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η-RICCI SOLITONS IN 3-DIMENSIONAL NORMAL ALMOST CONTACT METRIC MANIFOLDS

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Abstract

The object of the present paper is to study the normal almost contact metric manifolds admitting the η-Ricci Solitons. It is shown that a symmetric second order covariant tensor in a normal almost contact metric manifold is a constant multiple of metric tensor. Also an example of η-Ricci soliton in 3-dimensional normal almost contact metric manifolds is provided in the region where normal almost contact metric manifolds expanding. Also we obtain the η-Ricci soliton in quasi Sasakian manifolds which satisfies cyclic parallel Ricci tensor, then the manifold is of constant curvature. Finally, we show the existence of such a manifold by an example.

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1 Introduction

In recent years the pioneering works of R. Hamilton [10] and G. Perelman [16] towards the solution of the Poincare conjecture in dimension 3 have produced a flourishing activity in the research of self similar solutions, or solitons, of the Ricci flow. The study of the geometry of solitons, in particular their classification in dimension 3, has been essential in providing a positive answer to the conjecture; however in higher dimension and in the complete, possibly noncompact case, the understanding of the geometry and the classification of solitons seems to remain a desired goal for a not too proximate future. In the generic case a soliton structure on the Riemannian manifold (M, g) is the choice of a smooth vector field X on M and a real constant λ satisfying the structural requirement

$$Ric + \frac{1}{2}\mathcal{L}_X g = \lambda g, \tag{1.1}$$

where Ric is the Ricci tensor of the metric g and $\mathcal{L}_X g$ is the Lie derivative of this latter in the direction of X. In what follows we shall refer to λ as to the soliton

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constant. The soliton is called expanding, steady or shrinking if, respectively, $\lambda > 0$, $\lambda = 0$ or $\lambda > 0$. When X is the gradient of a potential $\psi \in C^\infty(M)$, the soliton is called a gradient Ricci soliton [20] and the previous equation (1.1) takes the form

$$\nabla\nabla\psi = S + \lambda g. \tag{1.2}$$

Both equations (1.1) and (1.2) can be considered as perturbations of the Einstein equation

$$Ric = \lambda g. \tag{1.3}$$

and reduce to this latter in case X or $\nabla\psi$ are Killing vector fields. When $X = 0$ or ψ is constant we call the underlying Einstein manifold a trivial Ricci soliton.

Definition 1.1. *A Ricci soliton (g, V, λ) on a Riemannian manifold is defined by*

$$\mathcal{L}_V g + 2S + 2\lambda g = 0, \tag{1.4}$$

where S is the Ricci tensor, $\mathcal{L}_V$ is the Lie derivative along the vector field V on M and λ is a real scalar. Ricci soliton is said to be shrinking, steady or expanding according as $\lambda < 0, \lambda = 0$ and $\lambda > 0$, respectively.

It is well know fact that, if the potential vector filed is zero or Killing, then the Ricci soliton is an Einstein real hypersurfaces on non-flat complex space forms [6]. Motivated by this in 2009, J. T. Cho and M. Kimura [7] introduced the notion of η-Ricci solitons and gave a classification of real hypersurfaces in non-flat complex space forms admitting η-Ricci solitons.

Definition 1.2. *An η-Ricci soliton (g, V, λ, μ) on a Riemannian manifold is defined by*

$$\mathcal{L}_X g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0, \tag{1.5}$$

where S is the Ricci tensor, $\mathcal{L}_X$ is the Lie derivative along the vector field X on M and λ, μ are real scalars. In particular $\mu = 0$ then the data (g, V, λ) is a Ricci soliton.

In 1925, Levy [12] proved that a second order parallel symmetric non-singular tensor in real space forms is proportional the metric tensor. Later, R. Sharma [18] initiated the study of Ricci solitons in contact Riemannian geometry . After that, Tripathi [20] Nagaraja et al. [14] and others like M. Turan et al. [21] extensively studied Ricci soliton in almost contact metric manifolds. In 2015, [17] S. K. Perktas and S. Keles was studied the Ricci soliton in normal almost paracontact metric manifolds. Recently, A. M. Blaga and various others authors also have been studied η-Ricci solitons in manifolds with different structures (see [4], [5], [15], [19]). In this paper we study the η-Ricci soliton in normal almost contact metric manifolds.

2 Preliminaries

A differentiable manifold M of dimension $(2n+1)$ is called almost contact manifold with the almost contact structure (φ, ξ, η) if it admits a tensor field φ of type $(1,1)$, a vector field ξ of type $(1,0)$ and a 1-form η of type $(0,1)$ satisfying the following conditions [13]

$$\varphi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta \circ \varphi = 0. \tag{2.1}$$

Let $\mathbb{R}$ be the real line and t a coordinate on $\mathbb{R}$. Define an almost complex structure J on $M \times \mathbb{R}$ by

$$J\left(X, \frac{\lambda d}{dt}\right) = \left(\varphi X - \lambda\xi, \eta(X)\frac{d}{dt}\right), \tag{2.2}$$

where the pair $\left(X, \frac{\lambda d}{dt}\right)$ denotes a tangent vector to $M \times \mathbb{R}$, X and $\frac{\lambda d}{dt}$ being tangent to M and $\mathbb{R}$ respectively.

M and (φ, ξ, η) are said to be normal if the structure J in integrable ([1], [2]). The necessary and sufficient condition for (φ, ξ, η) to be normal is

$$N_\varphi[X, Y] + 2d\eta \otimes \xi = 0, \tag{2.3}$$

where $N_\varphi[X, Y]$ is the Nijenhuis tensor of φ defined by

$$N_\varphi[X, Y] = [\varphi X, \varphi Y] + \varphi^2[X, Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] \tag{2.4}$$

for any $X, Y \in \chi(M)$; $\chi(M)$ being the Lie algebra of vector fields on M.

We say that the one form η has rank $r = 2s$ if $(d\eta)^s \neq 0$, and $\eta \wedge (d\eta)^s = 0$, and has rank $r = 2s+1$ if $\eta \wedge (d\eta)^s \neq 0$ and $(d\eta)^{s+1} = 0$. We also say that r is the rank of structure (φ, ξ, η).

A Riemannian metric g on M satisfies the condition

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.5}$$

for any $X, Y \in \chi(M)$, is said to be compatible with structure (φ, ξ, η). If g is such a metric, then the quadruple (φ, ξ, η, g) is called an almost contact metric structure on M and M is an almost contact metric manifold. On such a manifold we also have

$$g(X, \xi) = \eta(X)$$

for any $X \in \chi(M)$ and we can always define the 2-form Φ by

$$\Phi(X, Y) = g(X, \varphi Y),$$

where $X, Y \in \chi(M)$.
A normal almost contact metric structure (φ, ξ, η, g) satisfying additionally the condition $d\eta = \Phi$ is called Sasakian. Of course, any such structure on M has rank 3. Also a normal almost contact metric structure satisfying the condition $d\Phi = 0$

is said to be quasi Sasakian [3].

In [13], Olszak studied the some properties of normal almost contact manifold of dimension three.
For a normal almost contact metric structure (φ, ξ, η, g) on M, we have [13]

$$(\nabla_X \varphi)Y = g(\varphi \nabla_X \xi, Y)\xi - \eta(Y)\varphi \nabla_X \xi, \tag{2.6}$$

$$\nabla_X \xi = \alpha[X - \eta(X)\xi] - \beta \phi X, \tag{2.7}$$

where $2\alpha = div\xi$ and $2\beta = tr(\varphi \nabla \xi)$, $div\xi$ is the divergence of ξ defined by

$$div\xi = trace\{X \to \nabla_X \xi\}, \quad tr(\varphi \nabla \xi) = trace\{X \to \varphi \nabla_X \xi\}$$

A 3-dimensional normal almost contact metric manifold is said to be

- Cosympectic [1] if $\alpha = \beta = 0$,
- quasi-Sasakian [3] if and only if $\alpha = 0$ and $\beta \neq 0$,
- β-Sasakian [3] if and only if $\alpha = 0$, $\beta \neq 0$ and β is constant, in particular Sasakian if $\beta = -1$,
- α-Kenmotsu [11] if $\alpha \neq 0$ and α is constant and $\beta = 0$.

3 Basic curvature identities

In this section we discuss some curvature identities of 3-dimensional normal almost contact metric manifolds:

Let M be a 3-dimensional normal almost contact metric manifold. Then we have the following conditions [13]

$$R(X,Y)\xi = (Y\alpha + (\alpha^2 - \beta^2))\varphi^2 X - (X\alpha + (\alpha^2 - \beta^2))\varphi^2 Y \tag{3.1}$$

$$+(Y\beta + 2\alpha\beta\eta(Y))\varphi X - (X\beta + 2\alpha\beta\eta(X))\varphi Y,$$

$$S(Y,\xi) = -Y\alpha - (\varphi Y)\beta - (\xi\alpha + 2(\alpha^2 - \beta^2))\eta(Y), \tag{3.2}$$

$$\xi\beta + 2\alpha\beta = 0 \tag{3.3}$$

for all $X, Y \in TM$, where R denotes the curvature tensor and S is the Ricci tensor. On the other hand, the curvature tensor in 3-dimensional Riemannian manifold always satisfies:

$$\bar{R}(X,Y,Z,W) = g(X,W)S(Y,Z) - g(X,Z)S(Y,W) \tag{3.4}$$

$$+g(Y,Z)S(X,W) - g(Y,W)S(X,Z)$$

$$-\frac{r}{2}[g(X,W)g(Y,Z)-g(X,Z)g(Y,W)],$$

where $\bar{R}(X,Y,Z,W)=g(R(X,Y)Z,W)$ and r is the scalar curvature.
From (3.1)

$$\bar{R}(\xi,Y,Z,\xi)=-(\xi\alpha+(\alpha^2-\beta^2))g(\varphi Y,\varphi Z)-(\xi\beta+2\alpha\beta)g(Y,\varphi Z), \tag{3.5}$$

By (3.2), (3.4) and (3.5) we obtain for

$$\alpha=Constant \quad \text{and} \quad \beta=Constant,$$

$$S(X,Y)=\left(\frac{r}{2}+(\alpha^2-\beta^2)\right)g(\varphi X,\varphi Y)-2(\alpha^2-\beta^2)\eta(X)\eta(Y) \tag{3.6}$$

$$QX=\left(\frac{r}{2}+(\alpha^2-\beta^2)\right)X-2(\alpha^2-\beta^2)\eta(X)\xi, \tag{3.7}$$

$S(X,Y)=g(QX,Y)$, where S Ricci curvature tensor, Q Ricci operator.
Applying (3.5) in (3.4) we get

$$R(X,Y)Z=\left(\frac{r}{2}+2(\alpha^2-\beta^2)\right)[g(Y,Z)X-g(X,Z)Y] \tag{3.8}$$

$$+g(X,Z)\left[\left(\frac{r}{2}+3(\alpha^2-\beta^2)\right)\right]\eta(Y)\xi$$

$$-\left(\frac{r}{2}+3(\alpha^2-\beta^2)\right)\eta(Y)\eta(Z)X$$

$$-g(Y,Z)\left[\left(\frac{r}{2}+3(\alpha^2-\beta^2)\right)\right]\eta(X)\xi$$

$$+\left(\frac{r}{2}+3(\alpha^2-\beta^2)\right)\eta(X)\eta(Z)Y.$$

It is to be noted that the general formulas can be obtained by straightforward calculation

From (3.3), it follows that a 3-dimensional normal almost contact metric manifold with $\alpha,\beta=Constant$, then the manifold is either β-Sasakian [3] or α-Kenmotsu [11] or Cosymplectic [8].

4 η-Ricci solitons on normal almost contact metric manifolds

Fix h a symmetric tensor field of $(0,2)$-type which we suppose to be parallel with respect to the Levi-Civita connection ∇ that is $\nabla h = 0$. Applying the Ricci commutation identity [9].

$$\nabla^2 h(X,Y;Z,W) - \nabla^2 h(X,Y;W,Z) = 0, \tag{4.1}$$

we obtain the relation

$$h(R(U,V)X,Y) + h(X,R(U,V)Y) = 0, \tag{4.2}$$

where U,V,X and Y are arbitrary vectors on M.

As h is symmetric, putting $U = X = Y = \xi$ in (4.2), we obtain

$$h(\xi, R(\xi,X)\xi) = 0. \tag{4.3}$$

Let us assume that M is non-cosympletic.
Now applying (3.1) in (4.3) we have

$$(\alpha^2-\beta^2)h(X,\xi) - (\alpha^2-\beta^2)\eta(X)h(\xi,\xi) - 2\alpha\beta h(\varphi X,\xi) = 0. \tag{4.4}$$

Putting φX instead of X in (4.4) and Using (2.1) we get

$$(\alpha^2-\beta^2)[h(X,\xi) - \eta(X)h(\xi,\xi)] = 0.$$

Since M is non-cosymplectic, we have

$$h(X,\xi) - \eta(X)h(\xi,\xi) = 0. \tag{4.5}$$

Differentiating (4.5) covariantly along Y and applying (4.5) and (3.3) we find

$$\alpha[h(X,Y) - h(\xi,\xi)g(X,Y)] = \beta[h(X,\varphi Y) - h(\xi,\xi)g(X,\varphi Y)]. \tag{4.6}$$

Putting φY instead of Y in (4.6) and using (2.1) we have

$$(\alpha^2-\beta^2)[h(X,Y) - h(\xi,\xi)g(X,Y)] = 0. \tag{4.7}$$

This implies

$$h(X,Y) = h(\xi,\xi)g(X,Y), \qquad \text{since} \quad \alpha^2-\beta^2 \neq 0. \tag{4.8}$$

Hence, since h and g are parallel tensor field, $\lambda = h(\xi,\xi)$ is constant. By the parallelism of h and g it must be $h = \lambda g$ on M. Thus we have the following:

Theorem 4.1. *A parallel symmetric $(0,2)$ tensor field in a 3-dimensional non-cosymplectic normal almost contact metric manifold is a constant multiple of the associated metric tensor.*

Definition 4.2. *Let (M, ϕ, ξ, η, g) be a normal almost contact metric manifold. Consider the equation*

$$\mathcal{L}_\xi g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0, \tag{4.9}$$

where $\mathcal{L}_\xi$ is the Lie derivative operator along the vector field ξ, S is the Ricci curvature tensor field of the metric g and λ and μ are real constants. Writing $L_\xi g$ in terms of the Levi-Civita connection ∇, we obtain:

$$2S(X,Y) = -g(\nabla_X \xi, Y) - g(X, \nabla_Y \xi) - 2\lambda g(X,Y) - 2\mu\eta(X)\eta(Y), \tag{4.10}$$

for any $X, Y \in \chi(M)$.

The data (g, ξ, λ, μ) which satisfy the equation (4.9) is said to be η- Ricci solitons on M [11] ; in particular if $\mu = 0$ then (g, ξ, λ) is Ricci solitons [11] and its called shrinking, steady or expanding according as $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$ respectively [11].

Now, from (2.7) , the equation (4.10) becomes:

$$S(X,Y) = -(\lambda + \alpha)g(X,Y) + (\alpha - \mu)\eta(X)\eta(Y). \tag{4.11}$$

The above equations yields

$$S(X, \xi) = -(\lambda + \mu)\eta(X) \tag{4.12}$$

$$QX = -(\lambda + \alpha)X + (\alpha - \mu)\xi \tag{4.13}$$

$$Q\xi = -(\lambda + \mu)\xi \tag{4.14}$$

$$r = -\lambda n - (n-1)\alpha - \mu, \tag{4.15}$$

where r is the scalar curvature. Off the two natural situations regarding the vector field $V : V \in Span\,\{\xi\}$ and $V \perp \xi$, we investigate only the case $V = \xi$.

Our interest is in the expression of $\mathcal{L}\xi g + 2S + 2\mu\eta \otimes \eta$. A straightforward computations gives

$$\mathcal{L}_\xi g(X,Y) = 2\alpha[g(X,Y) - \eta(X)\eta(Y)]. \tag{4.16}$$

In a 3-dimensional normal almost contact metric manifold, we have

$$R(X,Y)Z = g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y \tag{4.17}$$

$$-\frac{r}{2}[g(Y,Z)X - g(X,Z)Y],$$

By using (3.6) and (3.7), we obtain

$$S(X,Y) = \left[\frac{r}{2} - (\alpha^2 - \beta^2)\right] g(X,Y) - \left[\frac{r}{2} - 3(\alpha^2 - \beta^2)\right] \eta(X)\eta(Y) \tag{4.18}$$

$$QX = \left[\frac{r}{2} - (\alpha^2 - \beta^2)\right] X - \left[\frac{r}{2} - 3(\alpha^2 - \beta^2)\right] \eta(X)\xi. \tag{4.19}$$

Next, we consider the equation

$$h(X,Y) = \mathcal{L}_\xi g(X,Y) + 2S(X,Y) + 2\mu\eta(X)\eta(Y). \tag{4.20}$$

By using (4.16) and (4.19) in (4.20), we have

$$h(X,Y) = \left[\frac{r}{2} - (\alpha^2 - \beta^2) + 2\alpha\right] g(X,Y) \tag{4.21}$$

$$- \left[\frac{r}{2} - (\alpha^2 - \beta^2) + 2\alpha + 2\mu\right] \eta(X)\eta(Y).$$

Putting $X = Y = \xi$ in (4.21), we get

$$h(\xi,\xi) = 2[2(\alpha^2 - \beta^2) - \mu]. \tag{4.22}$$

So, (4.8) becomes

$$h(X,Y) = 2[2(\alpha^2 - \beta^2) - \mu]g(X,Y). \tag{4.23}$$

From (4.20) and (4.23), it follows that g is an η-Ricci soliton.

Now, we can state the following theorem:

Theorem 4.3. *Let M be a 3-dimensional non-cosymplectic normal almost contact metric manifold. Then (g, ξ, μ) yields an η-Ricci soliton on M.*

Let V be pointwise collinear with ξ. i.e., $V = b\xi$, where b is a function on the 3-dimensional normal almost contact metric manifold. Then

$$g(\nabla_X b\xi, Y) + g(\nabla_Y b\xi, X) + 2S(X,Y) + 2\lambda g(X,Y) + 2\mu\eta(X)\eta(Y) = 0.$$

or

$$bg((\nabla_X \xi, Y) + (Xb)\eta(Y) + bg(\nabla_Y \xi, X) + (Yb)\eta(X)$$
$$+2S(X,Y) + 2\lambda g(X,Y) + 2\mu\eta(X)\eta(Y) = 0.$$

Using (2.7), we obtain

$$bg(\alpha(X - \eta(X)\xi - \beta\varphi X, Y) + (Xb)\eta(Y) + bg(\alpha(Y - \eta(Y)\xi - \beta\varphi Y, X)$$

$$+(Yb)\eta(X) + 2S(X,Y) + 2\lambda g(X,Y) + 2\mu\eta(X)\eta(Y) = 0.$$

which yields

$$-2b\alpha g(X,Y) - 2b\alpha\eta(X)\eta(Y) + (Xb)\eta(Y) \tag{4.24}$$

$$+(Yb)\eta(X) + 2S(X,Y) + 2\lambda g(X,Y) + 2\mu\eta(X)\eta(Y) = 0.$$

Replacing Y by ξ in (4.24), we obtain

$$(Xb) + (\xi b)\eta(X) + 2[2(\alpha^2 - \beta^2) + \lambda + \mu - 2b\alpha]\eta(X) = 0. \tag{4.25}$$

Again putting $X = \xi$ in (4.25), we obtain

$$\xi b = -2(\alpha^2 - \beta^2) - \lambda - \mu + 2b\alpha.$$

Plugging this in (4.25), we get

$$(Xb) + 2[2(\alpha^2 - \beta^2) + \lambda + \mu]\eta(X) = 0,$$

or

$$db = -\left\{\lambda + \mu + 2(\alpha^2 - \beta^2) - 2b\alpha\right\}\eta. \tag{4.26}$$

Applying d on (4.26), we get $\left\{\lambda + \mu + 2(\alpha^2 - \beta^2)\right\}d\eta$. Since $d\eta \neq 0$ we have

$$\lambda + \mu + 2(\alpha^2 - \beta^2) - 2b\alpha = 0. \tag{4.27}$$

Equation(4.27) in (4.26) yields b as a constant. Therefore from (4.25), it follows that

$$S(X, Y) = -(\lambda + b\alpha)g(X, Y) + (b\alpha - \mu)\eta(X)\eta(Y),$$

which implies that M is of constant scalar curvature for constant α. This leads to the following:

Theorem 4.4. *If in a 3-dimensional non-cosymplectic normal almost contact metric manifold the metric g is an η-Ricci soliton and V is positive collinear with ξ, then V is a constant multiple of ξ and g is of constant scalar curvature provided α, β is a constant.*

Let us consider the converse, that is let M be a 3-dimensional η-Einstein normal almost contact metric manifold with $\alpha, \beta = constant$ and $V = \xi$. Then we can write

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y), \tag{4.28}$$

where a, b are scalars and $X, Y \in TM$. From (2.7) we have

$$(\mathcal{L}_\xi g)(X, Y) = 2\alpha[g(X, Y) - \eta(X)\eta(Y)], \tag{4.29}$$

which implies that

$$(\mathcal{L}_\xi g)(X, Y) + 2S(X, Y) + 2\lambda g(X, Y) + 2\mu\eta(X)\eta(Y) \tag{4.30}$$

$$= 2(\alpha + a + \lambda)g(X, Y) - 2(\alpha - b + \mu)\eta(X)\eta(Y).$$

From the previous equation it is obvious that M admits η-Ricci soliton (g, ξ, λ, μ) if

$$\alpha + b + \lambda = 0 \tag{4.31}$$

and

$$b = -(\alpha + \mu) = constant. \tag{4.32}$$

Equating the right hand side of (2.6) and (4.28) and taking $X = Y = \xi$, we obtain

$$a + b = -2(\alpha^2 - \beta^2),$$

that is,

$$a = -2(\alpha^2 - \beta^2) + \alpha + \mu = constant$$

Thus, we get

Theorem 4.5. *Let M be a 3-dimensional non-cosymplectic normal almost contact metric manifold with $\alpha, \beta = constant$. If M is an η-Einstein manifold with $S = ag + b\eta \otimes \eta$, then the manifold admits a η-Ricci soliton $(g, \xi, -(a + b), \mu)$.*

Now taking $V = \xi$, the equation (4.9) becomes

$$(\mathcal{L}_\xi g)(X, Y) + 2S(X, Y) + 2\lambda g(X, Y) + 2\mu\eta(X)\eta(Y) = 0 \tag{4.33}$$

for all $X, Y \in TM$. By using (2.7), we get

$$(\mathcal{L}_\xi g)(X, Y) = 2\alpha[g(X, Y) - \eta(X)\eta(Y)]. \tag{4.34}$$

Using (4.34) and (2.6) we have

$$(\mathcal{L}_\xi g)(X, Y) + 2S(X, Y) = -\left\{r + 2(\alpha^2 - \beta^2)\right\} g(X, Y) \tag{4.35}$$

$$+ \left\{r - 2(\alpha^2 - \beta^2) - \alpha\right\} \eta(X)\eta(Y).$$

Replacing the last equation in (4.33) and taking $X = Y = \xi$, we obtain

$$\lambda = -2(\alpha^2 - \beta^2) + \mu \tag{4.36}$$

From (4.15) and (4.36), we obtain

$$r = 6(\alpha^2 - \beta^2) - 2\alpha + 2\mu. \tag{4.37}$$

Since λ is constant, it follows from (4.36) that $-(\alpha^2 - \beta^2)$ is a constant.

Theorem 4.6. *Let(g, ξ, μ) be an η-Ricci soliton in 3-dimensional non-cosymplectic normal almost contact metric manifold M. Then the scalar λ and the scalar curvature r satisfies the relations: $\lambda + \mu = -2(\alpha^2 - \beta^2)$, $r = 6(\alpha^2 - \beta^2) + 2\alpha + 2\mu$.*

Remark 4.7. *For $\mu = 0$, equation (4.36) reduces to $\lambda = -2(\alpha^2 - \beta^2)$, so we can state the following theorem:*

Theorem 4.8. *If a 3-dimensional non-cosymplectic normal almost contact metric manifold with $\alpha, \beta = constant$ admits an η-Ricci soliton (g, ξ, λ, μ) then the Ricci soliton is shrinking.*

Example of η-Ricci soliton in a 3-dimensional normal almost contact metric manifold.

Example 4.9. *Let $M = \{(x, y, z) \in \mathbb{R}^3 : z \neq 0\}$ where (x, y, z) are the standard coordinates of $\mathbb{R}^3$.*

The vector fields are

$$e_1 = z\frac{\partial}{\partial x}, \qquad e_2 = z\frac{\partial}{\partial y} \qquad e_3 = z\frac{\partial}{\partial z}$$

Let g be the Riemannian metric defined by

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1, \qquad g(e_1, e_3) = g(e_2, e_3) = g(e_1, e_2) = 0$$

that is, the form of the metric becomes

$$g = \frac{dx^2 + dy^2 + dz^2}{z^2}.$$

Let η be the 1-form defiend by $\eta(Z) = g(Z, e_3)$ for any $Z \in \chi(M)$.
Also, let φ be the $(1, 1)$ tensor field defined by

$$\varphi(e_1) = -e_2, \quad \varphi(e_2) = e_1, \quad \varphi(e_3) = 0.$$

Thus using the linearity of φ and g, we have

$$\eta(e_3) = 0, \quad , \eta(e_1) = 0, \quad \eta(e_2) = 0,$$

$$[e_1, e_2] = 0, \quad [e_2, e_3] = -e_2, \quad [e_1, e_3] = -e_1,$$

$$\varphi^2 Z = -Z + \eta(Z)e_3$$

,

$$g(\varphi Z, \varphi W) = g(Z, W) - \eta(Z)\eta(W)$$

for any $Z, W \in \chi(M)$.

Then for $e_3 = \xi$, the structure (φ, ξ, η, g) defines an almost contact metric structure on M.
Let ∇ be the Levi-Civita connection with respect to the metric g, then we have

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z])$$

$$-g(Y, [X, Z]) + g(Z, [X, Y]),$$

which is know as Koszul's formula.
Using Koszul's formula we have

$$\nabla_{e_1}e_1 = e_3, \quad \nabla_{e_1}e_2 = 0, \quad \nabla_{e_1}e_3 = -e_1,$$

$$\nabla_{e_2}e_1 = 0, \quad \nabla_{e_2}e_2 = e_3, \quad \nabla_{e_2}e_3 = -e_2,$$

$$\nabla_{e_3}e_1 = 0, \quad \nabla_{e_3}e_2 = 0, \quad \nabla_{e_3}e_3 = 0. \tag{4.38}$$

From (4.38) we find that the manifold satisfies (2.7) for $\alpha = -1$ and $\beta = 0$ and $\xi = e_3$. hence the manifold is a normal almost conatct metric manifold with $\alpha, \beta = constant$.

Then the Riemannian and Ricci curvature tensor fields are given by:

$$R(e_1, e_2)e_3 = 0, \quad R(e_2, e_3)e_3 = -e_3, \quad R(e_1, e_3)e_3 = -e_1,$$

$$R(e_1, e_2)e_2 = -e_1, \quad R(e_2, e_3)e_2 = e_3, \quad R(e_1, e_3)e_2 = 0,$$

$$R(e_1, e_2)e_1 = e_2, \quad R(e_2, e_3)e_1 = 0, \quad R(e_1, e_3)e = e_3.$$

From the above expressions of the curvature tensor we obtain

$$S(e_1, e_1) = g(R(e_1, e_2)e_2, e_1) + g(R(e_1, e_3)e_3, e_1) = -2$$

similarly we have

$$S(e_1, e_1) = S(e_2, e_2) = S(e_3, e_3) = -2.$$

In case of η-Ricci soliton, from the relation (4.10) it is sufficient to verify that

$$S(e_i, e_i) = -(\lambda + \alpha)g(e_i, e_i) + (\alpha - \mu)\eta(e_i)\eta(e_i) \tag{4.39}$$

for all $i = 1, 2, 3$ and $\alpha = -1$, $\beta = 0$, we get

$$S(e_1, e_1) = -(\lambda + \alpha)g(e_1, e_1)$$

which implies

$$-2 \;\; = \;\; -(\lambda - 1) \Rightarrow \quad \lambda = 3.$$

Also,

$$S(e_3, e_3) = -(\lambda + \alpha)g(e_3, e_3) + (\alpha - \mu)\eta(e_3)\eta(e_3) \tag{4.40}$$

By using $\lambda = 3$ and $\alpha = -1$ in (4.40) we obtain $\mu = 5$.
Therefore, the data (g, ξ, λ, μ) is an η-Ricci soliton in 3-dimensional normal almost contact metric manifold.

For this example we have $\lambda = 3$, i.e. $\lambda > 0$ so that the η-Ricci soliton is expanding.

5 η-Ricci soliton in Quasi-Sasakian manifolds

An almost contact metric manifold $M^{(2n+1)}$ with an almost contact structure (φ, ξ, η) is said to be quasi-Sasakian manifold if it is normal and the fundamental 2-form Φ is closed i.e.,

$$d\Phi = 0, \quad \Phi(X, Y) = g(X, \varphi Y).$$

For a particular choice of $\alpha = 0$, $\beta \neq 0$ in equations (2.7), (3.6) and (3.7) , we have quasi-Sasakian manifold. We may refer to [3] for more information about the quasi-Sasakian manifold. We also recall that quasi-Sasakian manifold satisfies the following conditions [13]:

$$\nabla_X \xi = -\beta \varphi X, \tag{5.1}$$

$$(\nabla_X \eta)Y = g(\nabla_X, \xi, Y) = -\beta g(\varphi X, Y), \tag{5.2}$$

$$R(X, Y)\xi = \beta^2[\eta(Y)X - \eta(X)Y], \tag{5.3}$$

$$S(X, \xi) = 2\beta^2 \eta(X) \tag{5.4}$$

$$S(X, Y) = \left(\frac{r}{2} - \beta^2\right) g(X, Y) + \left(3\beta^2 - \frac{r}{2}\right) \eta(X)\eta(Y) \tag{5.5}$$

$$QX = \left(\frac{r}{2} - \beta^2\right) X + \left(3\beta^2 - \frac{r}{2}\right) \eta(X)\xi. \tag{5.6}$$

Remark 5.1. *β-Sasakian manifold are quasi-Sasakian manifold. For particular value $\beta = -1$ we obtain the conditions for Sasakian manifold.*

Now, we have to show that a 3-dimensional quasi-Sasakian manifold admits the η-Ricci soliton.

Let $(M, \varphi, \xi, \eta, g)$ be a quasi-Sasakian manifold. Again, consider the equation (4.9)

$$\mathcal{L}_\xi g + 2S + 2\lambda g + 2\mu \eta \otimes \eta = 0, \tag{5.7}$$

It is know that [3] in a quasi-Sasakian maifold ξ is Killing therefor $\mathcal{L}_\xi g = 0$. Also using equation (2.7) in (5.7), the definition of η-Ricci-Soliton for 3-dimensional quasi-Sasakian manifold reduces in the following form

$$2S + 2\lambda g + 2\mu \eta \otimes \eta = 0, \tag{5.8}$$

or

$$S(X, Y) = -\lambda g(X, Y) - \mu \eta(X)\eta(Y). \tag{5.9}$$

The above equations yields

$$QX = -\lambda X - \mu\eta(X)\xi, \tag{5.10}$$

$$S(X,\xi) = -(\lambda+\mu)\eta(X), \tag{5.11}$$

$$Q\xi = -(\lambda+\mu)\xi, \tag{5.12}$$

$$r = -3\lambda - \mu, \tag{5.13}$$

Thus, we can state the following:

Theorem 5.2. *The 3-dimensional quasi-Sasakian manifold with scalar curvature, i.e., $r = -(3\lambda+\mu)$ admits the η-Ricci- solitons.*

For $\mu = 0$, immediately we have a corollary

Corollary 5.3. *The 3-dimensional quasi-Sasakian manifold with scalar curvature, i.e., $r = -3\lambda$ admits Ricci- solitons.*

Example of η-Ricci soliton in a 3-dimensional Quasi-Sasakian manifold.

Example 5.4. *Let $M = \{(x,y,z) \in \mathbb{R}^3 : z \neq 0\}$ where (x,y,z) are the standard coordinates of $\mathbb{R}^3$.*

The vector fields are

$$e_1 = \frac{\partial}{\partial z} - y\frac{\partial}{\partial x}, \qquad e_2 = \frac{\partial}{\partial y}, \qquad e_3 = 2\frac{\partial}{\partial x}$$

Let g be the Riemannian metric defined by

$$g(e_1,e_1) = g(e_2,e_2) = g(e_3,e_3) = 1, \qquad g(e_1,e_3) = g(e_2,e_3) = g(e_1,e_2) = 0$$

that is, the form of the metric becomes Let η be the 1-form defiend by $\eta(Z) = g(Z,e_3)$ for any $Z \in \chi(M)$.

Also, let φ be the $(1,1)$ tensor field defined by

$$\varphi(e_1) = -e_2, \quad \varphi(e_2) = e_1, \quad \varphi(e_3) = 0.$$

Thus using the linearity of φ and g, we have

$$\eta(e_3) = 0, \quad ,\eta(e_1) = 0, \quad \eta(e_2) = 0,$$

$$[e_1,e_2] = \frac{1}{2}e_3, \quad [e_2,e_3] = 0, \quad [e_1,e_3] = 0,$$

$$\varphi^2 Z = -Z + \eta(Z)e_3$$

,

$$g(\varphi Z, \varphi W) = g(Z, W) - \eta(Z)\eta(W)$$

for any $Z, W \in \chi(M)$.

Then for $e_3 = \xi$, the structure (φ, ξ, η, g) defines an almost contact metric structure on M.
Let ∇ be the Levi-Civita connection with respect to the metric g, then we have

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z])$$
$$-g(Y, [X, Z]) + g(Z, [X, Y]),$$

which is know as Koszul's formula.
Using Koszul's formula we have

$$\nabla_{e_1} e_1 = 0, \quad \nabla_{e_1} e_2 = -\frac{1}{4} e_3, \quad \nabla_{e_1} e_3 = \frac{1}{4} e_3,$$

$$\nabla_{e_2} e_1 = \frac{1}{4} e_3, \quad \nabla_{e_2} e_2 = 0, \quad \nabla_{e_2} e_3 = -\frac{1}{4} e_1,$$

$$\nabla_{e_3} e_1 = \frac{1}{4} e_2, \quad \nabla_{e_3} e_2 = -\frac{1}{4} e_1, \quad \nabla_{e_3} e_0 = 0. \tag{5.14}$$

From (5.14) we find that the structure (φ, ξ, η, g) satisfies the formula (5.3) for $\beta = \frac{1}{4}$ and $\xi = e_3$. Hence the manifold is a 3-dimensional quasi-Sasakian manifold with the constant structure function $\beta = \frac{1}{4}$.

Then the Riemannian and Ricci curvature tensor fields are given by:

$$R(e_1, e_2)e_3 = 0, \quad R(e_2, e_3)e_3 = \frac{1}{16} e_2, \quad R(e_1, e_3)e_3 = \frac{1}{16} e_1,$$

$$R(e_1, e_2)e_2 = -\frac{3}{16} e_1, \quad R(e_2, e_3)e_2 = -\frac{1}{16} e_3, \quad R(e_1, e_3)e_2 = 0,$$

$$R(e_1, e_2)e_1 = \frac{3}{16} e_2, \quad R(e_2, e_3)e_1 = 0, \quad R(e_1, e_3)e = -\frac{1}{16} e_3.$$

From the above expressions of the curvature tensor we obtain

$$S(e_1, e_1) = g(R(e_1, e_2)e_2, e_1) + g(R(e_1, e_3)e_3, e_1) = -\frac{1}{8}$$

similarly we have

$$S(e_1, e_1) = S(e_2, e_2) = -\frac{1}{8}, \quad \text{and} S(e_3, e_3) = \frac{1}{8}$$

. In case of η-Ricci soliton, from the relation (5.9) it is sufficient to verify that

$$S(e_i, e_i) = -\lambda g(e_i, e_i) - \mu\eta(e_i)\eta(e_i) \tag{5.15}$$

for all $i = 1, 2, 3$ and $\beta = \frac{1}{4}$, we get

$$S(e_1, e_1) = -\lambda g(e_1, e_1)$$

which implies

$$-\frac{1}{8} = -\lambda \Rightarrow \quad \lambda = \frac{1}{8}.$$

Also,

$$S(e_3, e_3) = -\lambda g(e_3, e_3) - \mu\eta(e_3)\eta(e_3) \tag{5.16}$$

By using $\lambda = \frac{1}{8}$ in (5.17) we obtain $\mu = -\frac{1}{4}$.
Therefore, the data (g, ξ, λ, μ) is an η-Ricci soliton in 3-dimensional quasi-Sasakian manifold.

For this example we have $\lambda = \frac{1}{8}$, i.e. $\lambda > 0$ so that the η-Ricci soliton in 3-dimensional quasi-Sasakian manifold is expanding.

In [20] and [13] the authors proved that the Ricci tensor fields satisfies

$$S(X, \xi) = (dim(M) - 1)\eta(X), \tag{5.17}$$

$$S(\varphi X, \varphi Y) = S(X, Y) + (dim(M) - 1)\eta(X)\eta(Y) \tag{5.18}$$

for any $X, Y \in \chi(M)$. From (5.9) and (5.17) we obtain

$$-\mu - \lambda = (n-1)\beta^2, \tag{5.19}$$

The next theorems formulate results in the case when the quasi-Sasakian manifold of the constant curvature, has cyclic tensor or cyclic η-recurrent Ricci tensor.

Remark that if the manifold M is of constant curvature, then M is elliptic manifold. Indeed, suppose that

$$R(X, Y)Z = k[g(Y, Z)X - g(X, Z)Y]$$

for $X, Y, Z \in \chi(M)$. Applying η to above equation and (5.3)- (5.4) we obtain $k = \beta^2$.

Example 5.5. *On the quasi-Sasakian manifold $(M, \varphi, \xi, \eta, g)$ considered in Example (5.14), the data $(g, \xi, \lambda,)$ for $\lambda = \frac{1}{8}$ and $\mu = -\frac{1}{4}$ defines an η-Ricci soliton. Indeed, scalar curvature r is given by*

$$r = S(e_1, e_1) + S(e_2, e_2) + S(e_3, e_3) = -\frac{1}{8}$$

Therefore the scalar curvature r is constant.
To verify the relation (5.4) it is sufficient to check

$$S(e_i, e_i) = 2\beta^2 g(e_i, e_i)$$

for all $i = 1, 2, 3$ and $\beta = \frac{1}{4}$. Hence the Ricci tensor of M is η-parallel, cyclic parallel and Einstein manifold.

Proposition 5.6. *Let $(M, \varphi, \xi, \eta, g)$ be a quasi-Sasakian manifold and let (g, ξ, λ, μ) be η-Ricci soliton on M. If the manifold M has cyclic Ricci tensor*

$$(\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) = 0$$

for any $X, Y, Z \in \chi(M)$, then $\mu = 0$ and $\lambda = -(n-1)\beta^2$.

Proof. We have

$$(\nabla_X S)(Y, Z) = X(S(Y, Z)) - S(\nabla_X Y, Z) - S(Y, \nabla_X Z). \tag{5.20}$$

Replacing the expression of S from (5.9) in (5.20) we obtain

$$(\nabla_X S)(Y, Z) = -\mu[\eta(Y)(\nabla_X \eta)Z + \eta(Z)(\nabla_X \eta)Y] \tag{5.21}$$

By using (5.2) in (5.21) we get

$$(\nabla_X S)(Y, Z) = \beta\mu[\eta(Y)g(\varphi X, Z) + \eta(Z)(\varphi X, Y)]. \tag{5.22}$$

Then

$$(\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) = \beta\mu g(\varphi X, Y) = 0,$$

for any $X, Y, Z \in \chi(M)$ and for $Y = \varphi Y$ we get

$$\beta\mu g(\varphi X, \varphi Y) = 0,$$

for any $X, Y, \in \chi(M)$. Adding the previous two relations we have

$$\beta\mu[g(\varphi X, Y) + g(\varphi X, \varphi Y)] = 0,$$

for any $X, Y \in \chi(M)$. Since $\beta \neq 0$ therefore $\mu = 0$. From (5.19) we get $\lambda = -(n-1)\beta^2$.

□

Proposition 5.7. *Let $(M, \varphi, \xi, \eta, g)$ be a quasi-Sasakian manifold and let (g, ξ, λ, μ) be η-Ricci soliton on M. If the manifold M Ricci symmetric $\nabla S = 0$, then $\mu = 0$ and $\lambda = -(n-1)\beta^2$.*

Proof. If $\nabla S = 0$, taking $Z = \xi$ in the expression of ∇S from (5.19) we get

$$\beta\mu g(\varphi X, Y) + g(\varphi X, \varphi Y) = 0,$$

for any $X, Y \in \chi(M)$ and the as in in the proof of Proposition (5.6) we get $\mu = 0$ and $\lambda = -(n-1)\beta^2$. □

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