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MAXIMUM FLOW IN A NETWORK WITH AN UNDERESTIMATED ARC CAPACITY

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Abstract

There are problems arising in real life that can be modeled and solved as maximum flow problems in several networks, some of these differing only by an arc capacity. Suppose that we have previously determined a maximum flow in a network G, but we also need to find a maximum flow in a network having the same structure as G and the same arc capacities excepting one: the capacity of a given arc (k, l) which has increased by a given amount a. In this paper, we will show how a maximum flow in the new network can be obtained in $O(am)$ time starting from the maximum flow in the initial network.

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1 Introduction

The maximum flow problem is included in the group of network flow problems with widespread and diverse applications. This is one of the reasons for which the literature on network flow problems is extensive. Over the past 6 decades, researchers have made continuous improvements to algorithms for determining maximum flows. There are many research contributions concerning the improvement of the running time of maximum flow algorithms by using enhanced data structures, techniques of scaling the problem data, etc.

The maximum flow algorithms developed until nowadays can be of one of the following two types:

1. augmenting path algorithms
2. preflow algorithms

Another method for improving the complexity of the maximum flow is to use wisely the particularities of the networks.

In this paper we focus on the problem of finding a maximum flow in a network differing only by an arc capacity from another network in which a maximum flow was already established.

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2 Notation and definition

Let $G = (N, A)$ be a directed graph, defined by a set N of n nodes and a set A of m arcs. Each arc $(x, y) \in A$ has a nonnegative capacity $c(x, y)$. In the directed network $G = (N, A, c, s, t)$, two special nodes are specified: s is the source node and t is the sink node.

Let X and Y be two subsets of the node set N. We define the set of arcs $(X, Y) = \{(x, y)|(x, y) \in A,\ x \in X, y \in Y\}$.

For any function $g : N \times N \to \mathbb{R}^+$ and for any function $h : N \to \mathbb{R}^+$ we define

$$g(X, Y) = \sum_{(X,Y)} g(x, y)$$

and

$$h(X) = \sum_{X} h(x).$$

If $X = \{x\}$ or $Y = \{y\}$ then we will use $g(x, Y)$ or $g(X, y)$ instead of $g(X, Y)$.

A *flow* from the source node s to the sink node t in the directed network $G = (N, A, c, s, t)$ is a function $f : A \to \mathbb{R}^+$ which meets the follwing conditions:

$$f(x, N) - f(N, x) = \begin{cases} v, x = s \\ 0, x \neq s, t \\ -v, x = t \end{cases} \tag{1}$$

$$0 \leq f(x, y) \leq c(x, y), \qquad \forall (x, y) \in A. \tag{2}$$

We refer to v as the *value* of the flow f. A flow whose value is maximum is called a *maximum flow.*

A *preflow* is a function $f : A \to \mathbb{R}^+$ satisfying relations (2) and the next conditions:

$$f(x, N) - f(N, x) \geq 0, \qquad \forall x \in N \backslash \{s, t\}. \tag{3}$$

Let f be a preflow. We define the *excess* of a node $x \in N$ in the following manner:

$$e(x) = f(x, N) - f(N, x)$$

Thus, for any preflow f, we have $e(x) \geq 0, \forall x \in N \backslash \{s, t\}$. We say that a node $x \in N \backslash \{s, t\}$ is active if $e(x) > 0$ and balanced if $e(x) = 0$. A preflow f for which $e(x) = 0, \forall x \in N \backslash \{s, t\}$ is a flow. Consequently, a flow is a particular case of preflow.

Let f be a flow from the source node s to the sink node t in the directed network $G = (N, A, c, s, t)$. The *residual capacity* of the arc (x, y) corresponding to the flow f is defined as $r(x, y) = c(x, y) - f(x, y) + f(y, x)$ and it is the maximum amount of additional flow that can be sent from x to y using both arcs (x, y) and

(y, x). By convention, if an arbitrary arc $(x, y) \notin A$, then we can add (x, y) to A and we will consider that $c(x, y) = 0$.

The *residual network* $G(f) = (N, A(f))$ corresponding to flow f contains all those arcs with strictly positive residual capacity.

3 Determining a maximum flow in a network with an underestimated arc capacity

Suppose that we have already established a maximum flow f of value v in the network $G = (N, A, c, s, t)$ and now we need to find a maximum flow in a network $G' = (N, A, c', s, t)$ having the same structure and the same capacities, except one, as G. So, the only difference between these two networks is the capacity of a given arc (k, l), which is greater in G' as in G. So, $c'(x, y) = c(x, y), \forall(x, y) \in A \backslash \{(k, l)\}$ and $c'(k, l) = c(k, l) + a$, where $a > 0$.

Obviously, the greater capacity of the arc (k, l) might imply the existence of additional augmenting paths and, consequently, the existence of maximum flow with a greater than v value in G'. But this isn't mandatory. If the residual network $G(f)$ with respect to the maximum flow f contains the arc (k, l), then f is also a maximum flow in G'. Otherwise, to increase by a the capacity of the arc (k, l) implies to add an arc (k, l) having the residual capacity equal to a to the residual network. This means that new augmening paths might appear, which implies that the maximum flow value in G' is at least v. In this case, for determining a maximum flow in G' starting with the maximum flow in G as an initial flow in G', there are two approaches:

1. an augmenting path algorithm approach
2. a preflow algorithm approach

Our approach is based on the generic augmenting path algorithm. Because in G' the capacity of the arc (k, l) is greater by a units than the capacity of the same arc in network G, the residual capacity of the arc (k, l) in $G'(f)$ is greater by a than the residual capacity of the same arc in $G(f)$. Here two cases might appear:

Case 1: The residual network $G(f)$ with respect to the maximum flow f contains the arc (k, l). In this case f is also a maximum flow in the network G'.

Case 2: The residual network $G(f)$ with respect to the maximum flow f doesn't contain the arc (k, l). In this case the residual network $G'(f)$ is obtained from $G(f)$ by adding the arc (k, l) having the residual capacity equal to a. The addition of the arc (k, l) to the residual network means that it is possible, but not mandatory, that the residual network $G'(f)$ contains augmenting paths through which additional flow can be sent. If there are such paths, they will all contain the arc (k, l). If we apply the generic augmenting path algorithm (see [1])in $G'(f)$, it will perform at most a flow augmentations. Since the time complexity of a flow augmentation is $O(m)$, it follows that a maximum flow in G' can be obtained in $O(am)$ time starting with a maximum flow in G.

The other approach for determining a maximum flow in G' starting with the maximum flow in G is a generic preflow algorithm approach and it is based on the incremental algorithm developed by S. Kumar and P. Gupta in [3]. Their algorithm solves the problem of determining a maximum flow in a network after inserting a new arc, which is equivalent to the problem that we study in this paper because the addition of a new arc (k, l) with capacity equal to a can be regarded as the augmentation of the capacity of the arc (k, l) from 0 to a. After inserting a new arc, new augmenting paths might appear. We will refer to a node that is contained in at least one of these new augmenting paths as an affected node. The set of all affected nodes will be denoted by AN. The affected nodes are contained in the directed paths from the source node s to the node k and in the directed paths from the node l to the sink node t in the residual network. Consequently, they can be determined by applying a modified Backward Breadth First Search algorithm from node k to node s and by applying a modified Breadth First Search algorithm starting from node l to node t.

The algorithm developed by S. Kumar and P. Gupta in [3] is obtained by modifying the generic preflow algorithm in the following ways: 1) it starts with a maximum flow f in G; 2) it saturates only the arcs outgoing from s and incoming into an affected node and 3) when relabeling a node it uses only its affected successors. Their algorithm establishes a maximum flow in G' in $O(|AN|^2 m)$ time.

Consequently, if the capacity of the arc (k, l) is incremented by a small amount a (if a is smaller than the square of the number of affected nodes) our algorithm outperforms the algorithm developed by S. Kumar and P. Gupta.

References

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