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MORE ON η-RICCI SOLITONS IN $(LCS)_n$-MANIFOLDS

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Abstract

The object of the present paper is to study generalized weakly symmetric $(LCS)_n$-manifolds whose metric tensor is η-Ricci soliton. The paper also aims to bring out curvature conditions for which η-Ricci solitons in $(LCS)_n$-manifolds are sometimes shrinking or expanding and some other time remain steady.

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1 Introduction

The notion of Lorentzian concircular structure manifolds (briefly $(LCS)_n$-manifolds) has been initiated by Shaikh [26]. Thereafter, a lot of study has been carried out. For details, we refer [27], [28], [30], [31] and the references therein.

The notion of weakly symmetric Riemannian manifolds has been introduced by Tamássy and Binh [32]. Thereafter, it became focus of interest for many geometers. For details, we refer to see [14], [18], [19], [21], [22], [24], [25], [29] and the references therein.

Keeping the spirit of Dubey [16], recently the present author has introduced a new type of space called generalized weakly symmetric manifold[6] which is abbreviated hereafter as $(GWS)_n$-manifold). An n-dimensional Riemannian manifold is said to be $(GWS)_n$-manifold, if it admits the equation

$$\begin{aligned} &(\nabla_X \bar{R})(Y, Z,U,V) \\ = \; & A(X)\bar{R}(Y, Z,U,V) + B(Y)\bar{R}(X, Z,U,V) + B(Z)\bar{R}(Y,X,U, V) \\ &+D(U)\bar{R}(Y, Z,X,V) + D(V)\bar{R}(Y,Z,U, X) + \alpha(X)\bar{G}(Y, Z,U,V) \\ &+\beta(Y)\bar{G}(X, Z,U,V) + \beta(Z)\,\bar{G}(Y,X,U, V) \\ &+\gamma(U)\,\bar{G}(Y,Z,X,V) + \gamma(V)\,\bar{G}(Y,Z,U,X) \end{aligned} \quad (1)$$

where

$$\bar{G}(Y,U,V, W) = [g(U,V)g(Y, W) - g(Y, V)g(U, W)] \quad (2)$$

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and $A, B, D, \alpha, \beta,$ & γ are non-zero 1-forms defined by $A(X) = g(X, \sigma_1)$, $B(X) = g(X, \varrho_1), D(X) = g(X, \pi_1), \alpha(X) = g(X, \theta_2), \beta(X) = g(X, \varrho_2)$ and $\gamma(X) = g(X, \phi_2)$. The beauty of such $(GWS)_n$-space is that it has the flavour of

(i) symmetric space[11] (for $A = B = D = \alpha = \beta = \gamma = 0$),

(ii) recurrent space[35] (for $A \neq 0$ and $B = D = \alpha = \beta = \gamma = 0$),

(iii) generalized recurrent space[16] (for $A \neq 0, \alpha \neq 0, B = D = \beta = \gamma = 0$),

(iv) pseudo symmetric space[12] (for $\frac{A}{2} = B = D = \delta \neq 0$, $\alpha = \beta = \gamma = 0$),

(v) generalized pseudo symmetric space[3] (for $\frac{A}{2} = B = D = \delta \neq 0$ & $\frac{\alpha}{2} = \beta = \gamma = \mu$),

(vi) semi-pseudo symmetric space[34] (for $B = D = \delta$, $A = \alpha = \beta = \gamma = 0$),

(vii) generalized semi-pseudo symmetric space[4] (for $A = 0 = \alpha, B = D = \delta$ & $\beta = \gamma = \mu$),

(viii) almost pseudo symmetric space[13] (for $A = E + H, B = D = H$ & $\alpha = \beta = \gamma = 0$),

(ix) almost generalized pseudo symmetric space[5] (for $A = E + H, B = D = H$ & $\alpha = \lambda + \psi, \beta = \gamma = \lambda$) and

(x) weakly symmetric space [32](for $A, B, D \neq 0$ & $\alpha = \beta = \gamma = 0$).

Analogously, we have defined generalized weakly Ricci symmetric $(LCS)_n$-manifold which is defined as follows

An n-dimensional Riemannian manifold is said to be generalized weakly Ricci symmetric if it admits the equation

$$\begin{aligned}(\nabla_X S)(Y, Z) &= A_1^*(X)S(Y, Z) + B_1^*(Y)S(X, Z) + D_1^*(Z)S(Y, X) \\ &\quad + A_2^*(X)g(Y, Z) + B_2^*(Y)g(X, Z) + D_2^*(Z)g(Y, X) \qquad (3)\end{aligned}$$

where $A_i^*,\ B_i^*$ & D_i^* are non-zero 1-forms which are defined as $A_i^*(X) = g(X, \theta_i)$, $B_i^*(X) = g(X, \phi_i)$, $D_i^*(X) = g(X, \pi_i)$ for $i = 1, 2$. The beauty of generalized weakly Ricci symmetric manifold is that it has the flavour of Ricci symmetric, Ricci recurrent, generalized Ricci recurrent, pseudo Ricci symmetric, generalized pseudo Ricci symmetric, semi-pseudo Ricci symmetric, generalized semi-pseudo Ricci symmetric, almost pseudo Ricci symmetric, almost generalized pseudo Ricci symmetric and weakly Ricci symmetric space as special cases.

The study of the Ricci solitons in contact geometry has begun with the work of Ramesh Sharma [17], Cornelia Livia Bejan and Mircea Crasmareanu [9] and the references therein. Ricci solitons are defined as triples (g, V, λ), where (M, g) is a Riemannian manifold and V is a vector field (the potential vector field) so that the following equation is satisfied

$$\frac{1}{2}\$_V g + S + \lambda g = 0 \qquad (4)$$

where $\$$ denotes the Lie derivative, S is the Ricci tensor and λ a constant on M. A Ricci soliton is said to be shrinking, steady or expanding according to λ negative, zero and positive respectively. A Ricci soliton with V zero is reduced to Einstein equation.

During the last two decades, the geometry of Ricci solitons has been the focus of attention of many mathematicians ([1], [2], [8], [15]).

η-Ricci solitons (M, g, λ, μ) is the generalization of Ricci solitons (M, g, λ) which is defined as[10]

$$L_\xi g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0, \tag{5}$$

where L_ξ is the Lie derivative operator along the vector field ξ, S is the Ricci curvature tensor field of the metric g, λ and μ are real constants. In [7], authors have studied η-Ricci solitons in $(LCS)_n$-manifold satisfying semi-symmetric type of curvature restriction.

Our paper is structured as follows. Section 2 is concerned with $(LCS)_n$-manifolds and some known results. Generalized weakly symmetric $(LCS)_n$- -manifold whose metric tensor is η-Ricci soliton, was studied in section 3. It is observed that (i) Ricci soliton in each of locally symmetric, locally recurrent and generalized recurrent $(LCS)_n$-manifold is expanding, steady or shrinking accordingly as $(n-1)(\alpha^2-\rho) \gtreqqless \alpha$, (ii) Ricci soliton in each of pseudo symmetric, semi-pseudo symmetric, almost pseudo symmetric and weakly symmetric $(LCS)_n$-manifold is expanding, steady or shrinking accordingly as $(n-2)\left(\alpha^2-\rho\right)\alpha \gtreqqless [\alpha + B(\xi)]\left[\alpha + (\alpha^2-\rho)\right]$ & (iii) Ricci soliton in each of generalized pseudo symmetric, generalized semi-pseudo symmetric and almost generalized pseudo symmetric $(LCS)_n$-manifold is expanding, steady or shrinking accordingly as $(n-2)\left[(\alpha^2-\rho)\alpha - \beta(\xi)\right] \gtreqqless \left[\alpha + (\alpha^2-\rho)\right][\alpha + B(\xi)]$. In section 4, we studied generalized weakly Ricci-symmetric $(LCS)_n$-manifolds whose metric tensor is η-Ricci soliton and we obtained some interesting results.

2 $(LCS)_n$-manifolds and some known results

An $n$dimensionally Lorentzian manifold M is a smooth connected paracompact Hausdorff manifold with a Lorentzian metric g, that is, M admits a smooth symmetric tensor field g of type $(0,2)$ such that for each point $p \in M$, the tensor $g_p : T_pM \times T_pM \to R$ is a non-degenerate inner product of signature $(-, +, ..., +)$, where T_pM denotes the tangent vector space of M at p and R is the real number space. A non-zero vector $v \in T_pM$ is said to be timelike (resp., non-spacelike, null, spacelike) if it satisfies $g_p(U,U) < 0$ (resp, $\leq 0, = 0, > 0$)[23]. The category into which a given vector falls is called its causal character.

Let M^n be a Lorentzian manifold admitting a unit timelike concircular vector field ξ, called the characteristic vector field of the manifold. Then we have

$$g(\xi, \xi) = -1. \tag{6}$$

Since ξ is a unit concircular vector field, there exists a non-zero 1-form η such that for

$$g(X, \xi) = \eta(X) \tag{7}$$

the equation of the following form holds

$$(\nabla_X \eta)(Y) = \alpha\{g(X,Y) + \eta(X)\eta(Y)\} \qquad (\alpha \neq 0) \tag{8}$$

for all vector fields X, Y where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g and α is a non-zero scalar function satisfies

$$\nabla_X \alpha = (X\alpha) = \alpha(X) = \rho\eta(X), \tag{9}$$

ρ being a certain scalar function. If we put

$$\phi X = \frac{1}{\alpha}\nabla_X \xi, \tag{10}$$

then from (8) and (10), we have

$$\phi X = X + \eta(X)\xi, \tag{11}$$

from which it follows that ϕ is a symmetric $(1,1)$ tensor. Thus the Lorentzian manifold M^ntogether with the unit timelike concircular vector field ξ, its associated 1-form η and (1,1) tensor field ϕ is said to be a Lorentzian concircular structure manifold (briefly $(LCS)_n$-manifold) [5]. In a $(LCS)_n$-manifold, the following relations hold [26]:

$$\begin{aligned}
\eta(\xi) &= -1, \ \ \phi \circ \xi = 0, & (12)\\
\eta(\phi X) &= 0, \qquad g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), & (13)\\
\eta(R(X, Y)Z) &= (\alpha^2 - \rho)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)], & (14)\\
R(X, Y)\xi &= (\alpha^2 - \rho)[\eta(Y)X - \eta(X)Y], & (15)\\
S(X, \xi) &= (n-1)(\alpha^2 - \rho)\eta(X) & (16)
\end{aligned}$$

for any vector fields X, Y, Z.

Let (M, ϕ, ξ, η, g) be an $(LCS)_n$ manifold satisfying (4). Writing $L_\xi g$ in terms of the Levi-Civita connection ∇, we obtain from (4) that,

$$2S(X, Y) = -g(\nabla_X \xi, Y) - g(X, \nabla_Y \xi) - 2\lambda g(X, Y) - 2\mu\eta(X)\eta(Y), \tag{17}$$

for any $X, Y \in \chi(M)$. As a consequence of (10) and (11), the above equation becomes

$$S(X, Y) = -(\lambda + \alpha)g(X, Y) - (\mu + \alpha)\eta(X)\eta(Y). \tag{18}$$

In the sequel we shall use the following Lemma.

Lemma 1. *Let (M^n, g) be an $(LCS)_n$-manifold. Then for any X; Y; Z the following relation holds:*

$$\begin{aligned}
(\nabla_U S)(X, \xi) \quad &= \quad (n-1)[\alpha(\alpha^2 - \rho)g(X, U) \\
&\quad +(2\alpha\rho - \beta)\eta(X)\eta(U)] - \alpha S(X, U). & (19)
\end{aligned}$$

Proof. By virtue of (8), (9) and (10) we can easily get (16). □

3 Generalized weakly symmetric $(LCS)_n$ -manifolds whose metric is η-Ricci soliton

A non-flat n-dimensional $(LCS)_n$-manifold $(M^n; g)$ $(n > 2)$, is termed as generalized weakly symmetric manifold, if its Riemannian curvature tensor $\bar{R}$ of type $(0; 4)$ is not identically zero and admits the identity

$$\begin{aligned}(\nabla_X \bar{R})(Y, Z, U, V) &= A(X)\bar{R}(Y, Z, U, V) + B(Y)\bar{R}(X, Z, U, V) \\ &+B(Z)\bar{R}(Y, X, U, V) + D(U)\bar{R}(Y, Z, X, V) \\ &+D(V)\bar{R}(Y, Z, U, X) + \alpha(X)G(Y, Z, U, V) \\ &+\beta(Y)G(X, Z, U, V) + \beta(Z)\, G(Y, X, U, V) \\ &+\gamma(U)\, G(Y, Z, X, V) + \gamma(V)\, G(Y, Z, U, X) \qquad (20)\end{aligned}$$

where

$$G(Y, Z, U, V) = [g(Z, U)g(Y, V) - g(Y, U)g(Z, V)] \qquad (21)$$

and $A,\ B, D, \alpha, \beta$ & γ are non-zero 1-forms which are defined as $A(X) = g(X, \theta_1)$, B(X)=g(X,ϕ_1), $D(X) = g(X, \pi_1)$, $\alpha(X) = g(X, \theta_2), \beta(X) = g(X, \phi_2)$ and $\gamma(X) = g(X, \pi_2)$. Now, contracting Z over U in both sides of (20) we find

$$\begin{aligned}(\nabla_X S)(Y, V) &= A(X)S(Y, V) + B(Y)S(X, V) + D(V)S(Y, X) \\ &-B(R(Y, X)V) + D(R(X, V)Y) + (n-1)[\alpha(X) \\ &g(Y, V) + \beta(Y)g(X, V) + \gamma(V)g(Y, X)] \\ &-\beta(Y)\, g(X, V) + [\beta(X) + \gamma(X)]\, g(Y, V) - \gamma(V)g(X, Y) \qquad (22)\end{aligned}$$

As a consequence of (16), (19) and (22) we obtain

$$\begin{aligned}&(n-1)[\alpha(\alpha^2 - \rho)g(X, V) + (2\alpha\rho - \beta)\eta(V)\eta(X)] - \alpha S(X, V) \\ =\ &(\alpha^2 - \rho)[(n-1)\{A(X)\eta(V) + D(V)\eta(X)\} + \eta(V)B(X) \\ &-g(X, V)B(\xi) + \eta(V)D(X) - \eta(X)D(V)] + B(\xi)S(X, V) \\ &+(n-1)[\alpha(X)\eta(V) + \beta(\xi)g(X, V) + \gamma(V)\eta(X)] \\ &-\beta(\xi)\, g(X, V) + [\beta(X) + \gamma(X)]\, \eta(V) - \gamma(V)\eta(X) \qquad (23)\end{aligned}$$

for $Y = \xi$. Furthermore, setting $X = V = \xi$ in the foregoing equation we get

$$\begin{aligned}&-(2\alpha\rho - \beta) = (\alpha^2 - \rho)[A(\xi) + B(\xi) + D(\xi)] \\ &+[\alpha(\xi) + \beta(\xi) + \gamma(\xi)]. \qquad (24)\end{aligned}$$

Again, in a weakly symmetric $(LCS)_n$-manifold we have the relation (23). Setting $X = \xi$ in (23) we get

$$\begin{aligned}&(n-2)[(\alpha^2 - \rho)D(V) + \gamma(V)] \\ =\ &[(n-1)\{(2\alpha\rho - \beta) + (\alpha^2 - \rho)\{A(\xi) + B(\xi)\}\} + (\alpha^2 \\ &-\rho)D(\xi)]\eta(V) + [(n-1)\{\alpha(\xi) + \beta(\xi)\} + \gamma(\xi)]\eta(V). \qquad (25)\end{aligned}$$

In view of (24), the relation (25) reduces to

$$[(\alpha^2-\rho)D(V)+\gamma(V)] = -[(\alpha^2-\rho)D(\xi)+\ \gamma(\xi)]\eta(V) \tag{26}$$

Again, contracting over Y and V in (**??**) we get

$$\begin{aligned}(\nabla_X S)(Z,U) &= A(X)S(Z,U)+B(R(X,Z)U)+B(Z)S(X,U)\\ &\quad +D(U)S(Z,X)+D(R(X,U)Z)+(n-1)[\{\alpha(X)g(Z,U)\\ &\quad +\beta(Z)g(X,U)+\gamma(U)g(X,Z)\}]+[\gamma(X)g(Z,U)\\ &\quad -\gamma(U)g(Z,X)+\beta(X)g(Z,U)-\beta(Z)g(X,U).\end{aligned} \tag{27}$$

Setting $U=\xi$ in (27) and using (16) and (19) we get

$$\begin{aligned}&(n-1)[\alpha(\alpha^2-\rho)g(X,Z)+(2\alpha\rho-\beta)\eta(X)\eta(Z)]-\alpha S(X,Z)\\ =\ &(\alpha^2-\rho)[(n-1)\{A(X)\eta(Z)+B(Z)\eta(X)\}+B(X)\eta(Z)-B(Z)\eta(X)\\ &+D(X)\eta(Z)-+D(\xi)g(X,Z)]+D(\xi)S(Z,X)+(n-1)[\{\alpha(X)\eta(Z)\\ &+\beta(Z)\eta(X)+\gamma(\xi)g(X,Z)\}]+[\gamma(X)\eta(Z)\\ &-\gamma(\xi)g(Z,X)+\beta(X)\eta(Z)-\beta(Z)\eta(X).\end{aligned} \tag{28}$$

which yields

$$[(\alpha^2-\rho)B(Z)+\beta(Z)] = -[(\alpha^2-\rho)B(\xi)+\beta(\xi)]\eta(Z) \tag{29}$$

for $X=\xi$ and

$$[(\alpha^2-\rho)A(X)+\alpha(X)] = -[(\alpha^2-\rho)A(\xi)+\alpha(\xi)]\eta(X) \tag{30}$$

for $Z=\xi$. In view of (24), (26), (29) and (30), we have

$$\begin{aligned}&(2\alpha\rho-\beta)\eta(X) = (\alpha^2-\rho)[A(X)+B(X)+D(X)]\\ &+[\alpha(X)+\beta(X)+\gamma(X)].\end{aligned} \tag{31}$$

Now, making use of (29)-(31) in (23), we find that

$$\begin{aligned}S(X,W) &= \left[(\alpha^2-\rho)+(n-2)\left(\frac{(\alpha^2-\rho)\alpha-\beta(\xi)}{\alpha+B(\xi)}\right)\right]g(X,W)\\ &\quad -\frac{(n-2)[(\alpha^2-\rho)D(\xi)+\gamma(\xi)]}{[\alpha+B(\xi)]}\eta(W)\eta(X)\end{aligned} \tag{32}$$

for any $Y,W\in\chi(M)$. Comparing (32) and (18), we obtain,

$$\lambda = -\left[\alpha+(\alpha^2-\rho)+(n-2)\left(\frac{(\alpha^2-\rho)\alpha-\beta(\xi)}{\alpha+B(\xi)}\right)\right] \tag{33}$$

This leads to the following:

Theorem 1. *The Ricci soliton in each of the locally symmetric, locally recurrent and generalized recurrent $(LCS)_n$-manifold is expanding, steady or shrinking accordingly as $(n-1)(\alpha^2-\rho) \gtreqless \alpha$.*

Theorem 2. *The Ricci soliton in each of the pseudo symmetric, semi-pseudo symmetric, almost pseudo symmetric and weakly symmetric $(LCS)_n$-manifold is expanding, steady or shrinking accordingly as $(n-2)\alpha\left(\alpha^2-\rho\right) \gtreqless [\alpha+B(\xi)] \times$ $\times\left[\alpha+(\alpha^2-\rho)\right]$.*

Theorem 3. *The Ricci soliton in each of the generalized pseudo symmetric, generalized semi-pseudo symmetric, almost generalized pseudo symmetric $(LCS)_n$ -manifold is expanding, steady or shrinking accordingly as $(n-2)\left[(\alpha^2-\rho)\alpha-\beta(\xi)\right]$ $\gtreqless [\alpha+B(\xi)]\left[\alpha+(\alpha^2-\rho)\right]$.*

4 Generalized weakly Ricci-symmetric $(LCS)_n$--manifolds whose metric is η-Ricci soliton

A non-flat n-dimensional $(LCS)_n$-manifold $(M^n;g)$ $(n>2)$, is said to be a generalized weakly Ricci symmetric manifold, if its Ricci tensor S of type $(0,2)$ is not identically zero and admits the identity

$$\begin{aligned}(\nabla_X S)(Y,Z) &= A_1^*(X)S(Y,Z)+B_1^*(Y)S(X,Z)+D_1^*(Z)S(Y,X)\\ &\quad +A_2^*(X)g(Y,Z)+B_2^*(Y)g(X,Z)+D_2^*(Z)g(Y,X) \qquad (34)\end{aligned}$$

where $A_i^*,\ B_i^*$ & D_i^* are non-zero 1-forms which are defined as $A_i^*(X)=g(X,\theta_i)$, $B_i^*(X)=g(X,\phi_i)$, $D_i^*(X)=g(X,\pi_i)$ for $i=1,2$.

Setting, $Y=\xi$ in (34) and then making use of (19), we have

$$\begin{aligned}&(n-1)[\alpha(\alpha^2-\rho)g(X,Z)+(2\alpha\rho-\beta)\eta(Z)\eta(X)]-\alpha S(X,Z)\\ =\ &(\alpha^2-\rho)(n-1)[A_1^*(X)\eta(Z)+D_1^*(Z)\eta(X)]+B_1^*(\xi)S(X,Z)\\ &+A_2^*(X)\eta(Z)+B_2^*(\xi)g(X,Z)+D_2^*(Z)\,\eta(X) \qquad (35)\end{aligned}$$

which yields

$$\begin{aligned}(\alpha^2-\rho)(n-1)[A_1^*(\xi)+B_1^*(\xi)+D_1^*(\xi)] &+ [A_2^*(\xi)+B_2^*(\xi)+D_2^*(\xi)\,]\\ &= -(n-1)(2\alpha\rho-\beta). \qquad (36)\end{aligned}$$

for $X=Z=\xi$

Setting $Z=\xi$ in (35) we obtain

$$(n-1)(\alpha^2-\rho)[A_1^*(X)+A_1^*(\xi)]=-[A_2^*(X)+A_2^*(\xi)\eta(X)] \qquad (37)$$

Proceeding in a similar manner we can find

$$(\alpha^2-\rho)(n-1)[B_1^*(X)+B_1^*(\xi)]=-[B_2^*(X)+B_2^*(\xi)\eta(X)] \qquad (38)$$

$$(\alpha^2-\rho)(n-1)[D_1^*(X)+D_1^*(\xi)] = -[D_2^*(\xi)+D_2^*(X)\eta(X)] \tag{39}$$

From the above relations, after a routine calculation one can easily bring out the following

$$\begin{aligned} S(X,Z) &= \left[\frac{(n-1)\alpha(\alpha^2-\rho)-B_2^*(\xi)}{\alpha+B_1^*(\xi)}\right]g(X,Z) \\ &\quad -\left[\frac{(\alpha^2-\rho)(n-1)B_1^*(\xi)+B_2^*(\xi)}{\alpha+B_1^*(\xi)}\right]\eta(X)\eta(Z) \end{aligned} \tag{40}$$

Now, comparing (18) and (40) we have

$$\lambda = -\left[\alpha+\frac{(n-1)\alpha(\alpha^2-\rho)-B_2^*(\xi)}{\alpha+B_1^*(\xi)}\right]$$

This leads to the following:

Theorem 4. *The Ricci soliton in each of the Ricci symmetric, Ricci recurrent and generalized Ricci recurrent $(LCS)_n$-manifold is expanding, steady or shrinking accordingly as $(n-1)(\alpha^2-\rho) \gtreqless \alpha$.*

Theorem 5. *The Ricci soliton in each of the pseudo Ricci symmetric, semi-pseudo Ricci symmetric, almost pseudo Ricci symmetric and weakly Ricci symmetric $(LCS)_n$-manifold is expanding, steady or shrinking accordingly as $(n-1)\alpha(\alpha^2-\rho) \gtreqless \alpha\left[\alpha+B_1^*(\xi)\right]$.*

Theorem 6. *The Ricci soliton in each of the generalized pseudo Ricci symmetric, generalized semi-pseudo Ricci symmetric, almost generalized pseudo Ricci symmetric $(LCS)_n$-manifold is expanding, steady or shrinking accordingly as $\left[(n-1)\alpha(\alpha^2-\rho)-B_2^*(\xi)\right] \gtreqless \alpha\left[\alpha+B_1^*(\xi)\right]$.*

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