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A NOTE ON p^λ-CONVEX SETS IN A COMPLETE RIEMANNIAN MANIFOLD

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Abstract

In this paper we have generalized the notion of λ-radial contraction in a complete Riemannian manifold and developed the concept of p^λ-convex functions. We have also given a counter example proving the fact that in general λ-radial contraction of a geodesic is not necessarily a geodesic. We have also deduced some relations between geodesic convex sets and p^λ-convex sets and showed that under certain condition they are equivalent.

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1 Introduction

The notion of convexity is a basic topic of modern mathematics, especially, in optimization theory and linear programming. But only convexity is not sufficient to study the behavior of a set. Hence there are many generalization of convexity not only in Euclidean space but also in manifold. Some related work on this topic can be found in [3, 4, 5, 6].

In 2010 Beltagy and Shenawy [1] defined the notion of λ-radial contraction in Euclidean space and proved that under such a contraction a line remains invariant. In this paper we have defined λ-radial contraction in a complete Riemannian manifold and showed that, in general, λ-radial contraction of a geodesic need not be a geodesic. In fact convexity property of a subset in a Riemannian manifold is not invariant under the λ-radial contraction and hence a new type of convexity is

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needed. Motivating by these ideas we have defined λ-convex set with respect to a point p, briefly called p^λ-convex set, and also p-convex set in a complete Riemannian manifold. It is proved that in a complete Riemannian manifold geodesic convexity and p-convexity are equivalent under certain conditions. We have also proved that if a set contains an interior point p then there exists some λ such that the set is p^λ-convex. We have also showed that every p^λ-convex set contains a geodesic convex set.

2 Radial contraction and p^λ-convexity

Let (M, g) be a complete n-dimensional Riemannian manifold with Levi-Civita connection ∇. For any two points $x, y \in M$, let $\gamma_{xy} : [0,1] \to M$ be the length minimizing geodesic [2] from x to y such that $\gamma_{xy}(0) = x$ and $\gamma_{xy}(1) = y$. For a fixed p and x in M, consider the set

$$G_p^x = \Big\{ \gamma'_{px}(0) : \forall \gamma_{px} : [0,1] \to M, \gamma_{px}(0) = p \text{ and } \gamma_{px}(1) = x \Big\}.$$

Now G_p^x is a subset of $T_p(M)$. Since (M, g) is complete, hence $G_p^x \neq \phi$. Let $\eta_p : M \to T_p(M)$ be a function such that $\eta_p(x) \in G_p^x \ \forall x \in M$. The function η_p is called direction function at p.

Definition 1. *[4] A subset A of M is called geodesic convex if for any two points $x, y \in A$ there exists a geodesic $\gamma_{xy} : [0,1] \to M$ such that $\gamma_{xy}(0) = x$ and $\gamma_{xy}(1) = y$ and $\gamma_{xy}(t) \in A, \forall t \in [0,1]$.*

Definition 2. *For a fixed point $p \in M$ and for fixed $\lambda \in (0,1]$ define $\eta_p C^\lambda : M \to M$ by*

$$\eta_p C^\lambda(x) = \gamma_{px}(\lambda) \quad \text{for } x \in M,$$

where $\gamma_{px} : [0,1] \to M$ is a geodesic such that $\gamma'_{px}(0) = \eta_p(x)$. This function is called λ-radial contraction *of x based at p with respect to η_p.*

Since there is exactly one length minimizing geodesic between any two points p and x in (M, g) whose initial vector is $\eta_p(x)$ hence $\eta_p C^\lambda$-function is well defined in (M, g). Beltagy and Shenawy [1] studied such type of function in an Euclidean space. They called it λ-*radial contraction* based at p.

Definition 3. *[1] Let A be a nonempty subset of $\mathbb{R}^n$. For a fixed point $p \in A$ and a fixed real number $\lambda \in (0,1]$, the λ-radial contraction of A based at p is denoted by $C_p^\lambda(A)$ and is defined by*

$$C_p^\lambda(A) = \{\lambda x + (1-\lambda)p : x \in A\}.$$

We have generalized this notion in a complete Riemannian manifold and developed a new type of convex set in (M, g).

For a fixed $\lambda \in (0,1]$, the λ-radial contraction of a non-empty subset A based at p with respect to η_p can be defined as

$$\eta_p C^\lambda(A) = \Big\{\eta_p C^\lambda(x) : x \in A\Big\}.$$

In $\mathbb{R}^n$, for each $x \in \mathbb{R}^n$, $\eta_p(x)$ has only one choice hence we denote $\eta_p C^\lambda$ by C_p^λ. In [1] it has been shown that λ-radial contraction of a line segment is also a line segment. But in general this does not hold, see example below.

Example 1. *Take $M = \mathbb{S}^2$ with the line element $ds^2 = d\theta^2 + \sin^2\theta d\phi^2$. Now take two points x, y on the equator and take p as the north pole of $\mathbb{S}^2$. Then if we choose $\lambda = \frac{1}{2}$, then $\alpha = \eta_p C^{\frac{1}{2}}(\gamma_{xy})$ is a curve joining x' and y' but it is not a geodesic, see Figure 1, since the circle containing α has radius less than one, hence it can not be a geodesic.*

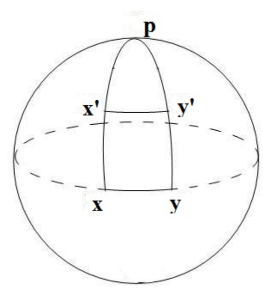

Figure 1:

Naturally the question arises. What is the sufficient condition that the map $\eta_p C^\lambda$ transforms geodesic to geodesic for a fixed η_p?

Definition 4. *Let $p \in M$ and $A \neq \phi$ be a subset of M with the direction function η_p. Then A is called p^λ-convex with respect to η_p for a fixed $\lambda \in (0,1]$ if for any two points $x, y \in A$ there is a length minimizing geodesic connecting $\eta_p C^\lambda(x)$ and $\eta_p C^\lambda(y)$ belongs to A, i.e, $\gamma_{x'y'}(t) \in A, \forall t \in [0,1]$, where $x' = \eta_p C^\lambda(x)$ and $y' = \eta_p C^\lambda(y)$.*

Unless there are some confusion, by a p^λ-convex set we always mean p^λ-convex with respect to some direction function η_p.

Definition 5. *If a non-empty subset A of M is p^λ-convex for all $\lambda \in (0,1]$, then A is called* totally p-convex.

Theorem 1. *Let A be a subset of M containing more than two points. Then A is geodesically convex set in (M, g) if and only if for each point $p \in A$ there exists a direction function η_p such that A is p^λ-convex with respect to η_p $\forall\lambda \in (0,1]$, i.e, A is totally p^λ-convex.*

Proof. Let A be a geodesically convex subset of M. Take a point $p \in A$. By convexity for any point $x \in A$ there exists a geodesic $\gamma_{px} : [0,1] \to M$ such that $\gamma_{px}(0) = p$, $\gamma_{px}(1) = x$ and $\gamma_{px}(t) \in A$ $\forall t \in [0,1]$. Define the function $\eta_p : M \to T_p(M)$ by

$$\eta_p(x) = \gamma'_{px}(0) \quad \forall x \in A.$$

Now for any two points x, y in A, the points $\eta_p C^\lambda(x), \eta_p C^\lambda(y) \in A$ $\forall \lambda \in (0,1]$. Hence there exists a unique geodesic connecting $\eta_p C^\lambda(x)$ and $\eta_p C^\lambda(y)$ that will belongs to A for each $\lambda \in (0,1]$. Hence A is totally p^λ-convex with respect to η_p.

Conversely let A be totally p-convex for each $p \in A$. Take any two points x, y in A. But $x = \eta_p C^\lambda(x)$ and $y = \eta_p C^\lambda(y)$, where $\lambda = 1$. So using the definition of totally p-convex we can say that there is a geodesic arc connecting x and y belongs to A. So A is geodesically convex. □

Corollary 1. *For a subset A in M containing more than two points the following statements are equivalent:*

1. *A is geodesically convex,*
2. *A is totally p-convex for any $p \in A$.*

Proposition 1. *In $\mathbb{R}^n$, a subset A is p^λ-convex for some $p \in A$ and $\lambda \in (0,1]$ if and only if $C_p^\lambda(A)$ is p^μ-convex for some $\mu \in (0,1]$.*

Proof. The proof can be easily deduced from the fact that in $\mathbb{R}^n$, the λ-contraction of a straight line is also a straight line [1]. □

But in case of manifold the above proposition is not true, see Example 2.

Example 2. *Take $M = \mathbb{S}^2$ and for two fixed points x, y on the equator let A be the set $\{\gamma_{xy}(t) : t \in [0,1]\} \cup \{p\}$, where p is the north pole, see Figure 1. Then A is p-convex. Now for some fixed η_p, the $p^{1/2}$-radial contraction of A is the set $= \{$the shortest latitude joining $\eta_p C^{\frac{1}{2}}(x)$ and $\eta_p C^{\frac{1}{2}}(y)\} \cup \{p\}$ but this set is not p^λ-convex for any $\lambda \in (0,1]$.*

One of the main difference between geodesically convex set and p^λ-convex set is that geodesically convex sets are always path-connected and hence, geodesically connected, but p^λ-convex set may be disconnected. For example, consider $M = \mathbb{S}^2$ and take upper hemisphere together with two distinct points in lower hemisphere, then this set is not path connected but it is p^λ-convex set for some $\lambda \in (0,1]$, where p is the noth pole of $\mathbb{S}^2$. We also mention that not all disconnected sets are p^λ-convex. For example, any finite set containing more than two points in $\mathbb{R}^n$ which is disconnected but not p^λ-convex for some $\lambda \in (0,1]$.

Proposition 2. *If a subset $A \subset M$ contains an interior point p then $\exists\, \zeta \in (0,1]$ such that A is p^λ-convex $\forall \lambda \in (0,\zeta]$.*

Proof. Let $p \in A$ be an interior point. Then $B(p,r) \subset A$ for some $r > 0$. Take $V_p = B(p,r) \cap N_p$, where B_p is the geodesic ball p. Now V_p looks like a flat n-dimensional Euclidean space. For each x in A take $\lambda_x = \inf\{\lambda \in (0,1] : \eta_p C^\lambda(x) \in V_p\}$. Now $\lambda_x \neq 0$ since V_p is open so sufficiently small portion of any geodesic emitting from p must lies in V_p. Again take $\zeta = \inf\{\lambda_x : x \in A\}$. Hence A is p^λ-convex for all $\lambda \in (0,\zeta]$. □

Proposition 3. *Let A be a nonempty subset of M and $\lambda, \beta \in (0,1]$. Then for $p \in M$*

$$C_p^\lambda(C_p^\beta(x)) = C_p^{\lambda\beta}(x) \text{ for } x \in A.$$

Proposition 4. *Let $A \subset M$ be a p^λ-convex set. Then A is also p^{λ^n}-convex $\forall n \in \mathbb{N}$.*

Proof. Fixed $m \in \mathbb{N}$ and choose two distinct points x, y from A. Since A is p^λ-convex, so the geodesic $\gamma_{x_1y_1}$ belongs to A where $x_1 = C_p^\lambda(x)$ and $y_1 = C_p^\lambda(y)$. Again by similar argument taking $x_2 = C_p^\lambda(x_1)$ and $y_2 = C_p^\lambda(y_1)$ we get $\gamma_{x_2y_2}$ belongs to A. Hence continuing this way we get $\gamma_{x_ny_n} \in A$ where $x_n = C_p^\lambda(x_{n-1})$ and $y_n = C_p^\lambda(y_{n-1})$. Thus from Proposition 3 we get $x_m = C_p^\lambda \circ C_p^\lambda \circ \cdots \circ C_p^\lambda(x)$ (m-times) and $y_m = C_p^\lambda \circ C_p^\lambda \circ \cdots \circ C_p^\lambda(y)$ (m-times). So, $C_p^{\lambda^m}(x)$, $C_p^{\lambda^m}(y)$ and the geodesic arc joining these two points belong to A. Hence A is p^{λ^m}-convex for any $m \in \mathbb{N}$. □

Theorem 2. *Let $\{A_i\}_{i\in\lambda}$ be an arbitrary collection of p^λ-convex sets and $\bigcap_{i\in\Lambda} A_i$ is nonempty. Then $\bigcap_{i\in\Lambda} A_i$ is also p^λ-convex.*

Proof. Let $x, y \in \bigcap_{i\in\Lambda} A_i$. Hence $x, y \in A_i$ for all $i \in \Lambda$. Now by definition of p^λ-convex set, we have $\eta_p C^\lambda(x), \eta_p C^\lambda(y) \in A_i$ $\forall i \in \Lambda$. Hence $\bigcap_{i\in\Lambda} A_i$ is p^λ-convex. □

Theorem 3. *For any p^λ-convex set $A \subset M$ there exists a geodesically convex subset V containing p such that $V \subset A$.*

Proof. Let A be a p^λ-convex set. Take

$$V = \bigcup_{x,y\in A} \Big\{\gamma_{x'y'} : x' = C_p^\lambda(x), y' = C_p^\lambda(y)\Big\}.$$

We shall show that V is geodesically convex. Choose any two points x, y from B. Then $x = C_p^\lambda(x')$ and $y = C_p^\lambda(y')$ for some $x', y' \in A$. Since A is p^λ-convex, the geodesic $\gamma_{xy} \in A$ implies γ_{xy} belongs to V. Hence V is geodesically convex. □

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