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THREE DIMENSIONAL SASAKIAN MANIFOLDS ADMITTING η-RICCI SOLITONS

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Abstract

In this paper we characterize the three dimensional Sasakian manifolds admitting η-almost Ricci solitons. After the introduction, in section 2, we study three dimensional Sasakian manifolds. In section 3, we prove that an η-Ricci soliton in Sasakian manifolds satisfying the curvature property $R \cdot Q = 0$ is shrinking and reduces to Ricci soliton. In section 4, we show that the necessary and sufficient condition for a Sasakian manifold not admitting a proper η-Ricci soliton is that it is Ricci symmetric. In sections 5 and 6, we study projectively flat and concircularly flat Sasakian manifold of dimension 3 respectively and find the type of an η-Ricci soliton on such manifold. The next section is devoted to the study of such a manifold admitting η-Ricci soliton and we prove some equivalent conditions. Finally, in section 8, we prove that in a three dimensional Sasakian manifold an η-Ricci soliton becomes Ricci soliton if and only if it is Ricci pseudo-symmetric.

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1 Introduction

In 1982, R. S. Hamilton [15] introduced the notion of Ricci flow to find a canonical metric on a smooth manifold. The Ricci flow is an evolution equation for metrics on a Riemannian manifold defined as follows:

$$\frac{\partial}{\partial t} g = -2S, \tag{1}$$

where S denotes the Ricci tensor. Ricci solitons are special solutions of the Ricci flow equation (1) of the form $g = \sigma(t)\psi_t^* g$ with the initial condition $g(0) = g$,

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where ψ_t are diffeomorphisms of M and $\sigma(t)$ is the scaling function. A Ricci soliton is a generalization of an Einstein metric. We recall the notion of Ricci soliton according to [10]. On the manifold M, a Ricci soliton is a triple (g, V, λ) with g, a Riemannian metric, V a vector field, called the potential vector field and λ a real scalar such that

$$\pounds_V g + 2S + 2\lambda g = 0, \tag{2}$$

where $\pounds$ is the Lie derivative. Metrics satisfying (2) are interesting and useful in physics and are often referred to as quasi-Einstein ([11],[12]). Compact Ricci solitons are the fixed points of the Ricci flow $\frac{\partial}{\partial t}g = -2S$ projected from the space of metrics onto its quotient modulo diffeomorphisms and scalings, and often arise blow-up limits for the Ricci flow on compact manifolds. Theoretical physicists have also been looking into the equation of Ricci soliton in relation with string theory. The initial contribution in this direction is due to Friedan [14] who discusses some aspects of it. Recently, the notion of almost Ricci soliton has been introduced [26] by Piagoli, Riogoli, Rimoldi and Setti.

The Ricci soliton is said to be shrinking, steady and expanding accordingly as λ is negative, zero and positive respectively. Ricci solitons have been studied by several authors such as ([13], [16], [17], [19], [30]) and many others.
As a generalization of Ricci soliton, the notion of η-Ricci soliton was introduced by Cho and Kimura [9]. This notion has also been studied in [10] for Hopf hypersurfaces in complex space forms. An η-Ricci soliton is a tuple (g, V, λ, μ), where V is a vector field on M, λ and μ are constants, and g is a Riemannian (or pseudo-Riemannian) metric satisfying the equation

$$\pounds_V g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0, \tag{3}$$

where S is the Ricci tensor associated to g. In this connection we may mention the works of Blaga ([1], [2], [3]), Prakasha et al. [25] and Kar et al. ([18], [21]). In particular, if $\mu = 0$, then the notion of η-Ricci soliton (g, V, λ, μ) reduces to the notion of Ricci soliton(g, V, λ). If $\mu \neq 0$, then the η-Ricci soliton is named proper η-Ricci soliton.

Motivated by the above studies we characterize three dimensional Sasakian manifolds admitting η-Ricci Solitons.
The present paper is organized as follows:
After the introduction, in section 2, we study three dimensional Sasakian manifolds. In section 3, we prove that an η-Ricci soliton in Sasakian manifolds satisfying the curvature property $R \cdot Q = 0$ is shrinking and reduces to Ricci soliton. In section 4, we show that the necessary and sufficient condition for a Sasakian manifold not admitting a proper η-Ricci soliton is that it is Ricci symmetric. In sections 5 and 6, we study projectively flat and concircularly flat Sasakian manifold of dimension 3 respectively and find the type of an η-Ricci soliton on such manifold. The next section is devoted to the study of such a manifold admitting

η-Ricci soliton and we prove some equivalent conditions. Finally, in section 8, we prove that in a three dimensional Sasakian manifold an η-Ricci soliton becomes Ricci soliton if and only if it is Ricci pseudo-symmetric.

2 Three dimensional Sasakian manifolds

An odd dimensional smooth manifold M^{2n+1} $(n \geq 1)$ is said to admit an almost contact structure, sometimes called a (ϕ, ξ, η)-structure, if it admits a tensor field ϕ of type $(1,1)$, a vector field ξ and a 1-form η satisfying ([5], [6])

$$\phi^2 = -I + \eta \otimes \xi, \;\; \eta(\xi) = 1, \;\; \phi\xi = 0, \;\; \eta \circ \phi = 0. \tag{4}$$

The first and one of the remaining three relations in (4) imply the other two relations in (4). An almost contact structure is said to be normal if the induced almost complex structure J on $M^n \times \mathbb{R}$ defined by

$$J(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt}) \tag{5}$$

is integrable, where X is any smooth vector field to M, t is the coordinate of $\mathbb{R}$ and f is a smooth function on $M^n \times \mathbb{R}$. Let g be a compatible Riemannian metric with (ϕ, ξ, η), structure, that is,

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{6}$$

or equivalently,

$$g(X, \phi Y) = -g(\phi X, Y) \tag{7}$$

and

$$g(X, \xi) = \eta(X), \tag{8}$$

for any smooth vector fields X, Y on M. Then M becomes an almost contact metric manifold equipped with an almost contact metric structure (ϕ, ξ, η, g).

An almost contact metric structure becomes a contact metric structure if

$$g(X, \phi Y) = d\eta(X, Y), \tag{9}$$

for any smooth vector fields X, Y on M. The 1-form η is then a contact form and ξ is its characteristic vector field.

Given the contact metric manifold (M, η, ξ, ϕ, g), we define a symmetric (1,1)-tensor field h as $h = \frac{1}{2}L_\xi\phi$, where $L_\xi\phi$ denotes Lie differentiation of ϕ in the direction of ξ. We have the following identities ([5], [6]):

$$h\xi = 0, \qquad h\phi + \phi h = 0, \tag{10}$$

$$\nabla_X \xi = -\phi X - \phi h X, \tag{11}$$

$$\nabla_\xi \phi = 0, \tag{12}$$

$$R(\xi, X)\xi - \phi R(\xi, \phi X)\xi = 2(h^2 + \phi^2)X, \tag{13}$$

$$(\nabla_\xi h)X = \phi X - h^2\phi X + \phi R(\xi, X)\xi, \tag{14}$$

$$S(\xi, \xi) = 2n - \operatorname{tr} h^2. \tag{15}$$

Here, ∇ is the Levi-Civita connection and R is the Riemannian curvature tensor of (M, g) with the sign convention defined by

$$R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, \tag{16}$$

for any smooth vector fields X, Y, Z on M. The tensor $l = R(., \xi)\xi$ is the Jacobi operator with respect to the characteristic field ξ.

If the characteristic vector field ξ is a Killing vector field , the contact metric manifold (M, η, ξ, ϕ, g) is called K-contact manifold. This is the case if and only if $h = 0$. The contact structure on M is said to be normal if the almost complex structure on $M \times \mathbb{R}$ defined by $J(X, \frac{fd}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt})$, where f is a real function on $M \times \mathbb{R}$, is integrable. A normal contact metric manifold is called a Sasakian manifold. Sasakian metrices are K-contact and K-contact 3-metrices are Sasakian. For a Sasakian manifold, the following hold ([5], [6]):

$$\nabla_X\xi = -\phi X, \tag{17}$$

$$(\nabla_X\phi)Y = g(X, Y)\xi - \eta(Y)X, \tag{18}$$

$$(\nabla_X\eta)Y = g(X, \phi Y), \tag{19}$$

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \tag{20}$$

$$Q\xi = 2n\xi, \tag{21}$$

where ∇, R and Q denote respectively, the Riemannian connection, curvature tensor and the $(1,1)$-tensor metrically equivalent to the Ricci tensor of g. The curvature tensor of a 3-dimensional Riemannian manifold is given by

$$\begin{aligned} R(X, Y)Z &= [S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY] \\ &\quad -\frac{r}{2}[g(Y, Z)X - g(X, Z)Y], \end{aligned} \tag{22}$$

where S and r are the Ricci tensor and scalar curvature respectively and Q is the Ricci operator defined by $g(QX, Y) = S(X, Y)$.

It is known that the Ricci tensor of a three dimensional Sasakian manifold is given by [7]

$$S(X, Y) = \frac{1}{2}\{(r - 2)g(X, Y) + (6 - r)\eta(X)\eta(Y)\}, \tag{23}$$

where r is the scalar curvature which needs not be constant, in general. So, g is Einstein (hence has constant curvature 1) if and only if $r = 6$.

As a consequence of (23), we have

$$S(X, \xi) = 2\eta(X). \tag{24}$$

Contact metric manifolds have also been studied by several authors such as ([7]-[13], [14]-[28]) and many others.

Definition 1. *A Riemannian manifold is said to be Ricci symmetric if the Ricci tensor satisfies the condition:*

$$(\nabla_Z S)(X, Y) = 0, \tag{25}$$

for any smooth vector fields X, Y and Z.

Definition 2. *In an n-dimensional Riemannian manifold the projective curvature tensor of type (0,3) is defined by*

$$\mathcal{P}(X, Y)Z = R(X, Y)Z - \frac{1}{n-1}[S(Y, Z)X - S(X, Z)Y], \tag{26}$$

for any smooth vector field X, Y and Z.

Definition 3. *A Riemannian manifold is said to be projectievely flat if the projective curvature tensor $\mathcal{P}$ vanishes.*

Definition 4. *In an n-dimensional Riemannian manifold the concircular curvature tensor of type (0,3) is defined by*

$$\mathcal{F}(X, Y)Z = R(X, Y)Z - \frac{r}{n(n-1)}[g(Y, Z)X - g(X, Z)Y], \tag{27}$$

for any smooth vector field X, Y and Z.

Definition 5. *A Riemannian manifold is said to be concircularly flat if the concircular curvature tensor $\mathcal{F}$ vanishes.*

Let (M, g) be a Riemannian manifold and let ∇ be the Levi-Civita connection of (M, g). A Riemannian manifold is called locally symmetric [8] if $\nabla R = 0$, where R is the Riemannian curvature tensor of (M, g). A Riemannian or a semi-Riemannian manifold $(M, g), n \geq 3$, is called semisymmetric if

$$R \cdot R = 0 \tag{28}$$

holds, where R denotes the curvature tensor of the manifold. It is well known that the class of semisymmetric manifolds includes the set of locally symmetric manifolds ($\nabla R = 0$) as a proper subset. Semisymmetric Riemannian manifolds were first studied by Cartan, Lichnerowich, Couty and Sinjukov. A fundamental study on Riemannian semisymmetric manifolds was made by Szabó [27], Boeckx et al [4] and Kowalski [20]. A semi-Riemannian manifold $(M, g), n \geq 3$, is said to be Ricci-semisymmetric if on M we have

$$R \cdot S = 0, \tag{29}$$

where S is the Ricci tensor.

The class of Ricci semisymmetric manifolds includes the set of Ricci symmetric manifolds ($\nabla S = 0$) as a proper subset. Ricci semisymmetric manifolds were investigated by several authors.
For a $(0, k+2)$-tensor field $Q(g, T)$ associated with any $(0, k)$-tensor field T on a Riemannian manifold (M, g) is defined as follows [29]:

$$\begin{aligned} (Q(g,T))(X_1, ..., X_k; X, Y) &= -((X \wedge Y).T)(X_1, ..., X_k) \\ &= T((X \wedge Y))X_1, X_2, ..., X_k) \\ &\quad +... + T(X_1, ...X_{k-1}, (X \wedge Y)X_k), \end{aligned} \tag{30}$$

where $X \wedge Y$ is the endomorphism given by

$$(X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y. \tag{31}$$

We define the subsets U_R, U_S of a Riemannian Manifold M by $U_R = \{x \in M : R - \frac{r}{n(n-1)}G \neq 0 \quad at \quad x\}$ and $U_S = \{x \in M : S - \frac{r}{n}g \neq 0 \quad at \quad x\}$ respectively, where $G(X, Y)Z = g(Y, Z)X - g(X, Z)Y$. Evidently we have $U_S \subset U_R$. A Riemannian manifold is said to be pseudo-symmetric [29] if at every point of M the tensor $R.R$ and $Q(g, R)$ are linearly dependent. This is equivalent to

$$R \cdot R = f_R Q(g, R)$$

on U_R, where f_R is some function on U_R. Clearly, every semi-symmetric manifold is pseudo-symmetric but the converse is not true [29].

A Riemannian manifold M is said to Ricci pseudo-symmetric if $R \cdot S$ and $Q(g, S)$ on M are linearly dependent. This is equivalent to

$$R \cdot S = f_S Q(g, S)$$

holds on U_S, where f_S is a function defined on U_S.

Lemma 1. *(Proposition 2.1 of [21]) The Ricci tensor of a three dimensional Sasakian manifold admitting η-Ricci soliton is of the form:*

$$S(X, Y) = -\lambda g(X, Y) - \mu\eta(X)\eta(Y). \tag{32}$$

As a consequence of the above Lemma we have

$$QX = -\lambda X - \mu\eta(X)\xi. \tag{33}$$

Lemma 2. *(Proposition 2.2 of [21]) For an η-Ricci soliton on a three dimensional Sasakian manifold we have*

$$\lambda + \mu = -2. \tag{34}$$

We use the above Lemmas in the next sections to develop our results.

3 η-Ricci solitons on a three dimensional Sasakian manifold satisfying $R \cdot Q = 0$

This section is devoted to the study of η-Ricci solitons on a three dimensional Sasakian manifold satisfying $R \cdot Q = 0$. Then we have

$$(R(X,Y) \cdot Q)Z = 0, \tag{35}$$

for any smooth vector fields X, Y, Z.
Using (32) in (35) we get

$$\mu\eta(R(X,Y)Z)\xi - \mu\eta(Z)\{\eta(Y)X - \eta(X)Y\} = 0. \tag{36}$$

Making (25) and (26) in (22) we obtain

$$\begin{aligned} R(X,Y)Z \;\; = \;\; & (2\lambda + \frac{r}{2})\{g(X,Z)Y - g(Y,Z)X\} \\ & -\mu\eta(Y)\eta(Z)X + \mu\eta(X)\eta(Z)Y \\ & -\mu g(Y,Z)\eta(X)\xi + \mu g(X,Z)\eta(Y)\xi. \end{aligned} \tag{37}$$

With the help of (37) we get

$$\eta(R(X,Y)Z) \;\; = \;\; (2\lambda + \mu + \frac{r}{2})\{g(X,Z)\eta(Y) - g(Y,Z)\eta(X)\}. \tag{38}$$

In view of (36) and (38) we have

$$\begin{aligned} & \mu\eta(Z)\eta(X)Y - \mu\eta(Z)\eta(Y)X \\ & +(\lambda + \frac{r}{2} - 2)\mu\{g(X,Z)\eta(Y) - g(Y,Z)\eta(X)\}\xi = 0. \end{aligned} \tag{39}$$

Replacing Z by $\phi^2 Z$ in (39) and using (4) we infer

$$(\lambda + \frac{r}{2} - 2)\mu\{g(X,Z)\eta(Y) - g(Y,Z)\eta(X)\}\xi = 0. \tag{40}$$

Putting $Y = \xi$ in the last equation gives

$$(\lambda + \frac{r}{2} - 2)\mu\{g(X,Z) - \eta(X)\eta(Z)\}\xi = 0. \tag{41}$$

Contracting X and Z in (41) we find

$$(\lambda + \frac{r}{2} - 2)\mu = 0, \tag{42}$$

from which it follows that either $\lambda = 2 - \frac{r}{2}$ or $\mu = 0$.
Contracting X, Y in (32) we get

$$r = -(3\lambda + \mu). \tag{43}$$

If $\lambda = 2 - \frac{r}{2}$, then using (43) we have

$$\lambda + \mu = -4, \tag{44}$$

which contradicts the **Lemma 2**. Hence $\mu = 0$ and then from (34) we obtain $\lambda = -2$. Thus, the η-Ricci soliton is shrinking. Hence, we can state the following:

Theorem 1. *An η-Ricci soliton on a three dimensional Sasakian manifold satisfying $R \cdot Q = 0$ is shrinking.*

Also $\mu = 0$ implies that an η-Ricci soliton becomes a Ricci soliton. Thus, we are in a position to state the following:

Theorem 2. *An η-Ricci soliton on a three dimensional Sasakian manifold satisfying $R \cdot Q = 0$ reduces to a Ricci soliton.*

4 Ricci parallel three dimensional Sasakian manifolds admitting η-Ricci solitons

In this section we consider three dimensional Ricci parallel Sasakian manifolds admitting η-Ricci solitons. Then the equation (25) holds good.

Differentiating (32) covariantly with respect to an arbitrary vector field Z and using (19) we obtain

$$(\nabla_Z S)(X, Y) = \mu[g(X, \phi Z)\eta(Y) + g(Y, \phi Z)\eta(X)]. \tag{45}$$

Comparing (25) and (45) and then substituting $X = \phi X$ we get

$$\mu\{g(X, Z) - \eta(X)\eta(Z)\} = 0. \tag{46}$$

Contracting X, Z in (46) we have

$$\mu = 0. \tag{47}$$

Also, in view of (45), it is easy to see that if $\mu = 0$, then (25) holds. Thus, we have the following:

Theorem 3. *A three dimensional Sasakian manifold does not admit proper η-Ricci soliton if and only if it is Ricci symmetric.*

5 Projectively flat three dimensional Sasakian manifolds admitting η-Ricci solitons

In this section, we classify the projectively flat three dimensional Sasakian manifolds admitting η-Ricci solitons. The projectiev curvature tensor $\mathcal{P}$ is given by (26).
Using (25) and (37) in (26) and on the hypothesis that the manifold is projectively flat we obtain

$$\begin{aligned}\mathcal{P}(X,Y)Z &= \frac{3\lambda+r}{2}\{g(X,Z)Y-g(Y,Z)X\}\\ &+2\mu\eta(Z)\{\eta(X)Y-\eta(Y)X\}\\ &+\mu\{g(X,Z)\eta(Y)-g(Y,Z)\eta(X)\}\xi=0.\end{aligned} \tag{48}$$

Substituting $X=\phi X$ and $Y=\phi Y$ in the last equation we have

$$\frac{3\lambda+r}{2}\{g(\phi X,Z)\phi Y-g(\phi Y,Z)\phi X\}=0. \tag{49}$$

Taking the inner product of (49) with respect to an arbitrary vector field W yields

$$\frac{3\lambda+r}{2}\{g(\phi X,Z)g(\phi Y,W)-g(\phi Y,Z)g(\phi X,W)\}=0. \tag{50}$$

Contracting X and Z in (50) and using $\operatorname{Tr}\phi=0$ and (6) we get

$$(3\lambda+r)\{g(Y,W)-\eta(Y)\eta(W)\}=0. \tag{51}$$

Contracting Y and W in the above equation we infer

$$\lambda=-\frac{r}{3}. \tag{52}$$

By the virtue of (34) and (52) we find

$$\mu=\frac{r}{3}-2. \tag{53}$$

Since, λ is constant, then from (52) it follows that r is constant.
Hence we can state the next theorem as follows:

Theorem 4. *An η-Ricci soliton on a projectievely flat three dimensional Sasakian manifold is of the type $(g,\xi,-\frac{r}{3},\frac{r}{3}-2)$.*

6 Concircularly flat three dimensional Sasakian manifolds admitting η-Ricci solitons

In this section, we classify the concircularly flat three dimensional Sasakian manifolds admitting η-Ricci solitons. The concircular curvature tensor $\mathcal{F}$ is given by (27).

Applying (37) in (27) and using the hypothesis that the manifold is concircularly flat it follows that

$$2(\lambda+\frac{r}{3})\{g(X,Z)Y-g(Y,Z)X\}+\mu\eta(Z)\{\eta(X)Y-\eta(Y)X\}$$
$$+\mu\{g(X,Z)\eta(Y)-g(Y,Z)\eta(X)\}\xi=0. \tag{54}$$

Substituting $X=\phi X$ and $Y=\phi Y$ in the last equation we have

$$2(\lambda+\frac{r}{3})\{g(\phi X,Z)\phi Y-g(\phi Y,Z)\phi X\}=0. \tag{55}$$

Taking the inner product of (56) with respect to an arbitrary vector field W yields

$$(\lambda+\frac{r}{3})\{g(\phi X,Z)g(\phi Y,W)-g(\phi Y,Z)g(\phi X,W)\}=0. \tag{56}$$

Contracting X and Z in (56) and using $\mathrm{Tr}\,\phi=0$ and (6) we get

$$(\lambda+\frac{r}{3})\{g(Y,W)-\eta(Y)\eta(W)\}=0. \tag{57}$$

Contracting Y and W in the above equation we infer

$$\lambda=-\frac{r}{3}. \tag{58}$$

By virtue of (34) and (51) we find

$$\mu=\frac{r}{3}-2. \tag{59}$$

Since, λ is constant, then from (58) it follows that r is constant.
Thus, we have the following:

Theorem 5. *An η-Ricci soliton on a concircularly flat three dimensional Sasakian manifold is of the type $(g,\xi,-\frac{r}{3},\frac{r}{3}-2)$.*

7 η-Ricci solitons on three dimensional Sasakian manifolds satisfying $\mathcal{Q}(g,S)=0$

This section is devoted to the study of η-Ricci solitons on three dimensional Sasakian manifolds satisfying the curvature property

$$\mathcal{Q}(g,S)(X,Y;U,V)=0, \tag{60}$$

where

$$\mathcal{Q}(g,S)(X,Y;U,V)=((X\wedge_g Y)\cdot S)(U,V). \tag{61}$$

In view of (60) and (61) we obtain

$$\mathcal{Q}(g,S)(X,Y;U,V)=-S((X\wedge_g Y)U,V)-S((X\wedge_g Y)V,U). \tag{62}$$

Making use of (31) in the preceeding equation entails that

$$\begin{aligned}\mathcal{Q}(g,S)(X,Y;U,V) &= g(Y,U)S(X,V)-g(X,U)S(Y,V)\\ &\quad +g(Y,V)S(X,U)-g(X,V)S(Y,U).\end{aligned} \tag{63}$$

From (32) and (63) it follows that

$$\begin{aligned}\mathcal{Q}(g,S)(X,Y;U,V) &= \mu\{g(X,U)\eta(Y)\eta(V)+g(X,V)\eta(Y)\eta(U)\\ &\quad -g(Y,U)\eta(X)\eta(V)-g(Y,V)\eta(X)\eta(U)\}.\end{aligned} \tag{64}$$

From (60) and (64) we have

$$\begin{aligned}&\mu\{g(X,U)\eta(Y)\eta(V)+g(X,V)\eta(Y)\eta(U)\\ &-g(Y,U)\eta(X)\eta(V)-g(Y,V)\eta(X)\eta(U)\}=0.\end{aligned} \tag{65}$$

Contracting X and V in (64) we get

$$\mu\{g(Y,U)-3\eta(Y)\eta(U)\}=0. \tag{66}$$

Replacing Y by ϕY in last equation we have

$$\mu g(\phi Y,U)=0, \tag{67}$$

from which it follows that

$$\mu\phi Y=0 \tag{68}$$

and hence

$$\mu=0, \qquad \text{if} \quad Y\neq\xi. \tag{69}$$

If $Y=\xi$, condition (60) implies that

$$\mu\{2\eta(X)\eta(U)\eta(V)-g(X,U)\eta(V)-g(X,V)\eta(U)\}=0. \tag{70}$$

Contracting X and V yields

$$\mu=0. \tag{71}$$

Therefore, η-Ricci soliton becomes Ricci soliton. Contrapositively, if $\mu=0$, then $\mathcal{Q}(g,S)(X,Y;U,V)=0$. Hence, we can conclude that

Theorem 6. *In a three dimensional Sasakian manifold, η-Ricci solitons become Ricci soliton if and only if $\mathcal{Q}(g,S)(X,Y;U,V)=0$.*

In view of **Theorem 3** and **Theorem 6** we have the following:

Theorem 7. *In a three dimensional Sasakian manifold M^3 the following statements are equivalent:*

(i) M^3 does not admit a proper η-Ricci soliton.

(ii) M^3 is Ricci symmetric.

(iii) M^3 satisfies $\mathcal{Q}(g,S)=0$.

8 Ricci pseudo-symmetric three dimensional Sasakian manifolds admitting η-Ricci soliton

In this section we study η-Ricci soliton on Ricci pseudo-symmetric three dimensional Sasakian manifolds. Then we have

$$(R(X,Y)\cdot S)(U,V) = f_S Q(g,S)(X,Y;U,V), \tag{72}$$

wherein f_S is a smooth function.
Let us assume that $f_S \neq \frac{r}{2} + \lambda - 2$.
With the help of (32) and (37) we deduce that

$$\begin{aligned}(R(X,Y)\cdot S)(U,V) &= (\lambda + \frac{r}{2} - 2)\mu\{g(Y,U)\eta(X)\eta(V) \\ &\quad +g(Y,V)\eta(X)\eta(U)g(X,U)\eta(Y)\eta(V) \\ &\quad +g(X,V)\eta(Y)\eta(U)\}. \end{aligned} \tag{73}$$

By the virtue of (64), (72) and (73) gives

$$\begin{aligned}&(\lambda + \frac{r}{2} - 2 + f_S)\mu\{g(Y,U)\eta(X)\eta(V) \\ &+g(Y,V)\eta(X)\eta(U)g(X,U)\eta(Y)\eta(V) \\ &+g(X,V)\eta(Y)\eta(U)\} = 0. \end{aligned} \tag{74}$$

Contracting X and U in above equation we find

$$(\lambda + \frac{r}{2} - 2 + f_S)\mu\{g(Y,V) - 3\eta(Y)\eta(V))\} = 0. \tag{75}$$

On contraction of Y and V in last equation we get

$$(\lambda + \frac{r}{2} - 2 + f_S)\mu = 0, \tag{76}$$

from which it follows that

$$\mu = 0. \tag{77}$$

On the other hand, $\mu = 0$ implies that the manifold is Ricci pseudo-symmetric which is followed from the equations (64), (72) and (73). Thus, we are in a position to state that

Theorem 8. *In a three dimensional Sasakian manifold the η-Ricci soliton becomes Ricci soliton if and only if it is Ricci psudo-symmetric.*

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