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ON GENERALIZED WEAKLY SYMMETRIC MANIFOLDS

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Abstract

The purpose of the study is to introduce a new type of space, called generalized weakly symmetric space. The existence of such space is ensured by a non-trivial example.

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1 Introduction

The notion of weakly symmetric Riemannian manifold has been introduced by Tamássy and Binh [25]. Thereafter, a lot of research has been carried out in this topic. For details, we refer to [10], [13], [14], [16], [19], [20], [21], [22], [24] and the references there in. In the spirit of Tamássy and Binh [25], a non-flat n-dimensional Riemannian manifold $(M^n, g)(n > 2)$, is said to be weakly symmetric manifold, if its curvature tensor $\bar{R}$ of type $(0, 4)$ is not identically zero and satisfies the relation

$$\begin{aligned}(\nabla_X \bar{R})(Y, U, V, W) = {} & A(X)\bar{R}(Y, U, V, W) + B(Y)\bar{R}(X, U, V, W) \\ & + B(U)\bar{R}(Y, X, V, W) + D(V)\bar{R}(Y, U, X, W) \\ & + D(W)\bar{R}(Y, U, V, X) \qquad (1)\end{aligned}$$

where A, B, & D are non-zero 1-forms defined by $A(X) = g(X, \theta_1)$, $B(X) = g(X, \varrho_1)$ and $D(X) = g(X, \phi_1)$ for all X and $\bar{R}(Y, U, V, W) = g(R(Y, U)V, W)$, ∇ being the operator of the covariant differentiation with respect to the metric tensor g. An n-dimensional manifold of this kind is denoted by $(WS)_n$.

Keeping in tune with Dubey [12], we introduce a new type of space called generalized weakly symmetric manifold which is abbreviated by $(GWS)_n$-space and defined as follows

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A non-flat n-dimensional Riemannian manifold $(M^n; g)$ $(n > 2)$, is termed as generalized weakly symmetric manifold, if its Riemannian curvature tensor $\bar{R}$ of type $(0; 4)$ is not identically zero and admits the identity

$$\begin{aligned}(\nabla_X \bar{R})(Y,U,V,\ W) \quad = \quad & A(X)\bar{R}(Y,U,V,\ W) + B(Y)\bar{R}(X,U,V,\ W) \\ & +B(U)X\bar{R}(Y,X,V,\ W) + D(V)\bar{R}(Y,U,X,\ W) \\ & +D(W)\bar{R}(Y,U,V,X\ ) + \alpha(X)G(Y,U,V,\ W) \\ & +\beta(Y)G(X,U,V,\ W) + \beta(U)\ G(Y,X,V,\ W) \\ & +\gamma(V)\ G(Y,U,X,\ W) + \gamma(W)\ G(Y,U,V,X\ ) \quad (2)\end{aligned}$$

where

$$G(Y,U,V,\ W) = [g(U,V)g(Y,\ W) - g(Y,\ V)g(U,\ W)] \quad (3)$$

and $A,\ B,\ D,\ \alpha,\ \beta\ \&\ \ \gamma$ are non-zero 1-forms which are defined as $A(X) = g(X, \theta_1), B(X) = g(X, \phi_1),\ D(X) = g(X, \pi_1),\ \alpha(X) = g(X, \theta_2), \beta(X) = g(X, \phi_2)$ and $\gamma(X) = g(X, \pi_2)$. The local expression of (2) is

$$\begin{aligned}R_{mnpq,k} \quad = \quad & A_k\bar{R}_{mnpq} + B_m\bar{R}_{knpq} + B_n\bar{R}_{mkpq} + D_p\bar{R}_{mnkq} + D_q\bar{R}_{mnpk} \\ & +\alpha_k G_{mnpq} + \beta_m G_{knpq} + \beta_n G_{mkpq} + \gamma_p G_{mnkq} + \gamma_q G_{mnpk}, \quad (4)\end{aligned}$$

where $A_i,\ B_i,\ D_i, \alpha_i,\ \beta_i$ and γ_i are non-zero co-vectors. The beauty of such $(GWS)_n$-space is that it has the flavour of

(i) symmetric space[6] (for $A = B = D = \alpha = \beta = \gamma = 0$),

(ii) recurrent space[29] (for $A \neq 0$ and $B = D = \alpha = \beta = \gamma = 0$),

(iii) generalized recurrent space[12] (for $A \neq 0, \alpha \neq 0,\ B = D = \beta = \gamma = 0$),

(iv) pseudo symmetric space[7] (for $\frac{A}{2} = B = D = \delta \neq 0,\ \alpha = \beta = \gamma = 0$),

(v) generalized pseudo symmetric space[1] (for $\frac{A}{2} = B = D = \delta \neq 0$ & $\frac{\alpha}{2} = \beta = \gamma = \mu \neq 0$),

(vi) semi-pseudo symmetric space[27] (for $B = D = \delta \neq 0,\ A = \alpha = \beta = \gamma = 0$),

(vii) generalized semi-pseudo symmetric space[2] (for $A = 0 = \alpha, B = D = \delta \neq 0$ & $\beta = \gamma = \mu \neq 0$),

(viii) almost pseudo symmetric space[8] (for $A = E + H, B = D = H$ & $\alpha = \beta = \gamma = 0$),

(ix) almost generalized pseudo symmetric space([3], [4], [5]) (for $A = E + H, B = D = H$ & $\alpha = \lambda + \psi, \beta = \gamma = \lambda$) and

(x) weakly symmetric space [25](for $A, B, D \neq 0$ & $\alpha = \beta = \gamma = 0$).

We organized this paper as follows; Section 2 is concerned with some results on $(GWS)_n$-manifold. Among other things, it is pointed out that a weakly concirculaly symmetric space can always be considered as a generalized weakly symmetric space. In section 3, the existence of a generalized weakly symmetric space is ensured by a non-trivial example.

2 Some results on $(GWS)_n$-manifold

In this section, we consider a Riemann manifold (M^n, g) $n > 2$ which is generalized weakly symmetric. Now, contracting (4) we find

$$\begin{aligned}(\nabla_X S)(Y,\ W) &= A(X)S(Y,\ W) + B(Y)S(X,\ W) + D(W)S(Y,\ X) \\ &\quad -B(R((Y,X)W) + D(R(X,\ W)Y) + (n-1)[\alpha(X) \\ &\quad g(Y,\ W) + \beta(Y)g(X,\ W) + \gamma(W)g(Y,X\ )] \\ &\quad -\beta(Y)\ g(X,W) +\ [\beta(X)+\gamma(X)]\ g(Y,W) - \gamma(W)g(X,\ Y)\end{aligned} \tag{5}$$

which yields after further contraction

$$\begin{aligned}dr(X) &= A(X)\ r +\ 2B_1(X) + 2\ D_1(X) \\ &\quad +(n-1)[n\alpha(X) + 2\beta(X)\ + 2\gamma(X)]\end{aligned} \tag{6}$$

where $\bar{A}_1(X) = S(X, \theta_1)$.

Next, if we suppose that the scalar curvature of a $(GWS)_n$-space is non-zero constant, then (6) becomes

$$[A(X)\ r +\ 2B_1(X) + 2\ D_1(X) = -(n-1)[n\alpha(X) + 2\beta(X)\ + 2\gamma(X)]. \tag{7}$$

This leads to the following:

Theorem 1. *The 1-forms of a $(GWS)_n$-manifold are related by the expression (7) provided that the scalar curvature is non-zero constant.*

Definition 1. *[1] A non-flat n-dimensional Riemannian manifold (M^n, g) $(n > 2)$, is said to be a generalized weakly Ricci-symmetric manifold (which is abbreviated by $(GWRS)_n$-space), if its Ricci tensor S of type (0, 2) is not identically zero and admits the identity*

$$\begin{aligned}(\nabla_X S)(Y,\ W) &= \bar{A}(X)S(Y,\ W) + \bar{B}(Y)S(X,\ W) \\ &\quad +\bar{D}(W)S(Y,\ X) + \bar{\alpha}(X)(Y,W) \\ &\quad +\bar{\beta}(Y)g(X,W) + \bar{\gamma}(W)\ g(Y,X\ )\end{aligned} \tag{8}$$

where $\bar{A}$, $\bar{B}$, $\bar{D}$, $\bar{\alpha}$, $\bar{\beta}$ & $\bar{\gamma}$ are non-zero 1-forms which are defined as $\bar{A}(X) = g(X, \bar{\theta}_1)$, $\bar{B}(X) = g(X, \bar{\phi}_1)$, $\bar{D}(X) = g(X, \bar{\pi}_1)$, $\bar{\alpha}(X) = g(X, \bar{\theta}_2)$, $\bar{\beta}(X) = g(X, \bar{\phi}_2)$ and $\bar{\gamma}(X) = g(X, \bar{\pi}_2)$. In particular, if $\bar{\alpha} = \bar{\beta} = \bar{\gamma} = 0$, the relation (8) reduces to weakly Ricci symmetric manifold[26].

Theorem 2. *A $(GWS)_n$-manifold is necessarily a $(GWPRS)_n$-manifold provided that the relation*

$$S(X, \phi_1) + S(X, \pi_1) = -(n-1)[\beta(X) + \gamma(X)] \tag{9}$$

holds for all X.

Proof. It follows directly from (5). □

Again, in analogous with the definition of a $(GWS)_n$-manifold, a non-flat concircular curvature tensor

$$\bar{E}(Y,U,V,\ W) = \bar{R}(Y,U,V,\ W) - \frac{r}{n(n-1)}G(Y,U,V,\ W), \tag{10}$$

is called a weakly concirculaly symmetric if it satisfies the identity

$$\begin{aligned}(\nabla_X \bar{E})(Y,U,V,\ W) &= A(X)\ \bar{E}(Y,U,V,\ W) + B(Y)\bar{E}(X,U,V,\ W) \\ &\quad +B(U)\bar{E}(Y,X,V,\ W) + D(V)\bar{E}(Y,U,X,\ W) \\ &\quad +D(W)\bar{E}(Y,U,V,X\ ).\end{aligned} \tag{11}$$

We observe that for the following choice of the 1-forms

$$\begin{aligned}\alpha(X) &= \frac{[dr(X) - rA(X)]}{n(n-1)},\ \beta(X) = -\frac{r}{n(n-1)}B(X) \\ \&\ \ \gamma(X) &= -\frac{r}{n(n-1)}D(X)\ \ \forall\ \ X.\end{aligned}$$

the equation (11) turns into (5). This motivates us to state

Theorem 3. *A weakly concircularly symmetric space is necessarily a $(GWS)_n$-space.*

However, the converse of the above Theorem may not true.

Again, contracting Y over W in (8) we obtain

$$\begin{aligned}(\nabla_X S)(U,V) - \frac{dr(X)}{n}g(U,V) &= A(X)[S(U,V) - \frac{r}{n}g(U,V)] \\ &\quad +B(U)[S(X,\ V) - \frac{r}{n}g(X,\ V)] \\ &\quad -B(E(U,X)V) + D(E(X,V)U) \\ &\quad +D(V)[S(U,X) - \frac{r}{n}g(U,\ X)]\end{aligned} \tag{12}$$

which yields

$$\begin{aligned}(\nabla_X Z)(Y,\ V) &= A(X)Z(Y,\ V) + B(Y)Z(X,\ V) \\ &\quad +D(V)Z(Y,\ X) \\ &\quad -B(E(Y,X)V) + D(E(X,V)Y)\end{aligned} \tag{13}$$

where Z stands for a well known Z-tensor ([17], [18]). This leads us to state

Theorem 4. *A weakly concircularly symmetric space is a weakly Z-symmetric space if*

$$Z(U,\phi) = Z(U,\pi). \tag{14}$$

Proof. It follows directly from (13). □

3 Existence of generalized weakly symmetric space

Example 1. *Let* $(\mathbb{R}^4, g)$ *be a* 4*-dimensional Riemannian space endowed with the Riemann metric* g *given by*

$$ds^2 = g_{ij}dx^i dx^j == (x^4)^{4/3}[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] - (dx^4)^2, \qquad (15)$$

(i, j = 1, 2, 3, 4).

The non-zero components of Riemannian curvature tensor are

$$\begin{aligned} R_{1212} &= \frac{4}{9}(x^4)^{2/3} = R_{1313} = R_{2323}, \\ R_{1414} &= R_{2424} = R_{3434} = \frac{2}{9(x^4)^{2/3}}, \end{aligned}$$

Making use of (15), we can easily bring out

$$\begin{aligned} G_{1212} &= G_{1313} = G_{2323} = -(x^4)^{8/3}, \\ G_{1414} &= G_{2424} = G_{3434} = -(x^4)^{4/3} \end{aligned}$$

Covariant derivatives of Riemannian curvature tensors are

$$\begin{aligned} R_{1212,4} &= R_{1313,4} = R_{2323,4} = -\frac{8}{9(x^4)^{1/3}}, \\ R_{1214,2} &= R_{1314,3} = R_{2324,3} = -\frac{4}{9(x^4)^{1/3}}, \\ R_{1224,1} &= R_{1334,1} = R_{2334,2} = \frac{4}{9(x^4)^{1/3}}, \\ R_{1414,4} &= R_{2424,4} = R_{3434,4} = -\frac{4}{9(x^4)^{5/3}}. \end{aligned}$$

For the following choice of the 1-forms

$$\begin{aligned} A_i &= \frac{1}{x^4}, \text{ for } i = 4 \\ &= 0, \text{ otherwise} \\ B_i &= -\frac{19}{3x^4}, \text{ for } i = 4 \\ &= 0, \text{ otherwise} \\ D_i &= -\frac{19}{3x^4}, \text{ for } i = 4 \\ &= 0, \text{ otherwise} \\ \alpha_i &= \frac{8}{9(x^4)^3}, \text{ for } i = 4 \\ &= 0, \text{ otherwise} \\ \beta_i &= -\frac{64}{27(x^4)^3}, \text{ for } i = 4 \\ &= 0, \text{ otherwise} \\ \gamma_i &= -\frac{64}{27(x^4)^3}, \text{ for } i = 4 \\ &= 0, \text{ otherwise} \end{aligned}$$

one can easily fit the followings

$$
\begin{aligned}
R_{1212,k} &= A_k R_{1212} + B_1 R_{k212} + B_2 R_{1k12} + D_1 R_{12k2} + D_2 R_{121k} \\
&\quad + \alpha_k\, G_{1212} + \beta_1 G_{k212} + \beta_2 G_{1k12} + \gamma_1 G_{12k2} + \gamma_2 G_{121k}, \\
R_{1313,k} &= A_k\, R_{1313} + B_1 R_{k313} + B_3 R_{1k13} + D_1 R_{13k3} + D_3 R_{131k} \\
&\quad + \alpha_k\, G_{1313} + \beta_1 G_{k313} + \beta_3 G_{1k13} + \gamma_1 G_{13k3} + \gamma_3 G_{131k}, \\
R_{1414,k} &= A_k R_{1414} + B_1 R_{k414} + B_4 R_{1k14} + D_1 R_{14k4} + D_4 R_{141k} \\
&\quad + \alpha_k\, G_{1414} + \beta_1 G_{k414} + \beta_4 G_{1k14} + \gamma_1 G_{14k4} + \gamma_4 G_{141k}, \\
R_{1214,k} &= A_k R_{1214} + B_1 R_{k214} + B_2 R_{1k14} + D_1 R_{12k4} + D_4 R_{121k} \\
&\quad + \alpha_k\, G_{1214} + \beta_1 G_{k214} + \beta_2 G_{1k14} + \gamma_1 G_{12k4} + \gamma_4 G_{121k}, \\
R_{1314,k} &= A_k\, R_{1314} + B_1 R_{k314} + B_3 R_{1k14} + D_1 R_{13k4} + D_4 R_{131k} \\
&\quad + \alpha_k\, G_{1314} + \beta_1 G_{k314} + \beta_3 G_{1k14} + \gamma_1 G_{13k4} + \gamma_4 G_{131k}, \\
R_{1224,k} &= A_k\, R_{1224} + B_1 R_{k224} + B_2 R_{1k24} + D_2 R_{12k4} + D_4 R_{122k} \\
&\quad + \alpha_k\, G_{1224} + \beta_1 G_{k224} + \beta_2 G_{1k24} + \gamma_2 G_{12k4} + \gamma_4 G_{122k}, \\
R_{1334,k} &= A_k\, R_{1334} + B_1 R_{k334} + B_3 R_{1k34} + D_3 R_{13k4} + D_4 R_{133k} \\
&\quad + \alpha_k\, G_{1334} + \beta_1 G_{k334} + \beta_3 G_{1k34} + \gamma_3 G_{13k4} + \gamma_4 G_{133k}, \\
R_{2323,k} &= A_k\, R_{2323} + B_2 R_{k323} + B_3 R_{2k23} + D_2 R_{23k3} + D_3 R_{232k} \\
&\quad + \alpha_k\, G_{2323} + \beta_2 G_{k323} + \beta_3 G_{2k23} + \gamma_2 G_{23k3} + \gamma_3 G_{232k}, \\
R_{2324,k} &= A_k\, R_{2324} + B_2 R_{k324} + B_3 R_{2k24} + D_2 R_{23k4} + D_4 R_{232k} \\
&\quad + \alpha_k\, G_{2324} + \beta_2 G_{k324} + \beta_3 G_{2k24} + \gamma_2 G_{23k4} + \gamma_4 G_{232k}, \\
R_{2334,k} &= A_k R_{2334} + B_2 R_{k334} + B_3 R_{2k34} + D_3 R_{23k4} + D_4 R_{233k} \\
&\quad + \alpha_k\, G_{2334} + \beta_2 G_{k334} + \beta_3 G_{2k34} + \gamma_3 G_{23k4} + \gamma_4 G_{233k}, \\
R_{2424,k} &= A_k R_{2424} + B_2 R_{k424} + B_4 R_{2k24} + D_2 R_{24k4} + D_4 R_{242k} \\
&\quad + \alpha_k\, G_{2424} + \beta_2 G_{k424} + \beta_4 G_{2k24} + \gamma_2 G_{24k4} + \gamma_4 G_{242k}, \\
R_{3434,k} &= A_k\, R_{3434} + B_3 R_{k434} + B_4 R_{3k34} + D_3 R_{34k4} + D_4 R_{343k} \\
&\quad + \alpha_k\, G_{3434} + \beta_3 G_{k434} + \beta_4 G_{3k34} + \gamma_3 G_{34k4} + \gamma_4 G_{343k},
\end{aligned}
$$

where, $k = 1, 2, 3, 4$. As a consequence of the above one can say that

Theorem 5. *There exists a* $(\mathbb{R}^4, g)$ *which is a generalized weakly symmetric space with non-zero and non-constant scalar curvature for the above mentioned choice of the i-forms.*

It is clear that the manifold under considered metric can't be symmetric, recurrent, generalized recurrent and almost pseudo symmetric space.

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