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SOME RESULTS ON PARASASAKIAN MANIFOLDS

U. C. DE^{*,1}, Gopal GHOSH[2], J. B. JUN[3] and Pradip MAJHI[4]

Abstract

In this paper at first we obtain a sufficient condition for a pseudo-Riemannian manifold to be a paraSasakian manifold. Next we study Ricci semisymmetric and Weyl semisymmetric paraSasakian manifolds. Also we study D-homothetic deformation of paraSasakian manifolds. Finally, we study gradient Ricci soliton on paraSasakian manifolds.

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1 Introduction

In 1977 Adati and Matsumoto [1] introduced the notion of paraSasakian manifolds or briefly P-Sasakian manifolds, which are considered as a special case of an almost paracontact manifold introduced by Sato [31]. In [27] Matsumoto and Mihai study P-Sasakian manifolds that admit W_2 or E-Tensor fields and also some curvature conditions. Moreover in ([14], [15], [25], [26], [28], [30], [40]) the authors study P-Sasakian manifolds satisfying certain curvature conditions. On the other hand in [20] Kaneyuki and Kozai defined the almost paracontact structure on pseudo-Riemannian manifold M of dimension $(2n+1)$ and constructed the almost paracomplex structure on $M^{2n+1} \times \mathbb{R}$. The main difference between the almost paracontact metric manifold in the sence of Sato [31] and Kaneyuki et al [19] is the signature of the metric. In 2009, Zamkovoy [41] defined paraSasakian manifolds as a normal paracontact manifold whose metric is pseudo-Riemannian. Thus a paraSasakian manifold is a subclass of paracontact metric manifolds. In [41], the author obtains a necessary and sufficient condition for a paracontact metric manifold to be a paraSasakian manifold. Also D-homothetic transformations

[1]Department of Pure Mathematics, University of Calcutta, 35, Ballygunge Circular Road, Kol- 700019, West Bengal, India, e-mail: uc_de@yahoo.com *Corresponding Author
[2]Department of Pure Mathematics, University of Calcutta, 35, Ballygunge Circular Road, Kol- 700019, West Bengal, India, e-mail: ghoshgopal.pmath@gmail.com
[3]Department of Mathematics, Faculty of Natural Science, Kookmin University, Seoul Korea, 136-702, e-mail: jbjun@.kookmin.ac.kr
[4]Department of Pure Mathematics, University of Calcutta, 35, Ballygunge Circular Road, Kol- 700019, West Bengal, India, e-mail: mpradipmajhi@gmail.com

have been studied in paraSasakian manifolds in [41]. In the present paper we characterize paraSasakian manifolds satisfying certain curvature conditions.

The paper is organized as follows: Section 2 is eqquiped with some prerequisites about paraSasakian manifolds. In section 3, we obtain a sufficient condition for a pseudo-Riemannin manifold to be a paraSasakian manifold. Section 4 and 5 are devoted to study Ricci semi-symmetric and conformally flat paraSasakian manifolds respectively. Section 6, deals with the study of Weyl semisymmetric paraSasakian manifolds and we prove that a Weyl semisymmetric paraSasakian manifold is of quasi-constant curvature. Section 7 is concerned with the study of D-homothetic deformation of paraSasakian manifolds. It is shown that ϕ-sectional curvature of a paraSasakian manifold is an invariant property under D-homothetic deformation. Next in section 8, we prove that η-parallelity of a paraSasakian manifold is invariant under D-homothetic deformation. Also we prove that Locally ϕ-Ricci symmetry on a paraSasakian manifold is an invariant property under D-homothetic deformation. Finally, it has been shown that a paraSasakian manifold admitting a gradient Ricci soliton is an Einstein manifold and the Ricci soliton is shrinking.

2 Preliminaries

Let M be an $(2n+1)$-dimensional differentiable manifold. If there exits a triplet (ϕ, ξ, η) of a tensor field ϕ of type $(1,1)$, a vector field ξ and a 1-form η on M^{2n+1} which satisfies the relation [31]

$$\phi^2 = I - \eta \otimes \xi, \eta(\xi) = 1,\ \phi\xi = 0,\ \eta \circ \phi = 0, \tag{1}$$

then we say the triplet (ϕ, ξ, η) is an almost paracontact structure and the manifold is an almost paracontact manifold.

If an almost para contact manifold M^{2n+1} with an almost paracontact structure (ϕ, ξ, η) admits a pseudo-Riemannian metric g such that [20]

$$g(X,Y) = -g(\phi X, \phi Y) + \eta(X)\eta(Y), \tag{2}$$

then we say that M^{2n+1} is an almost paracontact metric structure (ϕ, ξ, η, g) and such a metric g is called compatible metric. Any compatible metric g is necessarily of signature $(n+1, n)$. The fundamental 2-form of M^{2n+1} is defined by

$$\Phi(X,Y) = g(X, \phi Y). \tag{3}$$

An almost paracontact metric structure becomes a paracontact metric structure if

$$d\eta(X,Y) = g(X, \phi Y)$$

for all vector fields X, Y, where

$$d\eta(X,Y) = \frac{1}{2}[X\eta(Y) - Y\eta(X) - \eta([X,Y])].$$

Paracontact manifolds have been studied by several authors such as Kaneyuki and Willams [19], Calvaruso [6, 7], Cappelletti-Montano et al. [8, 9, 10], Mertin-Molina [23], Welyczko [39], Zamkovoy et al. [42] and many others.

An almost paracontact structure is said to be normal if and only if the tensor $N_\phi - 2d\eta \otimes \xi$ vanishes identically, where N_ϕ is the Nijenhuis tensor of ϕ : $N_\phi(X,Y) = [\phi,\phi](X,Y) = \phi^2[X,Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y]$ [41]. A normal paracontact metric manifold is known as paraSasakian manifold. It is known [41] that an almost paracontact manifold is paraSasakian manifold if and only if

$$(\nabla_X \phi)Y = -g(X,Y)\xi + \eta(Y)X, \tag{4}$$

for all vectors field X, Y, where ∇ is the Levi-Civita connection of the pseudo-Riemannian metric. From the above equation it follows that

$$\nabla_X \xi = -\phi X. \tag{5}$$

Moreover, in a paraSasakian manifold the curvature tensor R, the Ricci tensor S and the Ricci operator Q defined by $g(QX,Y) = S(X,Y)$ satisfy [41]

$$R(X,Y)\xi = -(\eta(Y)X - \eta(X)Y), \tag{6}$$

$$R(\xi,X)Y = -g(X,Y) + \eta(Y)X, \tag{7}$$

$$S(X,\xi) = -2n\eta(X), \tag{8}$$

$$Q\xi = -2n\xi, \tag{9}$$

$$(\nabla_X \eta)Y = g(X, \phi Y), \tag{10}$$

$$\eta(R(X,Y)Z) = -(g(Y,Z)\eta(X) - g(X,Z)\eta(Y)). \tag{11}$$

ParaSasakian manifolds have been studied by several authors such as Zamkovoy [41], Martin-Molina [22], Cappelletti Montano et al [4] and many others.

A paraSasakian manifold is said to be Einstein if

$$S(X,Y) = ag(X,Y),$$

where S is the Ricci tensor of type $(0,2)$ and a is a constant.

In 1972, Chen and Yano [11] introduced the notion of quasi-constant curvature. A pseudo-Riemannian manifold is said to be a manifold of quasi-constant curvature if the curvaure tensor R of M^{2n+1} of type $(0,4)$ satisfies the condition

$$\begin{aligned} R(X,Y)Z \;=\;& p[g(Y,Z)X - g(X,Z)Y] + q[A(Y)A(Z)X \\ & -A(X)A(Z)Y + g(Y,Z)A(X)\rho - g(X,Z)A(Y)\rho], \end{aligned} \tag{12}$$

where p and q are scalars, A is a non-zero 1-form and the vector field ρ corresponding to the 1-form A is a unit vector field. If $q = 0$, then the manifold reduces to a manifold of constant curvature. Thus a manifold of quasi-constant curvature is a generalization of the manifold of constant curvature. A manifold of quasi-constant curvature have been studied by several authors such as Adati and Wang [2], De and Ghosh [16], Wang [38] and many others.

A pseudo-Riemannian manifold (M, g) is called locally symmetric if its curvature tensor R is parallel, that is, $\nabla R = 0$. The notion of semisymmetric manifold, a proper generalization of locally symmetric manifold, is defined by $R(X, Y) \cdot R = 0$, where $R(X, Y)$ acts on R as a derivation. A complete intrinsic classification of these manifolds was given by Szabó [32].

A pseudo-Riemannian manifold is said to be Ricci semisymmetric if $R(X, Y) \cdot S = 0$, where S denotes the Ricci tensor of type $(0, 2)$. A general classification of these manifolds has been worked out by Mirzoyan [24].

A Ricci soliton is a generalization of an Einstein metric. In a pseudo-Riemannian manifold (M, g), g is called a Ricci soliton if [7]

$$(\$_V g + 2S + 2\lambda g)(X, Y) = 0, \tag{13}$$

where \$ is the Lie derivative, S is the Ricci tensor, V is a complete vector field on M and λ is a constant. The Ricci soliton is said to be shrinking, steady and expanding according as λ is negative, zero and positive respectively. If the vector field V is the gradient of a potential function $-f$, then g is called a gradient Ricci soliton and equation (1.1) assumes the form

$$\nabla\nabla f = S - \lambda g. \tag{14}$$

For more details we refer to the reader ([5], [12], [13], [29], [36], [37]).

3 Sufficient condition for a pseudo-Riemannian manifold to be a paraSasakian manifold

In this section we derive a sufficient condition for a paracontact metric manifold to be a paraSasakian manifold. Let M^{2n+1} be a pseudo-Riemannian manifold which admits a unit vector field ξ such that the 1 form η satisfies
(i) $\eta(X) = g(X, \xi)$,
(ii) η is closed and
(iii) $(\nabla_X \nabla_Y \eta)(Z) = \eta(Y)g(X, Z) - \eta(Z)g(X, Y)$.

Now we have $\eta(\xi) = 1$. Differentiating it covariently we get

$$(\nabla_X \eta)\xi = 0. \tag{15}$$

Now,

$$(\nabla_Y \nabla_X \eta)\xi = \nabla_Y (\nabla_X \eta)\xi - (\nabla_X \eta)(\nabla_Y \xi) - (\nabla_{\nabla_Y X} \eta)\xi. \tag{16}$$

Using (iii) and (15) in (16) yields

$$(\nabla_X \eta)(\nabla_Y \xi) = g(X, Y) - \eta(X)\eta(Y). \tag{17}$$

We put
$$\Phi(X,Y) = g(\phi X, Y) = -(\nabla_Y \eta)X, \tag{18}$$
and
$$\nabla_X \xi = -\phi X. \tag{19}$$
Using (18) and (19) in (17) we have
$$g(\phi^2 Y, X) = g(X,Y) - \eta(X)\eta(Y),$$
which implies
$$\phi^2 Y = Y - \eta(Y)\xi.$$
Since η is closed, it follows that
$$(\nabla_X \eta)Y - (\nabla_Y \eta)X = 0.$$
Now using (18) in the above equation yields
$$-g(\phi X, Y) = g(\phi Y, X). \tag{20}$$
Therefore, ϕ is skew-symmetric. Now
$$(\nabla_Z \Phi)(X,Y) = \nabla_Z \Phi(X,Y) - \Phi(\nabla_Z X, Y) - \Phi(X, \nabla_Z Y). \tag{21}$$
Using (18) in (21) yields
$$(\nabla_Z \Phi)(X,Y) = g((\nabla_Z \phi)X, Y). \tag{22}$$
On the other hand using (18) we obtain
$$(\nabla_X \Phi)(Y,Z) = -(\nabla_X \nabla_Z \eta)Y.$$
Hence using (iii) we get
$$(\nabla_X \Phi)(Y,Z) = \eta(Y)g(X,Z) - \eta(Z)g(X,Y).$$
Thus from (22) it follows that
$$(\nabla_X \phi)Y = -g(X,Y)\xi + \eta(Y)X,$$
which is the necessary and sufficient condition for a paracontact metric manifold to be a paraSasakian manifold.

From the above discussions we obtain the following:

Theorem 3.1. *Let M^{2n+1} be a pseudo Riemannian manifold which admits a unit vector field ξ such that the 1 form η satisfies*
(i) $\eta(X) = g(X,\xi)$*,*
(ii) η *is closed and*
(iii) $(\nabla_X \nabla_Y \eta)(Z) = \eta(Y)g(X,Z) - \eta(Z)g(X,Y)$.
Then M^{2n+1} is a paraSasakian manifold.

4 Ricci semisymmetric paraSasakian manifolds

Suppose the paraSasakian manifold is Ricci semisymmetric. Then

$$(R(X,Y)\cdot S)(Z,V)=0. \tag{23}$$

Putting $X=\xi$ in (23) we have

$$S(R(\xi,Y)Z,V)+S(Z,R(\xi,Y)V)=0. \tag{24}$$

Using (7) and (8) in (24) we get

$$\begin{aligned}&\eta(V)S(Y,Z)+\eta(Z)S(Y,V)+2n\eta(V)g(Y,Z)\\&+2n\eta(Z)g(Y,V)=0.\end{aligned} \tag{25}$$

Putting $V=\xi$ in (25) and with the help of (8), we get

$$S(Y,Z)=-2ng(Y,Z), \tag{26}$$

which implies that the manifold is an Einstein manifold.

Conversely, if the manifold is an Einstein manifold, then obviously it satisfies

$$R\cdot S=0.$$

This leads to the following:

Theorem 4.1. *A paraSasakian manifold is Ricci semisymmetric if and only if the manifold is an Einstein manifold.*

5 Conformally flat paraSasakian manifolds of dimension ≥ 5

In [41], the author has studied conformally flat paraSasakian manifolds and obtained the value of $S(X,X)-S(\phi X,\phi Y)$. In this section we characterize conformally flat paraSasakian manifolds. The Weyl conformal curvature tensor C is defined by

$$\begin{aligned}C(X,Y)Z &= R(X,Y)Z-\frac{1}{2n-1}[S(Y,Z)X-S(X,Z)Y\\&\quad+g(Y,Z)QX-g(X,Z)QY]\\&\quad+\frac{r}{2n(2n-1)}[g(Y,Z)X-g(X,Z)Y],\end{aligned} \tag{27}$$

where Q is the Ricci operator and r is the scalar curvature of the manifold M^{2n+1}.

Suppose the manifold is conformally flat. Then from (27) it follows that,

$$\begin{aligned}R(X,Y)Z &= \frac{1}{2n-1}[S(Y,Z)X-S(X,Z)Y+g(Y,Z)QX-g(X,Z)QY]\\&\quad-\frac{r}{2n(2n-1)}[g(Y,Z)X-g(X,Z)Y].\end{aligned} \tag{28}$$

Taking inner product with W in (28) yields

$$\begin{aligned} g(R(X,Y)Z,W) &= \frac{1}{2n-1}[S(Y,Z)g(X,W) - S(X,Z)g(Y,W) \\ &\quad + g(Y,Z)S(X,W) - g(X,Z)S(Y,W)] \\ &\quad - \frac{r}{2n(2n-1)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]. \end{aligned} \tag{29}$$

Putting $W = \xi$ in (29) we have

$$\begin{aligned} \eta(R(X,Y)Z) &= \frac{1}{2n-1}[S(Y,Z)\eta(X) - S(X,Z)\eta(Y) + g(Y,Z)S(X,\xi) \\ &\quad - g(X,Z)S(Y,\xi)] - \frac{r}{2n(2n-1)}[g(Y,Z)\eta(X) \\ &\quad - g(X,Z)\eta(Y)]. \end{aligned} \tag{30}$$

Using (8) and (11) in (30) we get

$$\begin{aligned} g(X,Z)\eta(Y) - g(Y,Z)\eta(X) &= \frac{1}{2n-1}[S(Y,Z)\eta(X) - S(X,Z)\eta(Y) \\ &\quad - 2ng(Y,Z)\eta(X) + 2ng(X,Z)\eta(Y)] \\ &\quad - \frac{r}{2n(2n-1)}[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]. \end{aligned} \tag{31}$$

Putting $Y = \xi$ in (31) and using (8) yields

$$S(X,Z) = [1 + \frac{r}{2n}]g(X,Z) - [(2n+1) + \frac{r}{2n}]\eta(X)\eta(Z). \tag{32}$$

From (32) it follows that

$$QX = [1 + \frac{r}{2n}]X - [(2n+1) + \frac{r}{2n}]\eta(X)\xi. \tag{33}$$

Using (32) and (33) in (28) we have

$$\begin{aligned} R(X,Y)Z &= \frac{r+4n}{2n(2n-1)}[g(Y,Z)X - g(X,Z)Y] - \frac{4n^2+4n+r}{2n(2n-1)}[\{g(Y,Z)\eta(X) \\ &\quad - g(X,Z)\eta(Y)\}\xi + \eta(Z)\{\eta(Y)X - \eta(X)Y\}] \end{aligned} \tag{34}$$

Thus we can state the following theorem:

Theorem 5.1. *A conformally flat paraSasakian manifold $M^{2n+1}(n > 1)$ is a manifold of quasi-constant curvature tensor.*

6 Weyl semisymmetric paraSasakian manifolds

A paraSasakian manifold M^{2n+1} is said to be Weyl semisymmetric if $R \cdot C = 0$. Suppose the manifold is Weyl semisymmetric. Then

$$(R(X,Y) \cdot C)(U,V)Z = 0,$$

for all smooth vector fields X, Y, U, V and W, which yields

$$R(X,Y)C(U,V)Z - C(R(X,Y)U,V)Z - C(U,R(X,Y)V)Z \\ -C(U,V)R(X,Y)Z = 0. \tag{35}$$

Taking $X = \xi$, we obtain by virtue of

$$R(\xi, X)Y = \eta(Y)X - g(X,Y)\xi,$$

$$\eta(C(U,V)Z)Y - g(Y,C(U,V)Z)\xi - \eta(U)C(Y,V)Z \\ +g(Y,U)C(\xi,V)Z - \eta(V)C(U,Y)Z + g(Y,V)C(U,\xi)Z \\ -\eta(Z)C(U,V)Y + g(Y,Z)C(U,V)\xi = 0. \tag{36}$$

Taking inner product on both sides by ξ and then using the skew-symmetry property of the conformal curvature tensor C, we get

$$\eta(Y)\eta(C(U,V)Z) - g(Y,C(U,V)Z) - \eta(U)\eta(C(Y,V)Z) \\ +g(Y,U)\eta(C(\xi,V)Z) - \eta(V)\eta(C(U,Y)Z) + g(Y,V)\eta(C(U,\xi)Z) \\ -\eta(Z)\eta(C(U,V)Y) + g(Y,Z)\eta(C(U,V)\xi) = 0. \tag{37}$$

Let $\{e_i : i = 1, 2, ..., 2n+1\}$ be an orthonormal basis of the tangent space at any point of the manifold. Then setting $U = Y = e_i$ and taking summation over $i(1 \leq i \leq (2n+1))$ we obtain by virtue of the properties of the conformal curvature tensor

$$\eta(C(\xi,V)Z) = 0, \tag{38}$$

for all vector fields V, Z. By virtue of (38), (37) reduces to

$$\eta(Y)\eta(C(U,V)Z) - g(Y,C(U,V)Z) - \eta(U)\eta(C(Y,V)Z) \\ -\eta(V)\eta(C(U,Z)Y) - \eta(Z)\eta(C(U,V)Y). \tag{39}$$

Now from the definition of conformal curvature tensor it can be easily seen that

$$\eta(C(X,Y)Z) = 0, \tag{40}$$

for all X, Y, Z. Using (40) in (39) implies

$$C(X,Y)Z = 0,$$

for all smooth vector fields X, Y, Z. Hence the manifold is confomally flat. Therefore by Theorem 5.1, we can state the following theorem:

Theorem 6.1. *A Weyl semisymmetric paraSasakian manifold of dimension ≥ 5 is a manifold of quasi-constant curvature tensor.*

7 D-homothetic deformation of paraSasakian manifolds

In this section we recall a notoin of D-homothetic deformation in paracontact geometry [41]. Let $M(\phi, \xi, \eta, g)$ be an almost paracontact metric manifold of dimension $2n+1$. The equation $\eta = 0$ defines an $2n$ -dimensional distribution D on M [34]. By a D-homothetic deformation we mean a change of structure tensors of the form $\bar{\eta} = a\eta$, $\bar{\xi} = \frac{1}{a}\xi$, $\bar{\phi} = \phi$, $\bar{g} = ag + a(a-1)\eta \circ \eta$, where a is a positive constant. If $M(\phi, \xi, \eta, g)$ is an almost paracontact metric structure with constant form η, then $M(\bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$ is also an almost paracontact metric structure [33]. Denoting by W^i_{jk} the difference $\bar{\Gamma}^i_{jk} - \Gamma^i_{jk}$ of Christoffel symbols we have in an almost paracontact metric manifold [33]

$$\begin{aligned} W(X,Y) &= (1-a)[\eta(Y)\phi X + \eta(X)\phi Y] \\ &\quad + \frac{1}{2}(1-\frac{1}{a})[(\nabla_X \eta)(Y) + (\nabla_Y \eta)(X)]\xi, \end{aligned} \tag{41}$$

for all $X, Y \in \chi(M)$, where $\chi(M)$ is the set of all differentiable vector fields on M. If R and $\bar{R}$ denote respectively the curvature tensor of the manifold $M(\phi, \xi, \eta, g)$ and $M(\bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$, then we have [33]

$$\begin{aligned} \bar{R}(X,Y)Z &= R(X,Y)Z + (\nabla_X W)(Z,Y) - (\nabla_Y W)(Z,X) \\ &\quad + W(W(Z,Y),X) - W(W(Z,X),Y), \end{aligned} \tag{42}$$

for all $X, Y, Z \in \chi(M)$. Using (10) in (41) yields

$$W(X,Y) = (1-a)[\eta(Y)\phi X + \eta(X)\phi Y]. \tag{43}$$

Differentiating it covariently we get

$$\begin{aligned} (\nabla_X W)(Y,Z) &= (1-a)[(\nabla_X \eta)(Z)\phi Y + (\nabla_X \eta)(Y)\phi Z + \eta(Z)(\nabla_X \phi)Y \\ &\quad + \eta(Y)(\nabla_X \phi)Z]. \end{aligned} \tag{44}$$

With the help of (4) and (10), (44) reduces to

$$\begin{aligned} (\nabla_X W)(Y,Z) &= (1-a)[g(X,\phi Z)\phi Y + g(X,\phi Y)\phi Z - g(X,Y)\eta(Z)\xi \\ &\quad - g(X,Z)\eta(Y)\xi + 2\eta(Y)\eta(Z)X]. \end{aligned} \tag{45}$$

Putting the value of (45) in (42) and using (41) we get

$$\begin{aligned} \bar{R}(X,Y)Z &= R(X,Y)Z + (1-a)[g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X \\ &\quad + 2g(X,\phi Y)\eta(Z)\xi + g(Y,Z)\eta(X)\xi \\ &\quad - g(X,Z)\eta(Y)\xi] + [2-(1-a)^2][\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y]. \end{aligned} \tag{46}$$

Taking inner product with W in (46) , then putting $X = W = e_i, 1 \leq i \leq (2n+1)$ and taking summation over i, we have

$$\bar{S}(Y,Z) = S(Y,Z) + 2n[2-(1-a)^2]\eta(Y)\eta(Z). \tag{47}$$

Then from (47) it follows that

$$\bar{Q}Y = QY + 2n[2-(1-a)^2]\eta(Y)\xi. \tag{48}$$

A plane section in M is called a ϕ-section if there exists a unit vector X in M orthogonal to ξ such that $\{X, \phi X\}$ is an orthonomal basis of the plane section. Then the sectional curvature $K(X, \phi X) = g(R(X, \phi X)\phi X, X)$ is called ϕ-sectional curvature [3].
Taking inner product with W and then putting $Y = Z = \phi X$ and $W = X$ in (46) we get

$$g(\bar{R}(X, \phi X)\phi X, X) = g(R(X, \phi X)\phi X, X).$$

Thus we can state the following:

Theorem 7.1. *Under a D-homothetic deformation, the ϕ-sectional curvature of a paraSasakian manifold is invariant .*

8 η-parallel Ricci tensor under D-homothetic deformation

The Ricci tensor S of a paraSasakian manifold is said to be η-parallel if it satisfies [21]

$$(\nabla_X S)(\phi Y, \phi Z) = 0,$$

for all vector fields X, Y and Z. Differentiating covariently (47) with respect to Z we get

$$\begin{aligned}(\nabla_Z \bar{S})(X, Y) &= (\nabla_Z S)(X, Y) + 2n[2-(1-a)^2][(\nabla_Z \eta)(Y)\eta(X) \\ &\quad + (\nabla_Z \eta)(X)\eta(Y)].\end{aligned} \tag{49}$$

Replacing X, Y by $\phi X, \phi Y$ respectively we have

$$(\nabla_Z \bar{S})(\phi X, \phi Y) = (\nabla_Z S)(\phi X, \phi Y).$$

This leads to the following theorem:

Theorem 8.1. *The η-parallelity of the Ricci tensor on paraSasakian manifolds is an invariant property under D-homothetic deformation.*

9 Locally ϕ-Ricci symmetric ParaSasakian manifolds under D-homothetic deformation

Takahasi [35] introduced the notion of ϕ-symmetry. Recently ϕ-Ricci symmetric Sasakian manifolds have been studied by De and Sarkar [17]. A Para-Sasakian manifold M^{2n+1} is said to be ϕ-Ricci symmetric if the Ricci operator satisfies

$$\phi^2((\nabla_X Q)Y) = 0,$$

for all vector fields $X, Y \in M^{2n+1}$ and $S(X,Y) = g(QX,Y)$. If X, Y are orthogonal to ξ, then the manifold is said to be locally ϕ-Ricci symmetric.

Now differentiating (48) covariently with respect to X yields

$$(\nabla_X \bar{Q})Y = (\nabla_X Q)Y + 2n[2-(1-a)^2][((\nabla_X \eta)(Y))\xi + \eta(Y)\nabla_X \xi]. \quad (50)$$

With the help of (5) and (10), (50) reduces to

$$(\nabla_X \bar{Q})Y = (\nabla_X Q)Y + 2n[2-(1-a)^2][g(X,\phi Y)\xi - \eta(Y)\phi X]. \quad (51)$$

Operating ϕ^2 on both sides of (51) yields

$$\phi^2((\nabla_X \bar{Q})Y) = \phi^2((\nabla_X Q)Y) - 2n[2-(1-a)^2]\eta(Y)\phi X. \quad (52)$$

If we consider X, Y orthogonal to ξ then the last equation becomes

$$\phi^2((\nabla_X \bar{Q})Y) = \phi^2((\nabla_X Q)Y).$$

Thus we can state the following theorem:

Theorem 9.1. *Locally ϕ-Ricci symmetry on a paraSasakian manifold is an invariant property under D-homothetic deformation.*

10 Gradient Ricci soliton on paraSasakian manifolds

Equation (14) can be written as

$$\nabla_X(grad\ f) = QX - \lambda X \quad (53)$$

for any vector fields $X \in \chi(M)$, where grad f is the gradient operator of g and $g(QX,Y) = S(X,Y)$. Using this we derive

$$R(X,Y)(grad\ f) = (\nabla_X Q)Y - (\nabla_Y Q)X, \quad (54)$$

for any $X, Y \in \chi(M)$. Putting $X = \xi$ in the above equation and then taking inner product with ξ yields

$$g(R(\xi,Y)(grad\ f),\xi) = g((\nabla_\xi Q)Y,\xi) - g((\nabla_Y Q)\xi,\xi) \quad (55)$$

for any $Y \in \chi(M)$. Using (1) and (9) in the last equation we have

$$g(R(\xi,Y)(grad\ f),\xi) = 0. \quad (56)$$

Using (7) it follows that

$$R(\xi,Y)grad\ f = -g(Y,grad\ f)\xi + \eta(grad\ f)Y. \quad (57)$$

With the help of (56) and (57) we can write

$$grad\ f = (\xi f)\xi. \quad (58)$$

From this equation, we get
$$df = (\xi f)\eta. \tag{59}$$
Its exterior derivative implies
$$0 = d(\xi f)\eta(X) + (\xi f)d\eta(X). \tag{60}$$
Putting $X = \xi$ in the above equation we have ξf is constant.
From (58) it follows that
$$\nabla_Y(grad\ f) = Y(\xi f)\xi + (\xi f)\nabla_Y\xi. \tag{61}$$
Using (10) in the above equation we get
$$g(\nabla_Y(grad\ f), X) = (\xi f)g(\phi Y, X), \tag{62}$$
since $Y(\xi f) = 0$.

Using (62), from (14) it follows that
$$S(X, Y) = \lambda g(X, Y) - (\xi f)g(\phi Y, X). \tag{63}$$
Interchanging X and Y in the last equation and then adding with (63) we have
$$S(X, Y) = \lambda g(X, Y).$$
Hence the Ricci soliton is trivial. Putting $X = Y = \xi$ and using (8) we get
$$\lambda = -2n.$$
Therefore, we can state the following:

Theorem 10.1. *If a paraSasakian manifold admits a gradient Ricci soliton, then the Ricci soliton is shrinking.*

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