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ON THE EXISTENCE OF COMMON FIXED POINTS OF TWO COMMUTING FUNCTIONS

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Abstract

In this note, we formulate a criterion for the existence of common fixed points of two commuting functions on complete lattices. In this way, we extend and explain some results in the literature.

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1 Introduction

The paper deals with the existence of common fixed points of commuting functions.

Definition 1. *We say that* $f, g : A \to A$ *are two commuting functions if* $f \circ g = g \circ f$.

This old problem has been extensively studied in the literature, in various mathematical contexts. The existence of common fixed points of commuting polynomial functions has been proved by Ritt (1923). Note that, the continuity of two commuting functions on a real compact interval does not provide the existence of common fixed points. A fine counterexample was given by Boyce (1969). A review of significant results in the field was carried out by McDowell (2009).

In the following we intend to highlight a simple and instructive result regarding the existence of common fixed points of two commuting functions. Our result is fundamentally based on the Knaster-Tarski's theorem.

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2 Main results

The following result provides a sufficient condition for the existence of common fixed points of two commuting functions.

Theorem 1. *Let $(L, \leq)$ be a complete lattice and let $f, g : L \to L$ be two commuting functions. Assume that f has at least a fixed point and the set of all fixed points of f is a complete lattice. If g is monotonically increasing, then f and g have a common fixed point.*

Proof. Let us denote by $P_f = \{x \in L : f(x) = x\}$ the set of all fixed points of f. From the hypothesis, $P_f \neq \emptyset$ and $(P_f, \leq)$ is a complete lattice, not necessarily a complete sublattice of $(L, \leq)$. For $x \in P_f$, we have $f(g(x)) = g(f(x)) = g(x)$, so $g(x) \in P_f$. Thus, we can define the restriction $g_f : P_f \to P_f$ of the function g to the set P_f. Since g_f is a monotone increasing function on a complete lattice, from the Knaster-Tarski's theorem there is $x_0 \in P_f$ such that $g_f(x_0) = x_0$. Therefore, $f(x_0) = x_0 = g_f(x_0) = g(x_0)$. Hence, f and g have a common fixed point. □

The above theorem explains and extends some known results on commuting functions on real compacts.

Corollary 1. *Let $f, g : [a, b] \to [a, b]$ be two commuting functions. If f is continuous and g is monotonically increasing, then f and g have a common fixed point.*

Proof. Since $([a, b], \leq)$ is a complete lattice, it is enough to verify that the set $P_f = \{x \in L : f(x) = x\}$ is also a complete sublattice of $[a, b]$. Thus, $P_f \neq \emptyset$ (as a consequence of the intermediate value theorem). Let U be a nonempty subset of P_f. If $u = \sup(U) \in [a, b]$, then there exists a sequence $(u_n)_{n\geq 1}$, with the terms in U, such that $u_n \to u$. Due to the continuity of f, we have $f(u) = \lim_{n\to\infty} f(u_n) = \lim_{n\to\infty} u_n = u$. Hence $u \in P_f$ is the least upper bound of U. Similarly, $\inf(U) \in P_f$. □

The above statement extends the result of Folkman (1966). Note that, an alternative proof can be found in Păltănea (2018).

Corollary 2. *Let $f, g : [a, b] \to [a, b]$ be two monotonically increasing commuting functions. Then f and g have a common fixed point.*

Proof. The Knaster-Tarski's theorem ensures that $(P_f, \leq)$ is a complete lattice, not necessarily a complete sublattice of $([a, b], \leq)$. As follows, f and g have a common fixed point. □

An elementary proof of this result is presented in Păltănea (2018).

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