogeneous on the fibres of Osc^2M.

Some geometrical properties of $\mathring{\mathbb{G}}$ are studied: the canonical N-linear metric connection, the induced linear connections, Bianchi identities.

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2 Preliminaries

As far we know the general theory of submanifolds (in particular the Finsler submanifolds or the complex Finsler submanifolds) is far from being settled ([10], [4], [11], [12]). In [9] and [10] R. Miron and M. Anastasiei give the theory of subspaces in generalized Lagrange spaces. Also, in [7] and [6] R. Miron presents the theory of subspaces in higher order Finsler and Lagrange spaces respectively.

Let M be a real differentiable manifold of dimension n, which has the local coordinates $(x) := (x^a)_{a=\overline{1,n}}$. The corresponding 2-osculator bundle Osc^2M (or 2-tangent bundle, [9],[2]) has the dimension equal to $3n$, and its local coordinates are[2]

$$\begin{aligned}\left(x, y^{(1)}, y^{(2)}\right) &:= \left(x^a, y^{(1)a}, y^{(2)a}\right)_{a=\overline{1,n}} \\ &= \left(\underbrace{x^1, ..., x^n}_{\text{space coordinates}}, \underbrace{y^{(1)1}, ..., y^{(1)n}}_{\text{tangent vector}}, \underbrace{y^{(2)1}, ..., y^{(2)n}}_{2-\text{tangent vector}}\right)\end{aligned}$$

If $\check{M}$ is an m-dimensional immersed manifold in manifold M, a nonlinear connection on Osc^2M induces a nonlinear connection $\check{N}$ on $Osc^2\check{M}$.

The d-tensor $\mathbb{G}$ from (*) is not homogeneous with respect to the variable $y^{(1)a}$. This in an incovenient from the point of view of analytical mechanics. Moreover, the physical dimensions of the terms of $\mathbb{G}$ are not the same. This disadvantage was corected by Gh. Atanasiu. He took a new kind of prolongation $\mathring{\mathbb{G}}$ to $\widetilde{Osc^2M}$ of the fundamental tensor of a Finsler space, [1], which depends only on the metric g_{ab}. Thus, $\mathring{\mathbb{G}}$ determines on the manifold $\widetilde{Osc^2M}$ a Riemannian structure which is 0-homogeneous on the fibres of Osc^2M and p is a positive constant required by applications in order that the physical dimensions of the terms of $\mathring{\mathbb{G}}$ be the same. He proved that there exist metrical N-linear connections with respect to the metric tensor $\mathring{\mathbb{G}}$.

We take this canonical N-linear metric connection D on the manifold Osc^2M and obtain the induced tangent and normal connections and the relative covariant derivatives in the algebra of d-tensor fields. It follows that we can get the Bianchi identities associated with the induced tangent connection with the coefficients $D^{\top}\Gamma\left(\check{N}\right) = \left(\underset{(i0)}{\overset{V_i}{L}}{}^{\alpha}_{\beta\delta}, \underset{(i1)}{\overset{V_i}{C}}{}^{\alpha}_{\beta\delta}, \underset{(i2)}{\overset{V_i}{C}}{}^{\alpha}_{\beta\delta}\right)$, $(i = 0, 1, 2; V_0 = H)$.

Let us consider the Finsler space $F^n = (M, F)$ ([10]) with the fundamental function $F : TM = OscM \to \mathbb{R}$ and the fundamental tensor $g_{ab}\left(x, y^{(1)}\right)$ on $\widetilde{OscM}$, given by

$$g_{ab}\left(x, y^{(1)}\right) = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^{(1)a}\partial y^{(1)b}}, \tag{1}$$

where $g_{ab}\left(x, y^{(1)}\right)$ is positively defined on $\widetilde{OscM}$.

[2]In this paper the Latin letters a, b, c, ... run from 1 to n and the Greek letters α, β, γ,... run from 1 to m The Einstein convention of summation is adopted all over this work.

The canonical 2-spray of F^n is given by

$$\frac{d^2x^a}{dt^2}+2G^a\left(x,\frac{dx}{dt}\right)=0\,,\;G^a=\frac{1}{2}\gamma^a_{bc}\left(x,y^{(1)}\right)y^{(1)b}y^{(1)c} \tag{2}$$

where $\gamma^a_{bc}\left(x,y^{(1)}\right)$ are the Christoffels symbols of the metric tensor $g_{ab}\left(x,y^{(1)}\right)$. The canonical nonlinear connection N of the space F^n has the dual coefficients [6]

$$\underset{(1)}{M}{}^a{}_b=\frac{\partial G^a}{\partial y^{(1)b}},\;\underset{(2)}{M}{}^a{}_b=\frac{1}{2}\left\{\Gamma\underset{(1)}{M}{}^a{}_b+\underset{(1)}{M}{}^a{}_c\underset{(1)}{M}{}^c{}_b\right\}, \tag{3}$$

where $\Gamma=y^{(1)a}\dfrac{\partial}{\partial x^a}+2y^{(2)a}\dfrac{\partial}{\partial y^{(1)a}}$.

We have the next decomposition

$$T_wOsc^2M=N_0\left(w\right)\oplus N_1\left(w\right)\oplus V_2\left(w\right),\forall w\in Osc^2M. \tag{4}$$

The adapted basis to (4) is given by $\left\{\dfrac{\delta}{\delta x^a},\dfrac{\delta}{\delta y^{(1)a}},\dfrac{\partial}{\partial y^{(2)a}}\right\}$, $(a=1,..,n)$ and its dual basis is $\left(dx^a,\delta y^{(1)a},\delta y^{(2)a}\right)$, where

$$\begin{cases}\dfrac{\delta}{\delta x^a}=\dfrac{\partial}{\partial x^a}-\underset{(1)}{N}{}^b{}_a\dfrac{\delta}{\delta y^{(1)b}}-\underset{(2)}{N}{}^b{}_a\dfrac{\partial}{\partial y^{(2)b}}\\[2ex] \dfrac{\delta}{\delta y^{(1)a}}=\dfrac{\partial}{\partial y^{(1)a}}-\underset{(1)}{N}{}^b{}_a\dfrac{\partial}{\partial y^{(2)b}}\end{cases} \tag{5}$$

and

$$\begin{cases}\delta y^{(1)a}=dy^{(1)a}+\underset{(1)}{M}{}^a{}_bdx^b\\[2ex] \delta y^{(2)a}=dy^{(2)a}+\underset{(1)}{M}{}^a{}_b\delta y^b+\underset{(2)}{M}{}^a{}_b\delta y^{(2)b},\end{cases} \tag{6}$$

where

$$\underset{(1)}{M}{}^a{}_b=\underset{(1)}{N}{}^a{}_b,\;\underset{(2)}{M}{}^a{}_b=\underset{(2)}{N}{}^a{}_b+\underset{(1)}{N}{}^a{}_c\underset{(1)}{N}{}^c{}_b.$$

We use the next notations:

$$\delta_a=\frac{\delta}{\delta x^a},\delta_{1a}=\frac{\delta}{\delta y^{(1)a}}\;\;\dot{\partial}_{2a}=\frac{\partial}{\partial y^{(2)a}}.$$

The fundamental tensor g_{ab} determines on the manifold $\widetilde{Osc^2M}$ the homogeneous tensor field $\overset{0}{\mathbb{G}}$, [1],

$$\begin{aligned}\overset{0}{\mathbb{G}}\;=\;&g_{ab}\left(x,y^{(1)}\right)dx^a\otimes dx^b+\underset{(1)}{g}{}_{ab}\left(x,y^{(1)}\right)\delta y^{(1)a}\otimes\delta y^{(1)b}\\ &+\underset{(2)}{g}{}_{ab}\left(x,y^{(1)}\right)\delta y^{(2)a}\otimes\delta y^{(2)b},\end{aligned} \tag{7}$$

where

$$\underset{(1)}{g}_{ab}\left(x, y^{(1)}\right) = \frac{p^2}{\left\|y^{(1)}\right\|^2} g_{ab}\left(x, y^{(1)}\right),$$

$$\underset{(2)}{g}_{ab}\left(x, y^{(1)}\right) = \frac{p^4}{\left\|y^{(1)}\right\|^4} g_{ab}\left(x, y^{(1)}\right),$$

$$\left\|y^{(1)}\right\|^2 = g_{ab} y^{(1)a} y^{(1)b}.$$

This is a homogeneous tensor field with respect to $y^{(1)a}$, $y^{(2)a}$ and p is a positive constant required by applications in order that the physical dimensions of the terms of $\mathring{\mathbb{G}}$ be the same.

Let $\check{M}$ be a real, m-dimensional manifold, immersed in M through the immersion $i : \check{M} \to M$. Localy, i can be given under the form

$$x^a = x^a\left(u^1, ..., u^m\right), \qquad rank \left\| \frac{\partial x^a}{\partial u^\alpha} \right\| = m.$$

We assume $1 \leq m < n$. We take the immersed submanifold $Osc^2\check{M}$ of the manifold $Osc^2 M$, by the immersion $Osc^2 i : Osc^2\check{M} \to Osc^2 M$. The parametric equations of the submanifold $Osc^2\check{M}$ are

$$\begin{cases} x^a = x^a\left(u^1, ..., u^m\right), rank \left\| \dfrac{\partial x^a}{\partial u^\alpha} \right\| = m \\ y^{(1)a} = \dfrac{\partial x^a}{\partial u^\alpha} v^{(1)\alpha} \\ 2y^{(2)a} = \dfrac{\partial y^{(1)a}}{\partial u^\alpha} v^{(1)\alpha} + 2\dfrac{\partial y^{(1)a}}{\partial v^{(1)\alpha}} v^{(2)\alpha}, \end{cases} \tag{8}$$

where

$$\begin{cases} \dfrac{\partial x^a}{\partial u^\alpha} = \dfrac{\partial y^{(1)a}}{\partial v^{(1)\alpha}} = \dfrac{\partial y^{(2)a}}{\partial v^{(2)\alpha}} \\ \dfrac{\partial y^{(1)a}}{\partial u^\alpha} = \dfrac{\partial y^{(2)a}}{\partial v^{(1)\alpha}}. \end{cases}$$

The restriction of the fundamental function F to the submanifold $\widetilde{Osc\check{M}}$ is

$$\check{F}\left(u, v^{(1)}\right) = F\left(x(u), y^{(1)a}\left(u, v^{(1)}\right)\right)$$

and we call $\check{F}^m = \left(\check{M}, \check{F}\right)$ the **induced Finsler subspaces** of F^n and $\check{F}$ the **induced fundamental function**.

Let $B^a_\alpha(u) = \dfrac{\partial x^a}{\partial u^\alpha}$ and $g_{\alpha\beta}$ the induced fundamental tensor,

$$g_{\alpha\beta}\left(u, v^{(1)}\right) = g_{ab}\left(x(u), y\left(u, v^{(1)}\right)\right) B^a_\alpha B^b_\beta. \tag{9}$$

We obtain a system of d-vectors $\{B^a_\alpha, B^a_{\bar{\alpha}}\}$ wich determines a moving frame

$$\mathcal{R} = \left\{ \left(u, v^{(1)}, v^{(2)}\right) ; B^a_\alpha \left(u\right), B^a_{\bar{\alpha}} \left(u, v^{(1)}, v^{(2)}\right) \right\}$$

in $Osc^2 M$ along with the submanifold $Osc^2 \check{M}$.

Its dual frame will be denoted by $\mathcal{R}^* = \left\{ B^\alpha_a \left(u, v^{(1)}, v^{(2)}\right), B^{\bar{\alpha}}_a \left(u, v^{(1)}, v^{(2)}\right) \right\}$. This is also defined on an open set $\check{\pi}^{-1}\left(\check{U}\right) \subset Osc^2 \check{M}$, $\check{U}$ being a domain of a local chart on the submanifold $\check{M}$.

The conditions of duality are given by:

$$B^a_\beta B^\alpha_a = \delta^\alpha_\beta \,, \; B^a_\beta B^{\bar{\alpha}}_a = 0 \,, \; B^\alpha_a B^a_{\bar{\beta}} = 0 \,, \; B^{\bar{\alpha}}_a B^a_{\bar{\beta}} = \delta^{\bar{\alpha}}_{\bar{\beta}} \,, \; B^a_\alpha B^\alpha_b + B^a_{\bar{\alpha}} B^{\bar{\alpha}}_b = \delta^a_b.$$

The restriction of the nonlinear connection $\underbrace{\text{N (3)}}$ to $\widetilde{Osc^2 \check{M}}$ uniquely determines an induced nonlinear connection $\check{N}$ on $\underbrace{Osc^2 \check{M}}$ with the dual coeficients ([3], [13], [14])

$$\begin{aligned} \underset{1}{\check{M}}{}^\alpha{}_\beta &= B^\alpha_a \left(B^a_{0\beta} + \underset{1}{M}{}^a{}_b B^b_\beta \right), \\ \underset{2}{\check{M}}{}^\alpha{}_\beta &= B^\alpha_a \left(\frac{1}{2} \frac{\partial B^a_{\delta\gamma}}{\partial u^\beta} v^{(1)\delta} v^{(1)\gamma} + B^a_{\delta\beta} v^{(2)\delta} + \underset{1}{M}{}^a{}_b B^b_{0\beta} + \underset{2}{M}{}^a{}_b B^b_\beta \right), \end{aligned} \tag{10}$$

where $\underset{1}{M}{}^a{}_b$, $\underset{2}{M}{}^a{}_b$ are the dual coeficients of the nonlinear connection N.

The adapted bases of the induced nonlinear connection Ň are defined by

$$\begin{cases} \dfrac{\delta}{\delta x^\alpha} = \dfrac{\partial}{\partial x^\alpha} - \underset{(1)}{N}{}^\beta{}_\alpha \dfrac{\delta}{\delta y^{(1)b}} - \underset{(2)}{N}{}^\beta{}_\alpha \dfrac{\partial}{\partial y^{(2)b}} \\ \dfrac{\delta}{\delta y^{(1)\alpha}} = \dfrac{\partial}{\partial y^{(1)\alpha}} - \underset{(1)}{N}{}^\beta{}_\alpha \dfrac{\partial}{\partial y^{(2)b}} \end{cases} \tag{11}$$

and

$$\begin{cases} \delta y^{(1)\alpha} = dy^{(1)\alpha} + \underset{(1)}{M}{}^\alpha{}_\beta dx^\beta \\ \delta y^{(2)\alpha} = dy^{(2)\alpha} + \underset{(1)}{M}{}^\alpha{}_\beta \delta y^\beta + \underset{(2)}{M}{}^\alpha{}_\beta \delta y^{(2)\beta} \end{cases} \tag{12}$$

We use the next notations:

$$\delta_\alpha = \frac{\delta}{\delta x^\alpha}, \delta_{1\alpha} = \frac{\delta}{\delta y^{(1)\alpha}} \;\; \dot{\partial}_{2\alpha} = \frac{\partial}{\partial y^{(2)\alpha}}.$$

Proposition 1. *The Lie brakets of the vector fields* $\left\{\delta_\alpha, \delta_{1\alpha}, \dot{\partial}_{2\alpha}\right\}$ *are given by*

$$[\delta_\beta, \delta_\gamma] = \underset{(01)}{R}{}^{\;\alpha}_{\beta\gamma} \delta_{1\alpha} + \underset{(01)}{R}{}^{\;\alpha}_{\beta\gamma} \dot{\partial}_{2\alpha} \,, \; [\delta_\beta, \delta_{1\gamma}] = \underset{(11)}{B}{}^{\;\alpha}_{\beta\gamma} \delta_{1\alpha} + \underset{(12)}{B}{}^{\;\alpha}_{\beta\gamma} \dot{\partial}_{2\alpha}, \tag{13}$$

$$\left[\delta_{\beta},\dot{\partial}_{2\gamma}\right]=\underset{(21)}{B}{}^{\alpha}_{\beta\gamma}\delta_{1\alpha}+\underset{(22)}{B}{}^{\alpha}_{\beta\gamma}\dot{\partial}_{2\alpha}\,,\ [\delta_{1\beta},\delta_{1\gamma}]=\underset{(12)}{R}{}^{\alpha}_{\beta\gamma}\dot{\partial}_{2\alpha}\,,\ \left[\delta_{1\beta},\dot{\partial}_{2\gamma}\right]=\underset{(21)}{B}{}^{\alpha}_{\beta\gamma}\dot{\partial}_{2\alpha},$$

where

$$\underset{(01)}{R}{}^{\alpha}_{\beta\gamma}=\delta_{\gamma}\underset{1}{N}{}^{\alpha}{}_{\beta}-\delta_{\beta}\underset{1}{N}{}^{\alpha}{}_{\gamma}\,,\ \underset{(02)}{R}{}^{\alpha}_{\beta\gamma}=\delta_{\gamma}\underset{2}{N}{}^{\alpha}{}_{\beta}-\delta_{\beta}\underset{2}{N}{}^{\alpha}{}_{\gamma}+\underset{1}{N}{}^{\alpha}{}_{\sigma}\underset{(01)}{R}{}^{\sigma}_{\beta\gamma},$$

$$\underset{(11)}{B}{}^{\alpha}_{\beta\gamma}=\delta_{1\gamma}\underset{1}{N}{}^{\alpha}{}_{\beta},\ \underset{(12)}{B}{}^{\alpha}_{\beta\gamma}=\delta_{1\gamma}\underset{2}{N}{}^{\alpha}{}_{\beta}-\delta_{\beta}\underset{1}{N}{}^{\alpha}{}_{\gamma}+\underset{1}{N}{}^{\alpha}{}_{\sigma}\underset{(11)}{B}{}^{\sigma}_{\beta\gamma},$$

$$\underset{(21)}{B}{}^{\alpha}_{\beta\gamma}=\dot{\partial}_{2\gamma}\underset{1}{N}{}^{\alpha}{}_{\beta}\,,\ \underset{(22)}{B}{}^{\alpha}_{\beta\gamma}=\dot{\partial}_{2\gamma}\underset{2}{N}{}^{\alpha}{}_{\beta}+\underset{1}{N}{}^{\alpha}{}_{f}\underset{(21)}{B}{}^{f}_{\beta\gamma}\,,\ \underset{(12)}{R}{}^{\alpha}_{\beta\gamma}=\delta_{1\gamma}\underset{1}{N}{}^{\alpha}{}_{\beta}-\delta_{1\beta}\underset{1}{N}{}^{\alpha}{}_{\gamma}. \tag{14}$$

The cobasis $\left(dx^{i},\delta y^{(1)a},\delta y^{(2)a}\right)$ restricted to $Osc^{2}\check{M}$ is uniquely represented in the moving frame $\mathcal{R}$ in the following form ([3], [13]):

$$\begin{cases} dx^{a}=B^{a}_{\beta}du^{\beta} \\ \delta y^{(1)a}=B^{a}_{\alpha}\delta v^{(1)\alpha}+B^{a}_{\bar{\alpha}}\underset{(1)}{K}{}^{\bar{\alpha}}_{\beta}du^{\beta} \\ \delta y^{(2)a}=B^{a}_{\alpha}\delta v^{(2)\alpha}+B^{a}_{\bar{\beta}}\underset{(1)}{K}{}^{\bar{\beta}}_{\alpha}\delta v^{(1)\alpha}+B^{a}_{\bar{\beta}}\underset{(2)}{K}{}^{\bar{\beta}}_{\alpha}du^{\alpha} \end{cases} \tag{15}$$

where

$$\underset{(1)}{K}{}^{\bar{\alpha}}_{\beta}=B^{\bar{\alpha}}_{a}\left(B^{a}_{0\beta}+\underset{(1)}{M}{}^{a}_{b}B^{b}_{\beta}\right)$$

$$\underset{(2)}{K}{}^{\bar{\alpha}}_{\beta}=B^{\bar{\alpha}}_{a}\left(\frac{1}{2}\frac{\partial B^{a}_{\delta\gamma}}{\partial u^{\beta}}v^{(1)\delta}v^{(1)\gamma}+B^{b}_{\delta\beta}v^{(2)\delta}+\underset{(1)}{M}{}^{a}_{b}B^{b}_{0\beta}+\underset{(2)}{M}{}^{a}_{b}B^{b}_{\beta}-\right. \tag{16}$$

$$\left.-B^{\bar{\alpha}}_{f}B^{\gamma}_{d}\left(B^{f}_{\gamma}+\underset{(1)}{M}{}^{f}_{b}B^{b}_{\gamma}\right)\left(B^{d}_{0\beta}+\underset{(1)}{M}{}^{d}_{g}B^{g}_{\beta}\right)\right.$$

are mixed d-tensor fields.

A linear connection D on the manifold $Osc^{2}M$ is called **metrical N-linear connection** with respect to $\mathring{\mathbb{G}}$, if $D\mathring{\mathbb{G}}=0$ and D preserves by parallelism the distributions N_0, N_1 and V_2. The coefficients of the N-linear connections $D\Gamma\left(N\right)$ will be denoted with $\left(\underset{(i0)}{\overset{V_i}{L}}{}^{a}_{bc},\underset{(i1)}{\overset{V_i}{C}}{}^{a}_{bc},\underset{(i2)}{\overset{V_i}{C}}{}^{a}_{bc}\right),(i=0,1,2)$.

Theorem 1. *([1]) There exist N-linear metric connections $D\Gamma\left(N\right)$ on $\widetilde{Osc^{2}M}$, with respect to the homogeneous prolongation $\mathring{\mathbb{G}}$, wich depend only on the metric $g_{ab}\left(x,y^{(1)}\right)$. One of these connections has* *the "horizontal" coefficients*

$$\overset{H}{\underset{(00)}{L}}{}^{a}_{bc} = \frac{1}{2} g^{ad} \left(\delta_b g_{cd} + \delta_c g_{bd} - \delta_d g_{bc} \right)$$

$$\overset{V_1}{\underset{(10)}{L}}{}^{a}_{bc} = \frac{1}{2} \underset{(1)}{g}{}^{ad} \left(\delta_b \underset{(1)}{g}{}_{cd} + \delta_c \underset{(1)}{g}{}_{bd} - \delta_d \underset{(1)}{g}{}_{bc} \right) \tag{17}$$

$$\overset{V_2}{\underset{(20)}{L}}{}^{a}_{bc} = \frac{1}{2} \underset{(2)}{g}{}^{ad} \left(\delta_b \underset{(2)}{g}{}_{cd} + \delta_c \underset{(2)}{g}{}_{bd} - \delta_d \underset{(2)}{g}{}_{bc} \right)$$

the "1-vertical" coefficients

$$\overset{H}{\underset{(01)}{C}}{}^{a}_{bc} = \frac{1}{2} g^{ad} \left(\delta_{1b} g_{dc} + \delta_{1c} g_{bd} - \delta_{1d} g_{bc} \right)$$

$$\overset{V_1}{\underset{(11)}{C}}{}^{a}_{bc} = \frac{1}{2} \underset{(1)}{g}{}^{ad} \left(\delta_{1b} \underset{(1)}{g}{}_{cd} + \delta_{1c} \underset{(1)}{g}{}_{bd} - \delta_{1d} \underset{(1)}{g}{}_{bc} \right) \tag{18}$$

$$\overset{V_2}{\underset{(21)}{C}}{}^{a}_{bc} = \frac{1}{2} \underset{(2)}{g}{}^{ad} \left(\delta_{1b} \underset{(2)}{g}{}_{cd} + \delta_{1c} \underset{(2)}{g}{}_{bd} - \delta_{1d} \underset{(2)}{g}{}_{bc} \right)$$

and the "2-vertical" coefficients

$$\overset{H}{\underset{(02)}{C}}{}^{a}_{bc} = \overset{V_1}{\underset{(12)}{C}}{}^{a}_{bc} = \overset{V_2}{\underset{(22)}{C}}{}^{a}_{bc} = 0. \tag{19}$$

It is called the ***canonical N-linear metric connection***.

This linear connection will be used throughout this paper.

For this N-linear connection, we have the operators $\overset{V_i}{D}, (i = 0, 1, 2; V_0 = H)$ which are given by the following relations

$$\overset{V_i}{D} X^a = dX^a + \overset{V_i}{\omega}{}^a_b X^b, \ \forall X \in \mathcal{F}\left(\widetilde{Osc^2} M\right), \tag{20}$$

where

$$\overset{H}{\omega}{}^a_b = \overset{H}{\underset{(00)}{L}}{}^{a}_{bc} dx^c + \overset{H}{\underset{(01)}{C}}{}^{a}_{bc} \delta y^{(1)c} + \overset{H}{\underset{(02)}{C}}{}^{a}_{bc} \delta y^{(2)c}$$

$$\overset{V_1}{\omega}{}^a_b = \overset{V_1}{\underset{(10)}{L}}{}^{a}_{bc} dx^c + \overset{V_1}{\underset{(11)}{C}}{}^{a}_{bc} \delta y^{(1)c} + \overset{V_1}{\underset{(12)}{C}}{}^{a}_{bc} \delta y^{(2)c} \tag{21}$$

$$\overset{V_2}{\omega}{}^a_b = \overset{V_2}{\underset{(20)}{L}}{}^{a}_{bc} dx^c + \overset{V_2}{\underset{(21)}{C}}{}^{a}_{bc} \delta y^{(1)c} + \overset{V_2}{\underset{(22)}{C}}{}^{a}_{bc} \delta y^{(2)c}.$$

We call these operators the **horizontal,** 1- and 2-**vertical covariant differentials**. The 1-forms $\overset{H}{\omega}{}^a_b, \overset{V_1}{\omega}{}^a_b, \overset{V_2}{\omega}{}^a_b$ will be called the **horizontal**, 1- and 2-**vertical 1-form.** From (19) we get that the horizontal, 1- and 2- vertical 1-form are

$$\overset{H}{\omega}{}^a_b = \underset{(00)}{\overset{H}{L}}{}^a_{bc} dx^c + \underset{(01)}{\overset{H}{C}}{}^a_{bc} \delta y^{(1)c} + \underset{(02)}{\overset{H}{C}}{}^a_{bc} \delta y^{(2)c}$$

$$\overset{V_1}{\omega}{}^a_b = \underset{(10)}{\overset{V_1}{L}}{}^a_{bc} dx^c + \underset{(11)}{\overset{V_1}{C}}{}^a_{bc} \delta y^{(1)c} + \underset{(12)}{\overset{V_1}{C}}{}^a_{bc} \delta y^{(2)c}$$

$$\overset{V_2}{\omega}{}^a_b = \underset{(20)}{\overset{V_2}{L}}{}^a_{bc} dx^c + \underset{(21)}{\overset{V_2}{C}}{}^a_{bc} \delta y^{(1)c} + \underset{(22)}{\overset{V_2}{C}}{}^a_{bc} \delta y^{(2)c}.$$

3 The relative covariant derivatives

Let $D\Gamma(N)$, the canonical N-linear metric connection of the manifold Osc^2M. A classical method to determine the laws of derivation on a Finsler submanifold is the type of the coupling ([6],[7],[9],[10]).

Definition 1. *We call a* ***coupling*** *of the canonical N-linear metric connection D to the induced nonlinear connection $\check{N}$ along $Osc^2\check{M}$ the operators $\overset{V_i}{\check{D}}$,$(i = 0, 1, 2\,; V_0 = H)$ defined by the operators $\overset{V_i}{D}$,$(i = 0, 1, 2; V_0 = H)$ (20) with the property*

$$\overset{V_i}{\check{D}} X^a = \overset{V_i}{D} X^a, (i = 0, 1, 2; V_0 = H) \text{ (modulo 15)} \tag{22}$$

Here

$$\overset{V_i}{\check{D}} X^a = dX^a + \overset{V_i}{\check{\omega}}{}^a_b X^b,\ \forall X \in \mathcal{F}\left(\widetilde{Osc^2}M\right). \tag{23}$$

The 1-forms $\underset{(i)}{\check{\omega}}{}^a_b$,$(i = 0, 1, 2)$ are the **connection 1-forms of the coupling** Ď.

Theorem 2. *The $\widetilde{\text{coupling}}$ of the N-linear connection D to the induced nonlinear connection $\check{N}$ along $\widetilde{Osc^2}\check{M}$ is locally given by the set of coefficients $\check{D}\Gamma\left(\check{N}\right) = \left(\underset{(i0)}{\overset{V_i}{\check{L}}}{}^a_{b\delta}, \underset{(i1)}{\overset{V_i}{\check{C}}}{}^a_{b\delta}, \underset{(i2)}{\overset{V_i}{\check{C}}}{}^a_{b\delta}\right)$,*
$(i = 0, 1, 2; V_0 = H)$ where

$$\underset{(i0)}{\overset{V_i}{\check{L}}}{}^a_{b\delta} = \underset{(i0)}{\overset{V_i}{L}}{}^a_{bd} B^d_\delta + \underset{(i1)}{\overset{V_i}{C}}{}^a_{bd} B^d_{\bar{\delta}} \underset{(1)}{K}{}^{\bar{\delta}}_{\delta},\ \underset{(i1)}{\overset{V_i}{\check{C}}}{}^a_{b\delta} = \underset{(i1)}{\overset{V_i}{C}}{}^a_{bd} B^d_\delta,\ \underset{(i2)}{\overset{V_i}{\check{C}}}{}^a_{b\delta} = 0, \tag{24}$$

$(i = 0, 1, 2; V_0 = H)$

Proof. From (22), (23), (20), and (15) we obtain

$$\overset{V_i}{\underset{(i0)}{\check{L}}}{}^{a}_{b\delta} = \overset{V_i}{\underset{(i0)}{L}}{}^{a}_{bd} B^d_\delta + \overset{V_i}{\underset{(i1)}{C}}{}^{a}_{bd} B^d_{\bar{\delta}} \underset{(1)}{K}{}^{\bar{\delta}}_{\delta} + \overset{V_i}{\underset{(i2)}{C}}{}^{a}_{b\delta} B^d_{\bar{\delta}} \underset{(2)}{K}{}^{\bar{\delta}}_{\delta}$$

$$\overset{V_i}{\underset{(i1)}{\check{C}}}{}^{a}_{b\delta} = \overset{V_i}{\underset{(i1)}{C}}{}^{a}_{bd} B^d_\delta + \overset{V_i}{\underset{(i2)}{C}}{}^{a}_{bd} B^d_{\bar{\delta}} \underset{(1)}{K}{}^{\bar{\delta}}_{\delta}, \ \overset{V_i}{\underset{(i2)}{\check{C}}}{}^{a}_{b\delta} = \overset{V_i}{\underset{(i2)}{C}}{}^{a}_{bd} B^d_\delta, (i = 0, 1, 2; V_0 = H).$$

and from (19) we get (24). □

Definition 2. *We call the* ***induced tangent connection*** *on* $\widetilde{Osc^2 \check{M}}$ *by the canonical N-linear metric connection D, the couple of the operators* $\overset{V_i}{D}{}^{\top}$ *, (*$i = 0, 1, 2; V_0 = H$*) which are defined by*

$$\overset{V_i}{D}{}^{\top} X^\alpha = B^\alpha_b \overset{V_i}{\check{D}} X^b, \quad \textit{for } X^a = B^a_\gamma X^\gamma \tag{25}$$

where

$$\overset{V_i}{D}{}^{\top} X^\alpha = dX^\alpha + X^\beta \overset{V_i}{\omega}{}^\alpha_\beta \tag{26}$$

and $\overset{V_i}{\omega}{}^\alpha_\beta$ *, (*$i = 0, 1, 2; V_0 = H$*) are called the* ***tangent connection 1-forms****.*

We have

Theorem 3. *The tangent connections 1-forms are as follows:*

$$\overset{V_i}{\omega}{}^\alpha_\beta = \overset{V_i}{\underset{(i0)}{L}}{}^{\alpha}_{\beta\delta} du^\delta + \overset{V_i}{\underset{(i1)}{C}}{}^{\alpha}_{\beta\delta} \delta v^{(1)\delta} + \overset{V_i}{\underset{(i2)}{C}}{}^{\alpha}_{\beta\delta} \delta v^{(2)\delta}, \tag{27}$$

where

$$\overset{V_i}{\underset{(i0)}{L}}{}^{\alpha}_{\beta\delta} = B^\alpha_d \left(B^d_{\beta\delta} + B^f_\beta \overset{V_i}{\underset{(i0)}{\check{L}}}{}^{d}_{f\delta} \right), \ \overset{V_i}{\underset{(i1)}{C}}{}^{\alpha}_{\beta\delta} = B^\alpha_d B^f_\beta \overset{V_i}{\underset{(i1)}{\check{C}}}{}^{d}_{f\delta}, \ \overset{V_i}{\underset{(i2)}{C}}{}^{\alpha}_{\beta\delta} = 0, \tag{28}$$

$(i = 0, 1, 2; V_0 = H)$.

Proof. From (23),(26) and (25) we have

$$\begin{aligned} \overset{V_i}{\underset{(i0)}{L}}{}^{\alpha}_{\beta\delta} &= B^\alpha_d \left(B^d_{\beta\delta} + B^f_\beta \overset{V_i}{\underset{(i0)}{\check{L}}}{}^{d}_{f\delta} \right), \ \overset{V_i}{\underset{(i1)}{C}}{}^{\alpha}_{\beta\delta} = B^\alpha_d B^f_\beta \overset{V_i}{\underset{(i1)}{\check{C}}}{}^{d}_{f\delta}, \ \overset{V_i}{\underset{(i2)}{C}}{}^{\alpha}_{\beta\delta} \\ &= B^\alpha_d B^f_\beta \overset{V_i}{\underset{(i2)}{\check{C}}}{}^{d}_{f\delta}, (i = 0, 1, 2; V_0 = H). \end{aligned}$$

and from (19) we get (28). □

The relation (26) is equivalent with

$$\overset{V_i}{D}{}^{\top} X^{\alpha} = X^{\alpha}{}_{|i\varepsilon} dx^{\varepsilon} + X^{\alpha} \overset{(1)}{|}_{i\varepsilon} \delta y^{(1)\varepsilon} + X^{\alpha} \overset{(2)}{|}_{i\varepsilon} \delta y^{(2)\varepsilon}$$

where

$$X^{\alpha}{}_{|i\varepsilon} = \delta_{\varepsilon} X^{\alpha} + X^{\beta} \underset{(i0)}{\overset{V_i}{L}}{}^{\alpha}_{\beta\varepsilon}, \; X^{\alpha} \overset{(1)}{|}_{i\delta} = \delta_{1\varepsilon} X^{\alpha} + X^{\beta} \underset{(i1)}{\overset{V_i}{C}}{}^{\alpha}_{\beta\varepsilon}, \; X^{\alpha} \overset{(2)}{|}_{i\delta} = \dot{\partial}_{2\varepsilon} X^{\alpha} + X^{\beta} \underset{(i2)}{\overset{V_i}{C}}{}^{\alpha}_{\beta\varepsilon}. \tag{29}$$

The operators "$|_{i\varepsilon}$" ," $\overset{(1)}{|}_{i\varepsilon}$" and " $\overset{(2)}{|}_{i\varepsilon}$" are called the h_i-,v_{1i}- and v_{2i}-**covariante derivatives** with respect to the induced tangent connection $D^{\top}\Gamma\left(\check{N}\right)$.

Definition 3. *We call the* ***induced normal connection*** *on* $\widetilde{Osc^2 \check{M}}$ *by the canonical N-linear metric connection D, the couple of the operators* $\overset{V_i}{D}{}^{\perp}$ *, (*$i = 0, 1, 2; V_0 = H$*) which are defined by*

$$\overset{V_i}{D}{}^{\perp} X^{\overline{\alpha}} = B^{\alpha}_{b} \overset{V_i}{\check{D}} X^{b} \quad for\ X^{a} = B^{a}_{\bar{\gamma}} X^{\bar{\gamma}} \tag{30}$$

where

$$\overset{V_i}{D}{}^{\perp} X^{\overline{\alpha}} = dX^{\overline{\alpha}} + X^{\overline{\beta}} \overset{V_i}{\omega}{}^{\overline{\alpha}}_{\overline{\beta}} \tag{31}$$

and $\overset{V_i}{\omega}{}^{\overline{\alpha}}_{\overline{\beta}}$ *, (*$i = 0, 1, 2; V_0 = H$*) are called the* ***normal connection 1-forms****.*

We have

Theorem 4. *The normal connections 1-forms are as follows:*

$$\overset{V_i}{\omega}{}^{\overline{\alpha}}_{\overline{\beta}} = \underset{(i0)}{\overset{V_i}{L}}{}^{\overline{\alpha}}_{\overline{\beta}\delta} du^{\delta} + \underset{(i1)}{\overset{V_i}{C}}{}^{\overline{\alpha}}_{\overline{\beta}\delta} \delta v^{(1)\delta} + \underset{(i2)}{\overset{V_i}{C}}{}^{\overline{\alpha}}_{\overline{\beta}\delta} \delta v^{(2)\delta} \tag{32}$$

where

$$\underset{(i0)}{\overset{V_i}{L}}{}^{\bar{\alpha}}_{\bar{\beta}\delta} = B^{\bar{\alpha}}_{d} \left(\frac{\delta B^{d}_{\bar{\beta}}}{\delta u^{\delta}} + B^{f}_{\bar{\beta}} \underset{(i0)}{\overset{V_i}{\check{L}}}{}^{d}_{f\delta} \right), \; \underset{(i1)}{\overset{V_i}{C}}{}^{\bar{\alpha}}_{\bar{\beta}\delta} = B^{\bar{\alpha}}_{d} \left(\frac{\partial B^{d}_{\bar{\beta}}}{\partial u^{\delta}} + B^{f}_{\bar{\beta}} \underset{(i1)}{\overset{V_i}{\check{C}}}{}^{d}_{f\delta} \right), \; \underset{(i2)}{\overset{V_i}{C}}{}^{\bar{\alpha}}_{\bar{\beta}\delta} = 0, \tag{33}$$

$(i = 0, 1, 2; V_0 = H)$.

Proof. From (23),(30),(31) and (15) we obtain

$$\begin{aligned} \underset{(i0)}{\overset{V_i}{L}}{}^{\bar{\alpha}}_{\bar{\beta}\delta} &= B^{\bar{\alpha}}_{d} \left(\frac{\delta B^{d}_{\bar{\beta}}}{\delta u^{\delta}} + B^{f}_{\bar{\beta}} \underset{(i0)}{\overset{V_i}{\check{L}}}{}^{d}_{f\delta} \right), \; \underset{(i1)}{\overset{V_i}{C}}{}^{\bar{\alpha}}_{\bar{\beta}\delta} = B^{\bar{\alpha}}_{d} \left(\frac{\delta B^{d}_{\bar{\beta}}}{\delta v^{(1)\delta}} + B^{f}_{\bar{\beta}} \underset{(i1)}{\overset{V_i}{\check{C}}}{}^{d}_{f\delta} \right), \; \underset{(i2)}{\overset{V_i}{C}}{}^{\bar{\alpha}}_{\bar{\beta}\delta} \\ &= B^{\bar{\alpha}}_{d} \left(\frac{\partial B^{d}_{\beta}}{\partial v^{(2)\delta}} + B^{f}_{\bar{\beta}} \underset{(i2)}{\overset{V_i}{\check{C}}}{}^{d}_{f\delta} \right) \end{aligned}$$

$(i = 0, 1, 2; V_0 = H)$ and from (24) and $\dfrac{\partial B^{d}_{\beta}}{\partial v^{(1)\delta}} = \dfrac{\partial B^{d}_{\beta}}{\partial v^{(2)\delta}} = 0$ we have (33). □

Now, we can define the relative (or mixed) covariant differentials $\overset{V_i}{\nabla}, (i = 0, 1, 2; V_0 = H)$.

Theorem 5. *The relative covariant (mixed) differentials in the algebra of mixed d-tensor fields are the operators* $\overset{V_i}{\nabla}, (i = 0, 1, 2; V_0 = H)$ *for which the following properties hold:*

$$\overset{V_i}{\nabla} f = df, \quad \forall f \in \mathcal{F}\left(Osc^2 \check{M}\right)$$

$$\overset{V_i}{\nabla} X^a = \overset{V_i}{\check{D}} X^a, \quad \overset{V_i}{\nabla} X^\alpha = \overset{V_i}{D}{}^\top X^\alpha, \quad \overset{V_i}{\nabla} X^{\bar{\alpha}} = \overset{V_i}{D}{}^\perp X^{\bar{\alpha}}, \ \ (i = 0, 1, 2; V_0 = H)$$

$\overset{V_i}{\check{\omega}}{}^a_b, \overset{V_i}{\omega}{}^\alpha_\beta, \overset{V_i}{\omega}{}^{\overline{\alpha}}_{\overline{\beta}}$ are called the **connection 1-forms** of $\overset{V_i}{\nabla}, (i = 0, 1, 2; V_0 = H)$.

The operators "$|_{i\varepsilon}$" ,"$\overset{(1)}{|}_{i\varepsilon}$" and "$\overset{(2)}{|}_{i\varepsilon}$" $(i = 0, 1, 2)$ from (29) can be extended to a mixed tensor field $T^{a\gamma\overline{\alpha}}_{b\delta\overline{\beta}}$ in a natural way. Thus, we have

- the h$_i$- *covariant derivatives*

$$\begin{aligned} T^{a\gamma\overline{\alpha}\dots}_{b\delta\overline{\beta}\dots}|_{i\varepsilon} \ &= \ \delta_\varepsilon T^{a\gamma\overline{\alpha}\dots}_{b\delta\overline{\beta}\dots} + \underset{(i0)}{\overset{V_i}{L}}{}^a_{f\varepsilon} T^{f\gamma\overline{\alpha}\dots}_{b\delta\overline{\beta}\dots} + \underset{(i0)}{\overset{V_i}{L}}{}^\gamma_{\varphi\varepsilon} T^{a\varphi\overline{\alpha}\dots}_{b\delta\overline{\beta}\dots} + \underset{(i0)}{\overset{V_i}{L}}{}^{\overline{\alpha}}_{\overline{\varphi}\varepsilon} T^{a\gamma\overline{\varphi}\dots}_{b\delta\overline{\beta}\dots} - \\ &- \underset{(i0)}{\overset{V_i}{L}}{}^f_{b\varepsilon} T^{a\gamma\overline{\alpha}\dots}_{f\delta\overline{\beta}\dots} - \underset{(i0)}{\overset{V_i}{L}}{}^\varphi_{\delta\varepsilon} T^{a\gamma\overline{\alpha}\dots}_{b\varphi\overline{\beta}\dots} - \underset{(i0)}{\overset{V_i}{L}}{}^{\overline{\varphi}}_{\overline{\beta}\varepsilon} T^{a\gamma\overline{\alpha}\dots}_{b\delta\overline{\varphi}\dots} \end{aligned} \tag{34}$$

- the v$_{2i}$- *covariant derivatives*

$$\begin{aligned} T^{a\gamma\overline{\alpha}\dots}_{b\delta\overline{\beta}\dots}\overset{(1)}{|}_{i\varepsilon} \ &= \ \delta_{1\varepsilon} T^{a\gamma\overline{\alpha}\dots}_{b\delta\overline{\beta}\dots} + \underset{(i1)}{\overset{V_i}{C}}{}^a_{f\varepsilon} T^{f\gamma\overline{\alpha}\dots}_{b\delta\overline{\beta}\dots} + \underset{(i1)}{\overset{V_i}{C}}{}^\gamma_{\varphi\varepsilon} T^{a\varphi\overline{\alpha}\dots}_{b\delta\overline{\beta}\dots} + \underset{(i1)}{\overset{V_i}{C}}{}^{\overline{\alpha}}_{\overline{\varphi}\varepsilon} T^{a\gamma\overline{\varphi}\dots}_{b\delta\overline{\beta}\dots} - \\ &- \underset{(i1)}{\overset{V_i}{C}}{}^f_{b\varepsilon} T^{a\gamma\overline{\alpha}\dots}_{f\delta\overline{\beta}\dots} - \underset{(i1)}{\overset{V_i}{C}}{}^\varphi_{\delta\varepsilon} T^{a\gamma\overline{\alpha}\dots}_{b\varphi\overline{\beta}\dots} - \underset{(i1)}{\overset{V_i}{C}}{}^{\overline{\varphi}}_{\overline{\beta}\varepsilon} T^{a\gamma\overline{\alpha}\dots}_{b\delta\overline{\varphi}\dots} \end{aligned} \tag{35}$$

- the v$_{2i}$- *covariant derivatives*

$$\begin{aligned} T^{a\gamma\overline{\alpha}\dots}_{b\delta\overline{\beta}\dots}\overset{(2)}{|}_{i\varepsilon} \ &= \ \dot{\partial}_{2\varepsilon} T^{a\gamma\overline{\alpha}\dots}_{b\delta\overline{\beta}\dots} + \underset{(i2)}{\overset{V_i}{C}}{}^a_{f\varepsilon} T^{f\gamma\overline{\alpha}\dots}_{b\delta\overline{\beta}\dots} + \underset{(i2)}{\overset{V_i}{C}}{}^\gamma_{\varphi\varepsilon} T^{a\varphi\overline{\alpha}\dots}_{b\delta\overline{\beta}\dots} + \underset{(i2)}{\overset{V_i}{C}}{}^{\overline{\alpha}}_{\overline{\varphi}\varepsilon} T^{a\gamma\overline{\varphi}\dots}_{b\delta\overline{\beta}\dots} - \\ &- \underset{(i2)}{\overset{V_i}{C}}{}^f_{b\varepsilon} T^{a\gamma\overline{\alpha}\dots}_{f\delta\overline{\beta}\dots} - \underset{(i2)}{\overset{V_i}{C}}{}^\varphi_{\delta\varepsilon} T^{a\gamma\overline{\alpha}\dots}_{b\varphi\overline{\beta}\dots} - \underset{(i2)}{\overset{V_i}{C}}{}^{\overline{\varphi}}_{\overline{\beta}\varepsilon} T^{a\gamma\overline{\alpha}\dots}_{b\delta\overline{\varphi}\dots}. \end{aligned} \tag{36}$$

4 Adapted components of torsion and curvature tensors

The study of the adapted components of the torsion and curvature tensors of an arbitrary N-linear connection $D\Gamma(N)$ on Osc^2M was done in [2]. In what follows, we study the adapted components of the torsion and curvature tensors for the induced tangent connection $D^\top\Gamma\left(\check{N}\right) = \left(\underset{(i0)}{\overset{V_i}{L}}{}^{\alpha}_{\beta\delta}, \underset{(i1)}{\overset{V_i}{C}}{}^{\alpha}_{\beta\delta}, \underset{(i2)}{\overset{V_i}{C}}{}^{\alpha}_{\beta\delta}\right)$, $(i = 0, 1, 2; V_0 = H)$, (28).

Theorem 6. *The torsion tensor $\mathbb{T}$ of the induced tangent connection $D^\top\Gamma\left(\check{N}\right)$ is characterized by the following local adapted d-tensors:*

$$\underset{(00)}{\overset{H}{T}}{}^{\alpha}_{\beta\gamma} = \underset{(00)}{\overset{H}{L}}{}^{\alpha}_{\beta\gamma} - \underset{(00)}{\overset{H}{L}}{}^{\alpha}_{\gamma\beta}\,,\ \underset{(01)}{\overset{V_1}{T}}{}^{\alpha}_{\beta\gamma} = \underset{(01)}{R}{}^{\alpha}_{\beta\gamma} = \delta_\gamma \underset{1}{N}{}^{\alpha}{}_{\beta} - \delta_\beta \underset{1}{N}{}^{\alpha}{}_{\gamma},$$

$$\underset{(02)}{\overset{V_2}{T}}{}^{\alpha}_{\beta\gamma} = \underset{(02)}{R}{}^{\alpha}_{\beta\gamma} = \delta_\gamma \underset{2}{N}{}^{\alpha}{}_{\beta} - \delta_\beta \underset{2}{N}{}^{\alpha}{}_{\gamma} + \underset{1}{N}{}^{\alpha}_{\varepsilon}\left(\delta_\gamma \underset{1}{N}{}^{\varepsilon}{}_{\beta} - \delta_\beta \underset{1}{N}{}^{\varepsilon}{}_{\gamma}\right),$$

$$\begin{aligned}
&\underset{(10)}{\overset{H}{P}}{}^{\alpha}_{\beta\gamma} = \underset{(01)}{\overset{H}{C}}{}^{\alpha}_{\beta\gamma}\\
&\underset{(11)}{\overset{V_1}{P}}{}^{\alpha}_{\beta\gamma} = \delta_{1\gamma} \underset{1}{N}{}^{\alpha}{}_{\beta} - \underset{(10)}{\overset{V_1}{L}}{}^{\alpha}_{\gamma\beta}\\
&\underset{(20)}{\overset{H}{P}}{}^{\alpha}_{\beta\gamma} = 0\\
&\underset{(21)}{\overset{V_1}{P}}{}^{\alpha}_{\beta\gamma} = \dot{\partial}_{2\gamma} \underset{1}{N}{}^{\alpha}{}_{\beta}\\
&\underset{(12)}{\overset{V_2}{P}}{}^{\alpha}_{\beta\gamma} = \delta_{1\gamma} \underset{2}{N}{}^{\alpha}{}_{\beta} - \delta_\beta \underset{1}{N}{}^{\alpha}{}_{\gamma} + \underset{1}{N}{}^{\alpha}{}_{\varepsilon}\left(\delta_{1\gamma} \underset{1}{N}{}^{\varepsilon}{}_{\beta}\right)\\
&\underset{(22)}{\overset{V_2}{P}}{}^{\alpha}_{\beta\gamma} = \dot{\partial}_{2\gamma} \underset{2}{N}{}^{\alpha}{}_{\beta} + \underset{1}{N}{}^{\alpha}{}_{\varepsilon}\left(\dot{\partial}_{2\gamma} \underset{1}{N}{}^{\varepsilon}{}_{\beta}\right) - \underset{(20)}{\overset{V_2}{L}}{}^{\alpha}_{\gamma\beta},
\end{aligned} \tag{37}$$

$$\underset{(21)}{\overset{V_1}{Q}}{}^{\alpha}_{\beta\gamma} = 0\,,\ \underset{(22)}{\overset{V_2}{Q}}{}^{\alpha}_{\beta\gamma} = \dot{\partial}_{2\gamma} \underset{1}{N}{}^{\alpha}{}_{\beta} - \underset{(21)}{\overset{V_2}{C}}{}^{\alpha}_{\gamma\beta},$$

$$\underset{(11)}{\overset{V_1}{S}}{}^{\alpha}_{\beta\gamma} = 0\,,\ \underset{(12)}{\overset{V_2}{S}}{}^{\alpha}_{\beta\gamma} = \underset{(12)}{R}{}^{\alpha}_{\beta\gamma} = \delta_{1\gamma} \underset{1}{N}{}^{\alpha}{}_{\beta} - \delta_{1\beta} \underset{1}{N}{}^{\alpha}{}_{\gamma},\ \underset{(21)}{\overset{V_1}{S}}{}^{\alpha}_{\beta\gamma} = 0\,,\ \underset{(22)}{\overset{V_2}{S}}{}^{\alpha}_{\beta\gamma} = 0.$$

Proof. Using the general local expressions from [2] which generally give the d-components of the torsion tensor of an N-linear connection, $D\Gamma(N)$, we deduce that the adapted components of the torsion tensor of $D^\top\Gamma\left(\check{N}\right)$ are given by the formulas from the theorem. □

In the next calculus we need the following d-tensor fields:

$$
\begin{aligned}
&\overset{V_1}{\underset{(0)}{T}}{}^{\alpha}_{\beta\gamma} = \overset{V_1}{\underset{(10)}{L}}{}^{\alpha}_{\beta\gamma} - \overset{V_1}{\underset{(10)}{L}}{}^{\alpha}_{\gamma\beta}, \qquad &&\overset{V_2}{\underset{(0)}{T}}{}^{\alpha}_{\beta\gamma} = \overset{V_2}{\underset{(20)}{L}}{}^{\alpha}_{\beta\gamma} - \overset{V_2}{\underset{(20)}{L}}{}^{\alpha}_{\gamma\beta},\\
&\overset{H}{\underset{(11)}{P}}{}^{\alpha}_{\beta\gamma} = \underset{(11)}{B}{}^{\alpha}_{\beta\gamma} - \overset{H}{\underset{(00)}{L}}{}^{\alpha}_{\gamma\beta}, &&\overset{V_2}{\underset{(11)}{P}}{}^{\alpha}_{\beta\gamma} = \underset{(11)}{B}{}^{\alpha}_{\beta\gamma} - \overset{V_2}{\underset{(20)}{L}}{}^{\alpha}_{\gamma\beta},\\
&\overset{H}{\underset{(22)}{P}}{}^{\alpha}_{\beta\gamma} = \underset{(22)}{B}{}^{\alpha}_{\beta\gamma} - \overset{H}{\underset{(00)}{L}}{}^{\alpha}_{\gamma\beta}, &&\overset{V_1}{\underset{(22)}{P}}{}^{\alpha}_{\beta\gamma} = \underset{(22)}{B}{}^{\alpha}_{\beta\gamma} - \overset{V_1}{\underset{(10)}{L}}{}^{\alpha}_{\gamma\beta},\\
&\overset{H}{\underset{(22)}{Q}}{}^{\alpha}_{\beta\gamma} = \underset{(21)}{B}{}^{\alpha}_{\beta\gamma} - \overset{H}{\underset{(01)}{C}}{}^{\alpha}_{\gamma\beta}, &&\overset{V_1}{\underset{(22)}{Q}}{}^{\alpha}_{\beta\gamma} = \underset{(21)}{B}{}^{\alpha}_{\beta\gamma} - \overset{V_1}{\underset{(11)}{C}}{}^{\alpha}_{\gamma\beta},\\
&\overset{H}{\underset{(1)}{S}}{}^{\alpha}_{\beta\gamma} = \overset{H}{\underset{(01)}{C}}{}^{\alpha}_{\beta\gamma} - \overset{H}{\underset{(01)}{C}}{}^{\alpha}_{\gamma\beta}, &&\overset{V_2}{\underset{(1)}{S}}{}^{\alpha}_{\beta\gamma} = \overset{V_2}{\underset{(21)}{C}}{}^{\alpha}_{\beta\gamma} - \overset{V_2}{\underset{(21)}{C}}{}^{\alpha}_{\gamma\beta},\\
&\overset{H}{\underset{(2)}{S}}{}^{\alpha}_{\beta\gamma} = \overset{H}{\underset{(02)}{C}}{}^{\alpha}_{\beta\gamma} - \overset{H}{\underset{(02)}{C}}{}^{\alpha}_{\gamma\beta}, &&\overset{V_1}{\underset{(2)}{S}}{}^{\alpha}_{\beta\gamma} = \overset{V_1}{\underset{(12)}{C}}{}^{\alpha}_{\beta\gamma} - \overset{V_1}{\underset{(12)}{C}}{}^{\alpha}_{\gamma\beta}.
\end{aligned}
\tag{38}
$$

Theorem 7. *The curvature tensor $\mathbb{R}$ of the induced tangent connection $D^{\top}\Gamma\left(\check{N}\right)$ is characterized by the following local adapted d-tensors:*

$$
\begin{aligned}
&\overset{H}{\underset{(00)}{R}}{}_{\beta}{}^{\alpha}{}_{\gamma\delta} = \delta_{\delta}\overset{H}{\underset{(00)}{L}}{}^{\alpha}_{\beta\gamma} - \delta_{\gamma}\overset{H}{\underset{(00)}{L}}{}^{\alpha}_{\beta\delta} + \overset{H}{\underset{(00)}{L}}{}^{\varepsilon}_{\beta\gamma}\overset{H}{\underset{(00)}{L}}{}^{\alpha}_{\varepsilon\delta} - \overset{H}{\underset{(00)}{L}}{}^{\varepsilon}_{\beta\delta}\overset{H}{\underset{(00)}{L}}{}^{\alpha}_{\varepsilon\gamma} + \overset{H}{\underset{(01)}{C}}{}^{\alpha}_{\beta\sigma}\underset{01}{R}{}^{\sigma}_{\gamma\delta}\\
&\overset{H}{\underset{(10)}{P}}{}_{\beta}{}^{\alpha}{}_{\gamma\delta} = \delta_{1\delta}\overset{H}{\underset{(00)}{L}}{}^{\alpha}_{\beta\gamma} - \overset{H}{\underset{(01)}{C}}{}^{\alpha}_{\beta\delta|0\gamma} + \overset{H}{\underset{(01)}{C}}{}^{\alpha}_{\beta\sigma}\overset{H}{\underset{(11)}{P}}{}^{\sigma}_{\gamma\delta}\,,\ \overset{H}{\underset{(20)}{P}}{}_{\beta}{}^{\alpha}{}_{\gamma\delta} = \dot{\partial}_{2\delta}\overset{H}{\underset{(20)}{L}}{}^{\alpha}_{\beta\gamma} - \overset{H}{\underset{(01)}{C}}{}^{\alpha}_{\beta\sigma}\overset{V_1}{\underset{(21)}{P}}{}^{\sigma}_{\gamma\delta},\\
&\qquad\overset{H}{\underset{(20)}{Q}}{}_{\beta}{}^{\alpha}{}_{\gamma\delta} = \dot{\partial}_{2\delta}\overset{H}{\underset{(01)}{C}}{}^{\alpha}_{\beta\gamma}\,,\ \overset{H}{\underset{(20)}{S}}{}_{\beta}{}^{\alpha}{}_{\gamma\delta} = 0,\\
&\qquad\overset{H}{\underset{(10)}{S}}{}_{\beta}{}^{\alpha}{}_{\gamma\delta} = \delta_{1\delta}\overset{H}{\underset{(01)}{C}}{}^{\alpha}_{\beta\gamma} - \delta_{1\gamma}\overset{H}{\underset{(01)}{C}}{}^{\alpha}_{\beta\delta} + \overset{H}{\underset{(01)}{C}}{}^{\varepsilon}_{\beta\gamma}\overset{H}{\underset{(01)}{C}}{}^{\alpha}_{\varepsilon\delta} - \overset{H}{\underset{(01)}{C}}{}^{e}_{\beta\delta}\overset{H}{\underset{(01)}{C}}{}^{\alpha}_{e\gamma},
\end{aligned}
\tag{39}
$$

$$
\begin{aligned}
&\overset{V_i}{\underset{(0i)}{R}}{}_{\beta}{}^{\alpha}{}_{\gamma\delta} = \delta_{\delta}\overset{V_i}{\underset{(i0)}{L}}{}^{\alpha}_{\beta\gamma} - \delta_{\beta}\overset{V_i}{\underset{(i0)}{L}}{}^{\alpha}_{\beta\delta} + \overset{V_i}{\underset{(i0)}{L}}{}^{e}_{\beta\gamma}\overset{V_i}{\underset{(i0)}{L}}{}^{\alpha}_{e\delta} - \overset{V_i}{\underset{(i0)}{L}}{}^{e}_{\beta\delta}\overset{V_i}{\underset{(i0)}{L}}{}^{\alpha}_{e\gamma} + \overset{V_i}{\underset{(i1)}{C}}{}^{\alpha}_{\beta\sigma}\underset{(01)}{R}{}^{\sigma}_{\gamma\delta},\\
&\overset{V_i}{\underset{(1i)}{P}}{}_{\beta}{}^{\alpha}{}_{\gamma\delta} = \delta_{1\delta}\overset{V_i}{\underset{(i0)}{L}}{}^{\alpha}_{\beta\gamma} - \overset{V_i}{\underset{(i1)}{C}}{}^{\alpha}_{\beta\delta|i\gamma} + \overset{V_i}{\underset{(i1)}{C}}{}^{\alpha}_{\beta\sigma}\overset{V_i}{\underset{(11)}{P}}{}^{\sigma}_{\gamma\delta}\,,\ \overset{V_i}{\underset{(2i)}{P}}{}_{\beta}{}^{\alpha}{}_{\gamma\delta} = \dot{\partial}_{2\delta}\overset{V_i}{\underset{(2i)}{L}}{}^{\alpha}_{\beta\gamma} + \overset{V_i}{\underset{(i1)}{C}}{}^{\alpha}_{\beta\varepsilon}\overset{V_1}{\underset{(21)}{P}}{}^{\varepsilon}_{\gamma\delta},\\
&\overset{V_i}{\underset{(1i)}{Q}}{}_{\beta}{}^{\alpha}{}_{\gamma\delta} = 0\,,\ \overset{V_i}{\underset{(1i)}{S}}{}_{\beta}{}^{\alpha}{}_{\gamma\delta} = \dot{\partial}_{2\delta}\overset{V_i}{\underset{(i1)}{C}}{}^{\alpha}_{\beta\gamma} - \dot{\partial}_{2\gamma}\overset{V_i}{\underset{(i1)}{C}}{}^{\alpha}_{\beta\delta} + \overset{V_i}{\underset{(i1)}{C}}{}^{\varepsilon}_{\beta\gamma}\overset{V_i}{\underset{(i1)}{C}}{}^{\alpha}_{\varepsilon\delta} - \overset{V_i}{\underset{(i1)}{C}}{}^{\varepsilon}_{\beta\delta}\overset{V_i}{\underset{(i1)}{C}}{}^{\alpha}_{\varepsilon\gamma}\,,\ \overset{V_i}{\underset{(2i)}{S}}{}_{\beta}{}^{\alpha}{}_{\gamma\delta} = 0,
\end{aligned}
\tag{40}
$$

$$\left(i=1,2; j=1,2 \overset{V_i}{\underset{(12)}{P}}{}^{\alpha}_{\beta\delta} = \underset{(12)}{P}{}^{\alpha}_{\beta\delta}, \overset{V_i}{\underset{(21)}{P}}{}^{\alpha}_{\beta\delta} = \underset{(21)}{P}{}^{\alpha}_{\beta\delta}, \underset{(22)}{R}{}^{\alpha}_{\beta\delta}=0\right).$$

Proof. The general formulas that express the local curvature d-tensors of an arbitrary N-linear connection (for more details, see [2]), applied to the induced tangent connection $D^{\top}\Gamma\left(\check{N}\right)$, imply the above formulas. □

5 The Bianchi identities in the adapted basis

From the general theory of linear connections on a vector bundle, one knows that the torsions $\mathbb{T}$ and curvature $\mathbb{R}$ of a connection D on the 2-osculator space $E = Osc^2M$ are interrelated by the following general *Bianchi identities* (for any $X, Y, Z, U \in \mathfrak{X}(E)$):

$$\sum_{\{X,Y,Z\}} \{(D_X\mathbb{T})(Y,Z) - \mathbb{R}(X,Y)Z + \mathbb{T}(\mathbb{T}(X,Y),Z)\} = 0,$$
$$\sum_{\{X,Y,Z\}} (D_X\mathbb{R})(Y,Z,U) + \mathbb{R}(\mathbb{T}(X,Y),Z)U = 0,$$

where $\Sigma_{\{X,Y,Z\}}$ means a cyclic sum. Obviously, working with an N-linear connection and the local adapted basis of d-vector fields $(X_\alpha) \subset \mathfrak{X}\left(\check{E}\right)$, $\check{E} = Osc^2\check{M}$, (associated with the induced nonlinear connection $\check{N}$ on $\check{E}$), the above Bianchi identities are locally described by the equalities:

$$\begin{gathered}\sum_{\{A,B,C\}} \left\{\mathbb{R}^F_{ABC} - \mathbb{T}^F_{AB:C} - \mathbb{T}^G_{AB}\mathbb{T}^F_{CG}\right\} = 0,\\ \sum_{\{A,B,C\}} \left\{\mathbb{R}^F_{DAB:C} + \mathbb{T}^G_{AB}\mathbb{R}^F_{DCG}\right\} = 0,\end{gathered} \tag{41}$$

where $\mathbb{R}(X_A, X_B)X_C = \mathbb{R}^D_{CBA}X_D$, $\mathbb{T}(X_A, X_B) = \mathbb{T}^D_{BA}X_D$, and "$_{:C}$" represents one of the local covariant derivatives "$_{|i\alpha}$", "$\overset{(1)}{|}_{i\alpha}$" or "$\overset{(2)}{|}_{i\alpha}$" from (34), (35) and (36) (for similar details, see the works [6], [9]). Consequently, we find:

Theorem 8. *For the induced tangent connection with the coefficients* $D^{\top}\Gamma\left(\check{N}\right) = \left(\overset{V_i}{\underset{(i0)}{L}}{}^{\alpha}_{\beta\delta}, \overset{V_i}{\underset{(i1)}{C}}{}^{\alpha}_{\beta\delta}, \overset{V_i}{\underset{(i2)}{C}}{}^{\alpha}_{\beta\delta}\right)$, $(i = 0,1,2; V_0 = H)$, *the following Bianchi identities hold:*

$$\sum^{0}\left[\underset{(0i)}{T}{}^{\alpha}_{\beta\gamma|i\delta} + \overset{V_i}{\underset{(0)}{T}}{}^{\varphi}_{\beta\gamma}\underset{(0i)}{T}{}^{\alpha}_{\delta\varphi} + \underset{(01)}{T}{}^{\varphi}_{\beta\gamma}\underset{(1i)}{P}{}^{\alpha}_{\delta\varphi} + \underset{(02)}{T}{}^{\varphi}_{\beta\gamma}\underset{(2i)}{P}{}^{\alpha}_{\delta\varphi} - \overset{V_i}{\underset{(00)}{R}}{}_{\beta}{}^{\alpha}{}_{\gamma\delta}\right] = 0\,,\ (i=0,1,2)\,,$$

where

$$\overset{0}{\underset{(00)}{R}}{}_{\beta}{}^{\alpha}{}_{\gamma\delta} = \underset{(00)}{R}{}_{\beta}{}^{\alpha}{}_{\gamma\delta},\ \overset{V_j}{\underset{(00)}{R}}{}_{\beta}{}^{\alpha}{}_{\gamma\delta} = 0, (j=1,2)\,,$$

$$
\begin{aligned}
&\underset{(0i)}{T}{}^{\alpha}{}_{\beta\gamma}\overset{(1)}{|}{}_{i\delta} - \underset{(1i)}{P}{}^{\alpha}{}_{\beta\delta|i\gamma} + \underset{(1i)}{P}{}^{\alpha}{}_{\gamma\delta|i\beta} - \\
- \quad &\overset{V_i}{\underset{(0)}{T}}{}^{\varphi}{}_{\beta\gamma}\underset{(1i)}{P}{}^{\alpha}{}_{\varphi\delta} - \underset{(1i)}{C}{}^{\varphi}{}_{\beta\delta}\underset{(0i)}{T}{}^{\alpha}{}_{\gamma\varphi} + \underset{(1i)}{C}{}^{\varphi}{}_{\gamma\delta}\underset{(0i)}{T}{}^{\alpha}{}_{\beta\varphi} + \\
+ \quad &\underset{(01)}{T}{}^{\varphi}{}_{\beta\gamma}\underset{(1i)}{S}{}^{\alpha}{}_{\delta\varphi} - \overset{V_i}{\underset{(11)}{P}}{}^{\varphi}{}_{\beta\delta}\underset{(1i)}{P}{}^{\alpha}{}_{\gamma\varphi} + \overset{V_i}{\underset{(11)}{P}}{}^{\varphi}{}_{\gamma\delta}\underset{(1i)}{P}{}^{\alpha}{}_{\beta\varphi} + \\
+ \quad &\underset{(02)}{T}{}^{\varphi}{}_{\beta\gamma}\underset{(2i)}{Q}{}^{\alpha}{}_{\delta\varphi} - \underset{(12)}{P}{}^{\varphi}{}_{\beta\delta}\underset{(2i)}{P}{}^{\alpha}{}_{\gamma\varphi} + \underset{(12)}{P}{}^{\varphi}{}_{\gamma\delta}\underset{(2i)}{P}{}^{\alpha}{}_{\beta\varphi} - \overset{V_i}{\underset{(10)}{A}}{}^{\alpha}{}_{\beta\ \gamma\delta} = 0\,,\ (i=0,1,2)\,,
\end{aligned}
\tag{42}
$$

where

$$
\overset{H}{\underset{(10)}{A}}{}^{\alpha}{}_{\beta\ \gamma\delta} = \underset{(10)}{P}{}^{\alpha}{}_{\beta\ \gamma\delta} - \underset{(10)}{P}{}^{\alpha}{}_{\gamma\ \beta\delta},\ \overset{V_1}{\underset{(10)}{A}}{}^{\alpha}{}_{\beta\ \gamma\delta} = \underset{(01)}{R}{}^{\alpha}{}_{\delta\ \beta\gamma},\ \overset{V_2}{\underset{(10)}{A}}{}^{\alpha}{}_{\beta\ \gamma\delta} = 0,
$$

$$
\begin{aligned}
&\underset{(0i)}{T}{}^{\alpha}{}_{\beta\gamma}\overset{(2)}{|}{}_{i\delta} - \underset{(2i)}{P}{}^{\alpha}{}_{\beta\delta|i\gamma} + \underset{(2i)}{P}{}^{\alpha}{}_{\gamma\delta|i\beta} - \\
- \quad &\overset{V_i}{\underset{(0)}{T}}{}^{\varphi}{}_{\beta\gamma}\underset{(2i)}{P}{}^{\alpha}{}_{\varphi\delta} - \underset{(20)}{P}{}^{\varphi}{}_{\beta\delta}\underset{(0i)}{T}{}^{\alpha}{}_{\gamma\varphi} + \underset{(20)}{P}{}^{\varphi}{}_{\gamma\delta}\underset{(0i)}{T}{}^{\alpha}{}_{\beta\varphi} - \\
- \quad &\underset{(01)}{T}{}^{\varphi}{}_{\beta\gamma}\underset{(2i)}{Q}{}^{\alpha}{}_{\varphi\delta} - \underset{(21)}{P}{}^{\varphi}{}_{\beta\delta}\underset{(1i)}{P}{}^{\alpha}{}_{\gamma\varphi} + \underset{(21)}{P}{}^{\varphi}{}_{\gamma\delta}\underset{(1i)}{P}{}^{\alpha}{}_{\beta\varphi} - \\
- \quad &\underset{(02)}{T}{}^{\varphi}{}_{\beta\gamma}\underset{(2i)}{S}{}^{\alpha}{}_{\varphi\delta} - \overset{V_i}{\underset{(22)}{P}}{}^{\varphi}{}_{\beta\delta}\underset{(2i)}{P}{}^{\alpha}{}_{\gamma\varphi} + \overset{V_i}{\underset{(22)}{P}}{}^{\varphi}{}_{\gamma\delta}\underset{(2i)}{P}{}^{\alpha}{}_{\beta\varphi} - \overset{V_i}{\underset{(20)}{A}}{}^{\alpha}{}_{\beta\ \gamma\delta} = 0\,,\ (i=0,1,2)\,,
\end{aligned}
\tag{43}
$$

where

$$
\overset{H}{\underset{(20)}{A}}{}^{\alpha}{}_{\beta\ \gamma\delta} = \underset{(20)}{P}{}^{\alpha}{}_{\beta\ \gamma\delta} - \underset{(20)}{P}{}^{\alpha}{}_{\gamma\ \beta\delta},\ \overset{V_1}{\underset{(20)}{A}}{}^{\alpha}{}_{\beta\ \gamma\delta} = 0,\ \overset{V_2}{\underset{(20)}{A}}{}^{\alpha}{}_{\beta\ \gamma\delta} = \underset{(02)}{R}{}^{\alpha}{}_{\delta\ \beta\gamma},
$$

$$
\begin{aligned}
&\underset{(1i)}{P}{}^{\alpha}{}_{\beta\gamma}\overset{(1)}{|}{}_{i\delta} - \underset{(1i)}{P}{}^{\alpha}{}_{bd}\overset{(1)}{|}{}_{i\gamma} + \underset{(1i)}{S}{}^{\alpha}{}_{\gamma\delta|i\beta} - \\
- \quad &\underset{(i1)}{C}{}^{\varphi}{}_{\beta\gamma}\underset{(1i)}{P}{}^{\alpha}{}_{\varphi\delta} + \underset{(i1)}{C}{}^{\varphi}{}_{\beta\delta}\underset{(1i)}{P}{}^{\alpha}{}_{\varphi\gamma} + \\
+ \quad &\overset{V_i}{\underset{(11)}{P}}{}^{\varphi}{}_{\beta\gamma}\underset{(1i)}{S}{}^{\alpha}{}_{\delta\varphi} - \overset{V_i}{\underset{(11)}{P}}{}^{\varphi}{}_{\beta\delta}\underset{(1i)}{S}{}^{\alpha}{}_{\gamma\varphi} + \overset{V_i}{\underset{(1)}{S}}{}^{\varphi}{}_{\gamma\delta}\underset{(1i)}{P}{}^{\alpha}{}_{\beta\varphi} + \\
+ \quad &\underset{(12)}{P}{}^{\varphi}{}_{\beta\gamma}\underset{(2i)}{Q}{}^{\alpha}{}_{\delta\varphi} - \underset{(12)}{P}{}^{\varphi}{}_{\beta\delta}\underset{(2i)}{Q}{}^{\alpha}{}_{\gamma\varphi} + \underset{(12)}{S}{}^{\varphi}{}_{\gamma\delta}\underset{(2i)}{P}{}^{\alpha}{}_{\beta\varphi} - \overset{V_i}{\underset{(11)}{A}}{}^{\alpha}{}_{\beta\ \gamma\delta} = 0\,,\ (i=0,1,2)\,,
\end{aligned}
\tag{44}
$$

where

$$
\overset{H}{\underset{(11)}{A}}{}^{\alpha}{}_{\beta\ \gamma\delta} = \underset{(10)}{S}{}^{\alpha}{}_{\beta\ \gamma\delta},\ \overset{V_1}{\underset{(11)}{A}}{}^{\alpha}{}_{\beta\ \gamma\delta} = \underset{(11)}{P}{}^{\alpha}{}_{\delta\ \beta\gamma} - \underset{(11)}{P}{}^{\alpha}{}_{\gamma\ \beta\delta},\ \overset{V_2}{\underset{(11)}{A}}{}^{\alpha}{}_{\beta\ \gamma\delta} = 0,
$$

$$
\begin{aligned}
&\underset{(2i)}{P}{}^{\alpha}{}_{\beta\gamma}\overset{(1)}{|}{}_{i\delta} - \underset{(1i)}{P}{}^{\alpha}{}_{\beta\delta}\overset{(2)}{|}{}_{i\gamma} - \underset{(2i)}{Q}{}^{\alpha}{}_{\delta\gamma|i\beta} - \\
- \quad &\underset{(i2)}{C}{}^{\varphi}{}_{\beta\gamma}\underset{(1i)}{P}{}^{\alpha}{}_{fd} + \underset{(i1)}{C}{}^{\varphi}{}_{\beta\delta}\underset{(2i)}{P}{}^{\alpha}{}_{\varphi\gamma} - \underset{(i2)}{C}{}^{\varphi}{}_{\delta\gamma}\underset{(1i)}{P}{}^{\alpha}{}_{\beta\varphi} + \\
+ \quad &\underset{(21)}{P}{}^{\varphi}{}_{\beta\gamma}\underset{(1i)}{S}{}^{\alpha}{}_{\delta\varphi} + \overset{V_i}{\underset{(11)}{P}}{}^{\varphi}{}_{\beta\delta}\underset{(2i)}{Q}{}^{\alpha}{}_{\varphi\gamma} - \overset{V_i}{\underset{(22)}{Q}}{}^{\varphi}{}_{\delta\gamma}\underset{(2i)}{P}{}^{\alpha}{}_{\beta\varphi} + \\
+ \quad &\overset{V_i}{\underset{(22)}{P}}{}^{\varphi}{}_{\beta\gamma}\underset{(2i)}{Q}{}^{\alpha}{}_{\delta\varphi} + \underset{(12)}{P}{}^{\varphi}{}_{\beta\delta}\underset{(2i)}{S}{}^{\alpha}{}_{\varphi\gamma} - \overset{V_i}{\underset{(12)}{A}}{}^{\alpha}{}_{\beta\ \gamma\delta} = 0\,,\ (i=0,1,2)\,,
\end{aligned}
\tag{45}
$$

where

$$\overset{H}{\underset{(12)}{A}}{}^{\alpha}{}_{\beta\gamma\delta}=0,\ \overset{V_1}{\underset{(12)}{A}}{}^{\alpha}{}_{\beta\gamma\delta}=\underset{(21)}{P}{}^{\alpha}{}_{\delta\beta\gamma},\ \overset{V_2}{\underset{(12)}{A}}{}^{\alpha}{}_{\beta\gamma\delta}=-\underset{(12)}{P}{}^{\alpha}{}_{\gamma\beta\delta},$$

$$\begin{aligned}
&\underset{(2i)}{P}{}^{\alpha}{}_{\beta\gamma}\overset{(2)}{|}{}_{i\delta}-\underset{(2i)}{P}{}^{\alpha}{}_{\beta\delta}\overset{(2)}{|}{}_{i\gamma}+\underset{(2i)}{S}{}^{\alpha}{}_{\gamma\delta|i\beta}-\\
-\;&\underset{(i2)}{C}{}^{\varphi}{}_{\beta\gamma}\underset{(2i)}{P}{}^{\alpha}{}_{\varphi\delta}+\underset{(i2)}{C}{}^{\varphi}{}_{\beta\delta}\underset{(2i)}{P}{}^{\alpha}{}_{\varphi\gamma}+\overset{V_i}{\underset{(2)}{S}}{}^{\varphi}{}_{\gamma\delta}\underset{(2i)}{P}{}^{\alpha}{}_{\beta\varphi}-\\
-\;&\underset{(21)}{P}{}^{\varphi}{}_{\beta\gamma}\underset{(2i)}{Q}{}^{\alpha}{}_{\varphi\delta}+\underset{(21)}{P}{}^{\varphi}{}_{\beta\delta}\underset{(2i)}{Q}{}^{\alpha}{}_{\varphi\gamma}-\\
-\;&\underset{(22)}{P}{}^{\varphi}{}_{\beta\gamma}\underset{(2i)}{S}{}^{\alpha}{}_{\varphi\delta}+\underset{(22)}{P}{}^{\varphi}{}_{\beta\delta}\underset{(2i)}{S}{}^{\alpha}{}_{\varphi\gamma}-\overset{V_i}{\underset{(22)}{A}}{}^{\alpha}{}_{\beta\gamma\delta}=0\,,\ (i=0,1,2)\,,
\end{aligned}\tag{46}$$

where

$$\overset{H}{\underset{(22)}{A}}{}^{\alpha}{}_{\beta\gamma\delta}=\underset{(20)}{S}{}^{\alpha}{}_{\beta\gamma\delta},\ \overset{V_1}{\underset{(22)}{A}}{}^{\alpha}{}_{\beta\gamma\delta}=0,\ \overset{V_2}{\underset{(22)}{A}}{}^{\alpha}{}_{\beta\gamma\delta}=\underset{(22)}{P}{}^{\alpha}{}_{\delta\beta\gamma}-\underset{(22)}{P}{}^{\alpha}{}_{\gamma\beta\delta},$$

$$\sum^{0}\left[\underset{(1j)}{S}{}^{\alpha}{}_{\beta\gamma}\overset{(1)}{|}{}_{j\delta}+\overset{V_j}{\underset{(1)}{S}}{}^{\varphi}{}_{\beta\gamma}\underset{(1j)}{S}{}^{\alpha}{}_{\delta\varphi}+\underset{(12)}{S}{}^{\varphi}{}_{\beta\gamma}\underset{(2i)}{Q}{}^{\alpha}{}_{\delta\varphi}-\overset{V_j}{\underset{(11)}{S}}{}^{\alpha}{}_{\beta\gamma\delta}\right]=0\,,\ (j=1,2)\,,\tag{47}$$

where

$$\overset{1}{\underset{(11)}{S}}{}^{\alpha}{}_{\beta\gamma\delta}=\underset{(11)}{S}{}^{\alpha}{}_{\beta\gamma\delta},\ \overset{2}{\underset{(11)}{S}}{}^{\alpha}{}_{\beta\gamma\delta}=0,$$

$$\begin{aligned}
&\underset{(1j)}{S}{}^{\alpha}{}_{\beta\gamma}\overset{(2)}{|}{}_{\beta\delta}-\underset{(2j)}{Q}{}^{\alpha}{}_{\beta\delta}\overset{(1)}{|}{}_{\beta\gamma}+\underset{(2j)}{Q}{}^{\alpha}{}_{\gamma\delta}\overset{(1)}{|}{}_{\beta b}-\\
-\;&\overset{V_j}{\underset{(1)}{S}}{}^{\varphi}{}_{\beta\gamma}\underset{(2j)}{Q}{}^{\alpha}{}_{\varphi\delta}-\underset{(j2)}{C}{}^{\varphi}{}_{\beta\delta}\underset{(1j)}{S}{}^{\alpha}{}_{\gamma\varphi}+\underset{(j2)}{C}{}^{\varphi}{}_{\gamma\delta}\underset{(1j)}{S}{}^{\alpha}{}_{\beta\varphi}-\\
-\;&\underset{(12)}{S}{}^{\varphi}{}_{\beta\gamma}\underset{(2j)}{S}{}^{\alpha}{}_{\varphi\delta}-\overset{V_j}{\underset{(22)}{Q}}{}^{\varphi}{}_{\beta\delta}\underset{(2j)}{Q}{}^{\alpha}{}_{\gamma\varphi}+\overset{V_j}{\underset{(22)}{Q}}{}^{\varphi}{}_{\gamma\delta}\underset{(2j)}{Q}{}^{\alpha}{}_{\beta\varphi}-\overset{V_j}{\underset{(21)}{B}}{}^{\alpha}{}_{\beta\gamma\delta}=0\,,\ (j=1,2)\,,
\end{aligned}\tag{48}$$

where

$$\overset{V_1}{\underset{(21)}{B}}{}^{\alpha}{}_{\beta\gamma\delta}=\underset{(21)}{Q}{}^{\alpha}{}_{\beta\gamma\delta}-\underset{(21)}{Q}{}^{\alpha}{}_{\gamma\beta\delta},\ \overset{V_2}{\underset{(21)}{B}}{}^{\alpha}{}_{\beta\gamma\delta}=\underset{(12)}{S}{}^{\alpha}{}_{\delta\beta\gamma},$$

$$\begin{aligned}
&\underset{(2j)}{Q}{}^{\alpha}{}_{\beta\gamma}\overset{(2)}{|}{}_{j\delta}-\underset{(2j)}{Q}{}^{\alpha}{}_{\beta\delta}\overset{(2)}{|}{}_{j\gamma}+\underset{(2j)}{S}{}^{\alpha}{}_{\gamma\delta}\overset{(1)}{|}{}_{j\beta}-\\
-\;&\underset{(j2)}{C}{}^{\varphi}{}_{\beta\gamma}\underset{(2j)}{Q}{}^{\alpha}{}_{\varphi\delta}+\underset{(j2)}{C}{}^{\varphi}{}_{\beta\delta}\underset{(2j)}{Q}{}^{\alpha}{}_{\varphi\gamma}+\overset{V_j}{\underset{(2)}{S}}{}^{\varphi}{}_{\gamma\delta}\underset{(2j)}{Q}{}^{\alpha}{}_{\beta\varphi}-\\
-\;&\underset{(22)}{Q}{}^{\varphi}{}_{\beta\gamma}\underset{(2j)}{S}{}^{\alpha}{}_{\varphi\delta}+\underset{(22)}{Q}{}^{\varphi}{}_{\beta\delta}\underset{(2j)}{S}{}^{\alpha}{}_{\varphi\gamma}-\overset{V_j}{\underset{(22)}{B}}{}^{\alpha}{}_{\beta\gamma\delta}=0\,,\ (j=1,2)\,,
\end{aligned}\tag{49}$$

where

$$\overset{V_1}{\underset{(22)}{B}}{}^{\alpha}{}_{\beta\gamma\delta}=\underset{(21)}{S}{}^{\alpha}{}_{\beta\gamma\delta},\ \overset{V_2}{\underset{(22)}{B}}{}^{\alpha}{}_{\beta\gamma\delta}=\underset{(22)}{Q}{}^{\alpha}{}_{\delta\beta\gamma}-\underset{(22)}{Q}{}^{\alpha}{}_{\gamma\beta\delta},$$

$$\sum^{0}\left[\underset{(22)}{S}{}^{\alpha}{}_{\beta\gamma}\overset{(2)}{|}{}_{2\delta}+\underset{(22)}{S}{}^{\varphi}{}_{\beta\gamma}\underset{(22)}{S}{}^{\alpha}{}_{\delta\varphi}-\underset{(22)}{S}{}^{\alpha}{}_{\beta\gamma\delta}\right]=0,\tag{50}$$

and

$$\sum^{0}\left[\underset{(0i)}{R}{}_{\alpha}{}^{\varepsilon}{}_{\beta\gamma|i\delta} + \underset{(0i)}{R}{}_{\alpha}{}^{\varepsilon}{}_{\beta\varphi}\overset{V_i}{\underset{(0)}{T}}{}^{\varphi}{}_{\gamma\delta} + \underset{(1i)}{P}{}_{\alpha}{}^{\varepsilon}{}_{\beta\varphi}\underset{(01)}{R}{}^{\varphi}{}_{\gamma\delta} + \underset{(2i)}{P}{}_{\alpha}{}^{\varepsilon}{}_{\beta\varphi}\underset{(02)}{R}{}^{\varphi}{}_{\gamma\delta}\right] = 0, (i=0,1,2)\,, \tag{51}$$

$$\begin{aligned}
&\underset{(0i)}{R}{}_{\alpha}{}^{\varepsilon}{}_{\beta\gamma}\overset{(1)}{|}{}_{i\delta} - \underset{(1i)}{P}{}_{\alpha}{}^{\varepsilon}{}_{\beta\delta|i\gamma} + \underset{(1i)}{P}{}_{\alpha}{}^{\varepsilon}{}_{\gamma\delta|i\beta} - \\
&- \overset{V_i}{\underset{(0)}{T}}{}^{\varphi}{}_{\beta\gamma}\underset{(1i)}{P}{}_{\alpha}{}^{\varepsilon}{}_{\delta\varphi} - \underset{(i1)}{C}{}^{\varphi}{}_{\beta\delta}\underset{(0i)}{R}{}_{\alpha}{}^{\varepsilon}{}_{\gamma\varphi} + \underset{(i1)}{C}{}^{\varphi}{}_{\gamma\delta}\underset{(0i)}{R}{}_{\alpha}{}^{\varepsilon}{}_{\beta\varphi} + \\
&\underset{(0i)}{R}{}_{\alpha}{}^{\varepsilon}{}_{\beta\gamma}\overset{(2)}{|}{}_{i\delta} - \underset{(2i)}{P}{}_{\alpha}{}^{\varepsilon}{}_{\beta\delta|i\gamma} + \underset{(2i)}{P}{}_{\alpha}{}^{\varepsilon}{}_{\gamma\delta|i\beta} - \\
&- \overset{V_i}{\underset{(0)}{T}}{}^{\varphi}{}_{\beta\gamma}\underset{(2i)}{P}{}_{\alpha}{}^{\varepsilon}{}_{\varphi\delta} - \underset{(i2)}{C}{}^{\varphi}{}_{\beta\delta}\underset{(0i)}{R}{}_{\alpha}{}^{\varepsilon}{}_{\gamma\varphi} + \underset{(i2)}{C}{}^{\varphi}{}_{\gamma\delta}\underset{(0i)}{R}{}_{\alpha}{}^{\varepsilon}{}_{\beta\varphi} - \\
&- \underset{(01)}{R}{}^{\varphi}{}_{\beta\gamma}\underset{(2i)}{Q}{}_{\alpha}{}^{\varepsilon}{}_{\varphi\delta} - \underset{(21)}{P}{}^{\varphi}{}_{\beta\delta}\underset{(1i)}{P}{}_{\alpha}{}^{\varepsilon}{}_{\gamma\varphi} + \underset{(21)}{P}{}^{\varphi}{}_{\gamma\delta}\underset{(1i)}{P}{}_{\alpha}{}^{\varepsilon}{}_{\beta\varphi} + \\
&+ \underset{(02)}{R}{}^{\varphi}{}_{\beta\gamma}\underset{(2i)}{S}{}_{\alpha}{}^{\varepsilon}{}_{\delta\varphi} - \overset{V_i}{\underset{(22)}{P}}{}^{\varphi}{}_{\beta\delta}\underset{(2i)}{P}{}_{\alpha}{}^{\varepsilon}{}_{\gamma\varphi} + \overset{V_i}{\underset{(22)}{P}}{}^{\varphi}{}_{\gamma\delta}\underset{(2i)}{P}{}_{\alpha}{}^{\varepsilon}{}_{\beta\varphi} = 0\,,\ (i=0,1,2)\,,
\end{aligned} \tag{52}$$

$$\begin{aligned}
&\underset{(1i)}{P}{}_{\alpha}{}^{\varepsilon}{}_{\beta\gamma}\overset{(1)}{|}{}_{i\delta} - \underset{(1i)}{P}{}_{\alpha}{}^{\varepsilon}{}_{\beta\delta}\overset{(1)}{|}{}_{i\gamma} + \underset{(1i)}{S}{}_{\alpha}{}^{\varepsilon}{}_{\gamma\delta}\overset{(1)}{|}{}_{i\beta} - \\
&- \underset{(i1)}{C}{}^{\varphi}{}_{\beta\gamma}\underset{(1i)}{P}{}_{\alpha}{}^{\varepsilon}{}_{\varphi\delta} + \underset{(i1)}{C}{}^{\varphi}{}_{\beta\delta}\underset{(1i)}{P}{}_{\alpha}{}^{\varepsilon}{}_{\varphi\gamma} + \\
&+ \overset{V_i}{\underset{(11)}{P}}{}^{\varphi}{}_{\beta\gamma}\underset{(1i)}{S}{}_{\alpha}{}^{\varepsilon}{}_{\delta\varphi} + \overset{V_i}{\underset{(11)}{P}}{}^{\varphi}{}_{\beta\delta}\underset{(1i)}{S}{}_{\alpha}{}^{\varepsilon}{}_{\varphi\gamma} + \overset{V_i}{\underset{(1)}{S}}{}^{\varphi}{}_{\gamma\delta}\underset{(1i)}{P}{}_{\alpha}{}^{\varepsilon}{}_{\beta\varphi} + \\
&+ \underset{(12)}{P}{}^{\varphi}{}_{\beta\gamma}\underset{(2i)}{Q}{}_{\alpha}{}^{\varepsilon}{}_{\delta\varphi} - \underset{(12)}{P}{}^{\varphi}{}_{\beta\delta}\underset{(2i)}{Q}{}_{\alpha}{}^{\varepsilon}{}_{\gamma\varphi} + \underset{(12)}{S}{}^{\varphi}{}_{\gamma\delta}\underset{(2i)}{P}{}_{\alpha}{}^{\varepsilon}{}_{\beta\varphi} = 0\,,\ (i=0,1,2)\,,
\end{aligned} \tag{53}$$

$$\begin{aligned}
&\underset{(2i)}{P}{}_{\alpha}{}^{\varepsilon}{}_{\beta\gamma}\overset{(1)}{|}{}_{i\delta} - \underset{(1i)}{P}{}_{\alpha}{}^{\varepsilon}{}_{\beta\delta}\overset{(2)}{|}{}_{i\gamma} - \underset{(2i)}{Q}{}_{\alpha}{}^{\varepsilon}{}_{\delta\gamma|i\beta} - \\
&- \underset{(i2)}{C}{}^{\varphi}{}_{\beta\gamma}\underset{(1i)}{P}{}_{\alpha}{}^{\varepsilon}{}_{\varphi\delta} + \underset{(i1)}{C}{}^{\varphi}{}_{\beta\delta}\underset{(2i)}{P}{}_{\alpha}{}^{\varepsilon}{}_{\varphi\gamma} + \\
&+ \underset{(21)}{P}{}^{\varphi}{}_{\beta\gamma}\underset{(1i)}{S}{}_{\alpha}{}^{\varepsilon}{}_{\delta\varphi} + \overset{V_i}{\underset{(11)}{P}}{}^{\varphi}{}_{\beta\delta}\underset{(2i)}{Q}{}_{\alpha}{}^{\varepsilon}{}_{\varphi\gamma} - \underset{(i2)}{C}{}^{\varphi}{}_{\delta\gamma}\underset{(1i)}{P}{}_{\alpha}{}^{\varepsilon}{}_{\beta\varphi} + \\
&+ \overset{V_i}{\underset{(22)}{P}}{}^{\varphi}{}_{\beta\gamma}\underset{(2i)}{Q}{}_{\alpha}{}^{\varepsilon}{}_{\delta\varphi} + \underset{(12)}{P}{}^{\varphi}{}_{\beta\delta}\underset{(2i)}{S}{}_{\alpha}{}^{\varepsilon}{}_{\varphi\gamma} - \overset{V_i}{\underset{(22)}{Q}}{}^{\varphi}{}_{\delta\gamma}\underset{(2i)}{P}{}_{\alpha}{}^{\varepsilon}{}_{\beta\varphi} = 0\,,\ (i=0,1,2)\,,
\end{aligned} \tag{54}$$

$$\begin{aligned}
&\underset{(2i)}{P}{}_{\alpha}{}^{\varepsilon}{}_{\beta\gamma}\overset{(2)}{|}{}_{i\delta} - \underset{(2i)}{P}{}_{\alpha}{}^{\varepsilon}{}_{\beta\delta}\overset{(2)}{|}{}_{i\gamma} + \underset{(2i)}{S}{}_{\alpha}{}^{\varepsilon}{}_{\gamma\delta|i\beta} - \\
&- \underset{(i2)}{C}{}^{\varphi}{}_{\beta\gamma}\underset{(2i)}{P}{}_{\alpha}{}^{\varepsilon}{}_{\varphi\delta} + \underset{(i2)}{C}{}^{\varphi}{}_{\beta\delta}\underset{(2i)}{P}{}_{\alpha}{}^{\varepsilon}{}_{\varphi\gamma} - \\
&- \underset{(21)}{P}{}^{\varphi}{}_{\beta\gamma}\underset{(2\alpha)}{Q}{}_{\alpha}{}^{\varepsilon}{}_{\varphi\delta} + \underset{(21)}{P}{}^{\varphi}{}_{\beta\delta}\underset{(2\alpha)}{Q}{}_{\alpha}{}^{\varepsilon}{}_{\varphi\gamma} - \\
&- \overset{V_i}{\underset{(22)}{P}}{}^{\varphi}{}_{\beta\gamma}\underset{(2i)}{S}{}_{\alpha}{}^{\varepsilon}{}_{\varphi\delta} + \overset{V_i}{\underset{(12)}{P}}{}^{\varphi}{}_{\beta\delta}\underset{(2i)}{S}{}_{\alpha}{}^{\varepsilon}{}_{\varphi\gamma} - \overset{V_i}{\underset{(2)}{S}}{}^{\varphi}{}_{\gamma\delta}\underset{(2i)}{P}{}_{\alpha}{}^{\varepsilon}{}_{\beta\varphi} = 0\,,\ (i=0,1,2)\,,
\end{aligned} \tag{55}$$

$$\sum^{0}\left[\underset{(1i)}{S}{}_{\alpha}{}^{\varepsilon}{}_{\beta\gamma}\overset{(1)}{|}_{i\delta}+\overset{V_i}{\underset{(1)}{S}}{}^{\varphi}_{\beta\gamma}\underset{(1i)}{S}{}_{\alpha}{}^{\varepsilon}{}_{\delta\varphi}+\underset{(12)}{S}{}^{\varphi}_{\beta\gamma}\underset{(2i)}{Q}{}_{\alpha}{}^{\varepsilon}{}_{\delta\varphi}\right]=0\,,\ (i=0,1,2)\,, \tag{56}$$

$$\begin{aligned}
&\underset{(1i)}{S}{}_{\alpha}{}^{\varepsilon}{}_{\beta\gamma}\overset{(2)}{|}_{i\delta}-\underset{(2i)}{Q}{}_{\alpha}{}^{\varepsilon}{}_{\beta\delta}\overset{(1)}{|}_{i\gamma}+\underset{(2i)}{Q}{}_{\alpha}{}^{\varepsilon}{}_{\gamma\delta}\overset{(1)}{|}_{i\beta}-\\
-\ &\overset{V_i}{\underset{(1)}{S}}{}^{\varphi}_{\beta\gamma}\underset{(2i)}{Q}{}_{\alpha}{}^{\varepsilon}{}_{\varphi\delta}-\underset{(i2)}{C}{}^{\varphi}_{\beta\delta}\underset{(1i)}{S}{}_{\alpha}{}^{\varepsilon}{}_{\gamma\varphi}+\underset{(i2)}{C}{}^{\varphi}_{\gamma\delta}\underset{(1i)}{S}{}_{\alpha}{}^{\varepsilon}{}_{\beta\varphi}-\\
-\ &\underset{(12)}{S}{}^{\varphi}_{\beta\gamma}\underset{(2i)}{S}{}_{\alpha}{}^{\varepsilon}{}_{\varphi\delta}-\overset{V_i}{\underset{(22)}{Q}}{}^{\varphi}_{\beta\delta}\underset{(2i)}{Q}{}_{\alpha}{}^{\varepsilon}{}_{\gamma\varphi}+\overset{V_i}{\underset{(22)}{Q}}{}^{\varphi}_{\gamma\delta}\underset{(2i)}{Q}{}_{\alpha}{}^{\varepsilon}{}_{\beta\varphi}=0\,,\ (i=0,1,2)\,,
\end{aligned} \tag{57}$$

$$\begin{aligned}
&\underset{(2i)}{Q}{}_{\alpha}{}^{\varepsilon}{}_{\beta\gamma}\overset{(2)}{|}_{i\delta}-\underset{(2i)}{Q}{}_{\alpha}{}^{\varepsilon}{}_{\beta\delta}\overset{(2)}{|}_{i\gamma}+\underset{(2i)}{S}{}_{\alpha}{}^{\varepsilon}{}_{\gamma\delta}\overset{(1)}{|}_{i\beta}-\\
-\ &\underset{(i2)}{C}{}^{\varphi}_{\beta\gamma}\underset{(2i)}{Q}{}_{\alpha}{}^{\varepsilon}{}_{\varphi\delta}+\underset{(i2)}{C}{}^{\varphi}_{\beta\delta}\underset{(2\alpha)}{Q}{}_{\alpha}{}^{\varepsilon}{}_{\varphi\gamma}-\overset{V_i}{\underset{(2)}{S}}{}^{\varphi}_{\gamma\delta}\underset{(2i)}{Q}{}_{\alpha}{}^{\varepsilon}{}_{\beta\varphi}-\\
-\ &\overset{V_i}{\underset{(22)}{Q}}{}^{\varphi}_{\beta\gamma}\underset{(2i)}{S}{}_{\alpha}{}^{\varepsilon}{}_{\varphi\delta}+\overset{V_i}{\underset{(22)}{Q}}{}^{\varphi}_{\beta\delta}\underset{(22)}{S}{}_{\alpha}{}^{\varepsilon}{}_{\varphi\gamma}=0\,,\ (i=0,1,2)\ ,
\end{aligned} \tag{58}$$

$$\sum^{0}\left[\underset{(2i)}{S}{}_{\alpha}{}^{\varepsilon}{}_{\beta\gamma}\overset{(2)}{|}_{i\delta}+\overset{V_i}{\underset{(22)}{S}}{}^{\varphi}_{\beta\gamma}\underset{(22)}{S}{}_{\alpha}{}^{\varepsilon}{}_{\delta\varphi}\right]=0\,,\ (i=0,1,2)\,. \tag{59}$$

Here, everywhere, $\overset{0}{\sum}$ means cyclic sum over (δ,γ,β).

Proof. Taking into account that the indices $A, B, C, D...$ are of type $\alpha,\beta,\gamma,\delta$ and the torsion $\mathbb{T}^{C}_{AB}$ and curvature $\mathbb{R}^{D}_{ABC}$ adapted components are given in (37), (39) and (40), after laborious local computations, the formulas (41) imply the required Bianchi identities.

□

Remark 1. *We point out that, the induced tangent connection* $D^{\top}\Gamma\left(\check{N}\right) = \left(\overset{V_i}{\underset{(i0)}{L}}{}^{\alpha}_{\beta\delta}, \overset{V_i}{\underset{(i1)}{C}}{}^{\alpha}_{\beta\delta}, \overset{V_i}{\underset{(i2)}{C}}{}^{\alpha}_{\beta\delta}\right), (i=0,1,2; V_0=H)$, *(28) does not coincide with the canonical intrinsic N-linear metric connection of the submanifold* $Osc^2\check{M}$, $D\Gamma\left(N\right) = \left(\overset{V_i}{\underset{(i0)}{L}}{}^{\alpha}_{\beta\delta}, \overset{V_i}{\underset{(i1)}{C}}{}^{\alpha}_{\beta\delta}, \overset{V_i}{\underset{(i2)}{C}}{}^{\alpha}_{\beta\delta}\right), (i=0,1,2; V_0=H)$. *For this reason, the Bianchi identities produced by these N-linear connections do not coincide.*

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