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COMPACT GEODESICALLY REVERSIBLE HARMONIC FINSLER MANIFOLDS

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Abstract

A Finsler manifold is said to be geodesically reversible if for every one of its oriented geodesic paths, the same path traversed in the opposite sense is also a geodesic. We prove that the compact geodesically reversible harmonic Finsler manifolds with finite fundamental groups have the Randers metrics.

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1 Introduction

A geodesic in a Finsler manifold (where the Finsler function is positively homogeneous) should be thought of as an oriented path, that is, an imbedded one dimensional submanifold with a sense of direction, or an equivalence class of curves determined up to reparametrization with positive derivative. There is, in general, no reason why a path which coincides with a geodesic as a point set but is traversed in the opposite direction should be a geodesic. A Finsler metric is said to be *geodesically reversible* if every oriented geodesic can be reparametrized as a geodesic with the reverse orientation. Any reversible Finsler metric is geodesically reversible. On the other hand, the non-Riemannian Finsler examples constructed in Section 4 of [4] are geodesically reversible but not reversible, so the reverse implication does not hold.

A Finsler metric is called the *Zoll* if all of its geodesics are closed and of the same length. The canonical round metric on the compact rank one symmetric spaces is a Zoll Riemannian metric. However, there exist Zoll Riemannian metrics on spheres which are not round. Contrariwise, a Riemannian metric on the real projective space is a Zoll metric if and only if it has constant curvature, which follows from Green's theorem, cf. [3, Theorem 5.59], since the orientable double

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cover of a real projective space is a canonical Riemannian sphere. However, this rigidity result fails in the Finsler case (cf. [13, Appendix]).

There are classes of Finsler manifolds related to Zoll manifolds. A Finsler manifold is called *harmonic* if the mean curvature of all geodesic spheres is a function depending only on the radius ([7, 8]). A compact Finsler manifold is called a *Blaschke* manifold if its diameter and its injectivity radius conincide. The result of Allamigeon (see Theorem 3) implies that compact and simply connected geodesically reversible harmonic Finsler manifolds are Blaschke Finsler manifolds.

The goal of this article is to study the geodesically reversible harmonic Finsler manifolds.

Theorem. *The compact geodesically reversible harmonic Finsler manifolds with finite fundamental groups have the Randers metrics.*

A Finsler manifold is *locally symmetric* if the geodesic reflection is a local isometry of the Finsler metric. It is obvious that the geodesic reflection induces the minus identity on the tangent spaces, therefore, complete locally symmetric Finsler manifolds have reversible harmonic Finsler metrics. Thus, the following result is a straightforward consequence of the main theorem (cf. [6]).

Corollary. *If (M, F) is a compact locally symmetric Finsler manifold with finite fundamental groups, then F is a Riemannian metric. In fact, the universal covering space of M is isometric to one of compact rank one symmetric Riemannian spaces.*

We do not know whether our results extend to non-reversible Finsler metrics as several arguments only work in the geodesically reversible case. It would be interesting to clarify this point.

2 Preliminaries

In this section we recall some basics in Finsler geometry and prove some auxiliary facts. We follow the presentation in [14], where most concepts are developed from the Riemannian point of view. We refer to [15] as more exhaustive references in Finsler geometry. Let M be an n-dimensional smooth manifold and TM denotes its tangent bundle. A *Finsler structure* on a manifold M is a function $F : TM \to [0, \infty)$ which has the following properties:

(1) F is smooth on $\widetilde{TM} := TM \setminus \{0\}$;

(2) $F(tv) = tF(v)$, for all $t > 0$, $v \in T_xM$ (positively homogeneous);

(3) $\frac{1}{2}F^2$ is strongly convex, i.e., $g_{ij}(x, y) := \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}(x, y)$ is positive definite for all $(x, y) \in \widetilde{TM}$.

A Finsler structure F is called *reversible* if $F(-v) = F(v)$ for all $v \in T_xM$ (absolutely homogeneous). For a fixed $v \in T_xM$ let $\gamma_v(t)$ be the geodesic from x with $\gamma_v'(0) = v$. Along $\gamma_y(t)$, we have the osculating Riemannian metrics

$$g^{\gamma_v'(t)} := g(\gamma_v(t), \gamma_v'(t))$$

in $T_{\gamma_v(t)}M$. The Chern connection on a Finsler manifold M is defined by the unique set of local 1-forms $\{{\omega_j}^i\}_{1\le i,j\le n}$ on $\widetilde{TM}$ such that

$$\begin{aligned} d\omega^i &= \omega^j \wedge {\omega_j}^i, \\ dg_{ij} &= g_{kj}{\omega_i}^k + g_{ik}{\omega_j}^k + 2A_{ijk}{\omega_n}^k, \text{ where } A_{ijk} = \frac{\partial g_{ij}}{\partial y^k}. \end{aligned}$$

Define the set of local curvature forms ${\Omega_j}^i$ by ${\Omega_j}^i := d{\omega_j}^i - {\omega_j}^k \wedge {\omega_k}^i$. Then, one can write ${\Omega_j}^i = \frac{1}{2}{R_j}^i{}_{kl}\omega^k \wedge \omega^l + {P_j}^i{}_{kl}\omega^k \wedge \omega^{n+l}$. Define the curvature tensor R by $R(U,V)W = u^k v^l w^j {R_j}^i{}_{kl}E_i$, where $U = u^i E_i, V = v^i E_i, W = w^i E_i$ are vectors in the pull-back bundle π^*TM of TM by $\pi : \widetilde{TM} \to M$. Define the *flag curvature* $R^t : T_{\gamma_v(t)}M \to T_{\gamma_v(t)}M$ by

$$R^t(u(t)) := R^{\dot\gamma_v(t)}(u(t)) := R(U(t), V(t))V(t),$$

where $U(t) = (\hat\gamma_v(t); u(t)), V(t) = (\hat\gamma_v(t); \gamma_v(t)) \in \pi^*TM$. We remark that if F is Riemannian, then the flag curvature coincides with the sectional curvature. Then the Ricci curvature is defined by

$$\mathrm{Ric}(v) := \sum_{i=1}^{n} g^v(R^v(e_i), e_i)), v \in T_xM,$$

where $\{e_i\}_{i=1}^n$ is a g^v-orthonormal basis for T_xM.

Let $\{\frac{\partial}{\partial x^i}\}_{i=1}^n$ be a local basis for TM and $\{dx^i\}_{i=1}^n$ be its dual basis for T^*M. Put $S_x(1) := \{y \in T_xM : F(x,y) = 1\}$. Let $\alpha(n-1)$ be the volume of the unit $(n-1)$-sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$. The volume form dv on M is defined by

$$dv(x) := \frac{\alpha(n-1)}{\mathrm{vol}(S_x(1))} dx^1 \wedge \cdots \wedge dx^n := \sigma(x)dx,$$

where $\mathrm{vol}(A)$ denotes the volume of a subset A with respect to the standard Euclidean structure on $\mathbb{R}^n$. Busemann proved that for any bounded open subset $U \subset M$, $\mathrm{vol}(U,F) := \int_U dv(x) = H_{d_F}(U)$, where $H_{d_F}(U)$ denotes the Hausdorff measure of U for the metric d_F on M induced by the Finsler norm.

For a tangent vector $v = (x,y) \in \widetilde{TM}$, define the *mean distortion* ρ by

$$\rho(v) := \frac{\sigma(x)}{\sqrt{\det(g_{ij}^v)}} = \frac{\alpha(n-1)}{\mathrm{vol}(S_x(1))}\frac{1}{\sqrt{\det(g_{ij}^v)}} = \frac{\alpha(n-1)}{\mathrm{vol}(S_x(1), g^v)},$$

and the *S-curvature* $S : \widetilde{TM} \to \mathbb{R}$ is defined by

$$S(v) := \frac{d}{dt}\bigg|_{t=0} \left\{ \ln \rho(\dot{\gamma_v}(t)) \right\}.$$

For a local smooth distance function φ, by the volume density $\sigma(x) = \rho(v)\sqrt{\det(g_{ij}^v)}$, we have $\Delta\varphi = \overline{\Delta}\varphi + S(\mathrm{grad}\varphi)$, where $\Delta\varphi$ and $\overline{\Delta}\varphi$ denote the Laplacian of φ with respect to F and $g^{\mathrm{grad}\varphi}$, respectively. An important property is that $S = 0$ for Finsler manifolds modeled on a single Minkowski space. In particular, $S = 0$ for Berwald spaces. Locally Minkowski spaces and Riemannian spaces are all Berwald spaces.

By the Chern connection, we obtain the decomposition

$$T^*(\widetilde{TM}) = \mathrm{span}\{dx^i\} \oplus \mathrm{span}\{\delta y^i\},$$

where δy^i is the vertical component dy^i and is given by $\delta y^i = dy^i + N^i_j dx^j$ for some N^s_l determined by the Chern connection. Then there is a naturally induced Sasaki metric $\hat{g}$ on $\widetilde{TM}$ defined by

$$\hat{g}(v) = g_{ij}(v)dx^i \otimes dx^j \oplus g_{ij}(v)\delta y^i \otimes \delta y^j,$$

and the volume form dV of $\hat{g}$ on $\widetilde{TM}$ is given by

$$\begin{aligned} dV(v) &:= \sqrt{\det(g_{ij}(v))}dx^1 \wedge \cdots \wedge dx^n \wedge \sqrt{\det(g_{ij}(v))}\delta y^1 \wedge \cdots \wedge \delta y^n \\ &= \det(g_{ij}(v))dx^1 \wedge \cdots \wedge dx^n \wedge dy^1 \wedge \cdots \wedge dy^n. \end{aligned}$$

Let $\omega = \frac{\partial F}{\partial y^i}dx^i$ be the Hilbert 1-form on $\widetilde{TM}$. In local coordinates,we have $dV = (d\omega)^n/n!$.

There is another interpretation of this volume on tangent space. Let SM be the unit tangent bundle on M and $i : SM \to \widetilde{TM}$ the natural embedding. Let X_ω be the Reeb field of the Hilbert 1-form ω. It is uniquely determined by the conditions $\omega(X_\omega) = 1, i_{X_\omega}(d\omega) = 0$. In particular we have $L_{X_\omega}\omega = 0$ and the geodesic flow of F, i.e., the flow with infinitesimal generator X_ω, consists of contact diffeomorphisms and the volume form $i^*(dV)$ on SM is

$$dV = \frac{1}{(n-1)!}\omega \wedge (d\omega)^{n-1}.$$

Since $L_{X_\omega}\omega = 0$, the volume form is invariants under the geodeisc flow of F. We shall use the same notation dV for the volume forms of TM and SM, so that no confusion is caused. Let $V(SM)$ be the volume of SM with respect to the volume form dV. In the case of Riemannian metrics, all unit tangent spaces are isometric to the Euclidean spheres, and we have $V(SM) = \alpha(n-1) \cdot \mathrm{vol}(M, F)$. On the other hand, in a general Finsler metric, unit tangent spaces may not be isometric to each other, and hence one cannot expect the equality. We instead have the following.

Theorem 1. *([5]) Let (M,F) be an n-dimensional compact reversible Finsler manifold. Then we have*

$$V(SM) \leq \alpha(n-1) \cdot \mathrm{vol}(M, F),$$

with equality if and only if (M, F) is a Riemannian metric.

If F is not a reversible Finsler metric, then the volume $V(SM)$ may be greater than $\alpha(n-1)\cdot \mathrm{vol}(M,F)$ (see [10]).

For the purposes of this paper, we need only remark that the symplectic character of the Holmes-Thompson volume makes it the volume of choice to trivially extend Riemannian results that only depend on the symplectic properties of the geodesic flow. The notion of symplectic structure came up in Weinstein's work on the Blaschke conjecture in [17]. He proved that for an n-dimensional Riemannian manifold M all of whose geodesics are closed and of the same length 2π, the ratio $\mathrm{vol}(M)/((2\pi)^n\cdot\alpha(n))$ is an integer.

Let CM be the spaces of oriented geodesic of a Finsler manifold M and $p: SM \to CM$ be the canonical projection which sends a given unit vector to the geodesic which has this vector as initial condition. Álvarez Paiva [1] proved that if dw is the standard symplectic form on TM, then there is a unique symplectic form Ω on CM which satisfies the equation $p^*\Omega = i^*(d\omega)$. Thus the symplectic nature of Weinstein's proof implies that they can be extended to Finsler manifolds with little modification. Since the Riemannian relation $V(SM)=\alpha(n-1)\cdot\mathrm{vol}(M,F)$ breaks down in the Finsler case, we rewrite Weinstein's result as follows. For the sake of completeness we sketch the proof.

Theorem 2. *([9, Theorem 3]) Let (M,F) be an n-dimensional Finsler manifold all of whose geodesics are closed and of the same length 2π. Then the ratio*

$$i(M)=\frac{V(SM)}{V(S\mathbb{S}^n)}$$

is an integer, where $\mathbb{S}^n$ is the standard Riemannian sphere of constant sectional curvature one.

Proof. Since the orbits of the geodesic spray are all periodic with 2π, the geodesic flow on the SM defines a fixed point free $S^1=\mathbb{R}/\mathbb{Z}$-action, whose orbits are identified with closed geodesics of length 2π. Therefore, the orbit space SM/S^1 may be considered as a $2(n-1)$-dimensional manifold CM of all closed geodesics of M. The projection $p: SM\to CM$ is a principle bundle with structure group S^1, and we get a symplectic form Ω on CM by the condition $p^*\Omega=i^*(d\omega/2\pi)=d\omega/2\pi$.

From the Fubini theorem for fibrations we get

$$\begin{aligned} V(SM) &= \int_{SM}\frac{1}{(n-1)!}\omega\wedge(d\omega)^{n-1}\\ &= \frac{1}{(n-1)!}\int_{SM}\omega\wedge p^*(2\pi\Omega)^{n-1}\\ &= \frac{(2\pi)^{n-1}}{(n-1)!}\int_{x\in CM}\left(\int_{p^{-1}(x)}\omega\right)\Omega^{n-1}. \end{aligned}$$

Now we set

$$j(M):=\int_{CM}\Omega^{n-1}.$$

Then $j(M)$ is a topological invariant of the fibration $p : SM \to CM$. We adapt Weinstein's argument (see [17]) to see that the integer $j(M)$ is an even integer $2 \cdot i(M)$. However we know $\int_{p^{-1}(x)} \omega = 2\pi$ and

$$V(SM) = \frac{(2\pi)^n}{(n-1)!} \int_{CM} \Omega^{n-1} = \frac{(2\pi)^n}{(n-1)!} 2 \cdot i(M).$$

Since $2 \cdot (2\pi)^n/(n-1)! = \alpha(n-1) \cdot \alpha(n) = V(S\mathbb{S}^n)$, we obtain the equality as stated in the theorem. $\square$

Remark 1. *Under the assumption of Theorem 2, if M is homeomorphic to one of the compact rank one symmetric spaces $(\mathbb{P}, g_0)$, i.e., $\mathbb{S}^n$, $\mathbb{C}P^{n/2}$, $\mathbb{H}P^{n/4}$, $\mathbb{C}aP^2$, Weinstein ([17]), Yang ([18], [19]), and Reznikov ([11], [12]) showed that $V(SM) = V(S\mathbb{P})$.*

3 Harmonic Finsler manifolds

A compact Finsler manifold is called a *Blaschke* manifold, if every minimal geodesic of length less than the diameter is the unique shortest path between any of its points. Equivalently, for which all cut loci are round spheres of constant radius and dimension. For a reversible Blaschke Finsler manifold the exponential map restricted to the unit tangent sphere defines a great sphere foliation. Since every great sphere foliation of sphere is homeomorphic to a Hopf fibration, simply connected reversible Blaschke Finsler manifolds are actually homeomorphic to compact rank-one symmetric spaces.

For a nonzero $v \in TM$ the mean curvature $m_t(v)$ of geodesic sphere $S(\gamma_v(0), t)$ of radius t about geodesic $\gamma_v(t)$ has the following Taylor expansion

$$m_t(v) = \frac{n-1}{t} - S(v) - \frac{1}{3}\Big(\mathrm{Ric}(v) + 3\dot{S}(v)\Big)t + O(t),$$

where S is S-curvature. Let $\hat{m}_t(v)$ denotes the mean curvature of geodesic sphere $S(\gamma_v(0), t)$ in g^{γ_v} with respect to normal vector $\gamma_v'(t)$. Then we have

$$m_t(v) = \hat{m}_t(v) - S\big(\gamma_v'(t)\big) = \frac{d}{dt}\Big[\ln \eta_t(v)\Big],$$

where $\eta_t(v)$ is the Busemann-Hausdorff volume density of geodesic sphere $S(\gamma_v(0), t)$ around $\gamma_v(t)$.

A complete Finsler manifold is called *harmonic* if the mean curvature of all geodesic spheres is a function depending only on the radius. Then S-curvature is zero. A historical break in the theory of harmonic Riemannian manifolds was made by Allamigeon when he proved the following: A simply connected harmonic Riemannian manifold is either diffeomorphic to Euclidean space or is a Blaschke manifold. The following theorem is to put them in a Finsler-geometric setting.

Theorem 3. *A simply connected geodesically reversible harmonic Finsler manifold M is either diffeomorphic to Euclidean space or is a Blaschke manifold.*

Proof. Suppose there is no conjugate points. Then the exponential map is a covering map and since M is simply connected, a diffeomorphism. So take a $0 \neq v_0 \in T_xM$ and an $r_0 \in \mathbb{R}$ such that the first conjugate point along γ_{v_0} is $\gamma_{v_0}(r_0)$. Then the first conjugate point along γ_v is $\gamma_v(r_0)$ for all $v \in T_xM$, since the mean curvature is radial. Note that r_0 is the same for every point in M. This means that M is a Blaschke manifold by the Allamigeon-Warner theorem (cf. [3, Corollary 5.31]). □

Let $a : TM \to TM$ be the map that sends each tangent vector v to its opposite $-v$ and the *symmetrization* $\overline{F} := (F + F \circ a)/2$ of Finsler metric F. The following theorem is the final ingredient needed for the main theorem.

Theorem 4. *([2, Main Result]) Let M be a manifold diffeomorphic to a compact rank one symmetric space. If F is a geodesically reversible Zoll metric on M, then F is the sum of a reversible Zoll metric $\overline{F}$ and an exact one-form β.*

Remark 2. *Since β is exact, the time change does not modify the lengths of closed geodesics. In fact, (M, F) and $(M, \overline{F})$ have the geodesic conjugacy.*

Now we are ready to prove the main theorem using Theorem 2 and Remark 1.

Theorem 5. *If (M, F) is a compact geodesically reversible harmonic Finsler manifold with finite fundamental group, then F is a Randers metric.*

Proof. Let $\widetilde{M}$ be the universal covering space of M. By Theorem 3, we know $\widetilde{M}$ is a Blaschke manifold all of whose geodesics are closed and of the same length 2π, up to a scaling of the metric, and then we have that $\widetilde{M}$ is diffeomorphic to one of the compact rank one symmetric spaces $\mathbb{P}$. Hence applying Theorem 2 and Remark 1 gives $V(S(\mathbb{P}, g_0)) = V(S(\widetilde{M}, F))$.

Since $(\widetilde{M}, F)$ is a harmonic Finsler manifold, for all nonzero $v \in T\widetilde{M}$, for all $t > 0$, we obtain $S\big(\gamma_v'(t)\big) = \hat{m}_t(v) - m_t(v) = 0$, and the osculating Riemannian metric $g^{\gamma_v'}$ on $\widetilde{M} \setminus \{\gamma_v(0)\}$ is a harmonic Riemannian metric. On the other hand, in the case of rank one symmetric Riemannian manifolds, we have $\mathbb{S}^n : \eta_t = \sin^{n-1} t$; $\mathbb{C}P^n : \eta_t = \sin t(1 - \cos t)^{\frac{n-2}{2}}$; $\mathbb{H}P^n : \eta_t = \sin^3 t(1 - \cos t)^{\frac{n-4}{2}}$; $\mathbb{C}aP^2 : \eta_t = \sin^7 t(1 - \cos t)^4$. Szabò ([16]) remarked that these are *only possibilities* for a compact harmonic Riemannian manifold. Since

$$\frac{d}{dt}\Big[\ln \eta_t(v)\Big] = m_t(v) = \hat{m}_t(v) = \frac{d}{dt}\Big[\ln \hat{\eta}_t(v)\Big],$$

we have $\eta_t(v) = \hat{\eta}_t(v)$. By the co-area formula, we obtain

$$\mathrm{vol}(\widetilde{M}, F) = \mathrm{vol}(\widetilde{M} \setminus \{\gamma_v(0)\}, g^{\gamma_v'}) = \mathrm{vol}(\mathbb{P}, g_0).$$

Let $\overline{F}$ be the symmetrization of F. Then by Theorem 4 and Remark 2, we have $V(S(\widetilde{M}, F)) = V(S(\widetilde{M}, \overline{F}))$. Therefore, we conclude

$$\begin{aligned} \alpha(n-1) \cdot \mathrm{vol}(\widetilde{M}, F) &= \alpha(n-1) \cdot \mathrm{vol}(\mathbb{P}, g_0) = V(S(\widetilde{M}, F)) \\ &= V(S(\widetilde{M}, \overline{F})) \leq \alpha(n-1) \cdot \mathrm{vol}(\widetilde{M}, \overline{F}). \end{aligned}$$

We note that the second line is obtained from Theorem 1, and hence we obtain

$$\mathrm{vol}(\widetilde{M}, F) \le \mathrm{vol}(\widetilde{M}, \overline{F}).$$

Then by the equality case of Theorem 1, $\overline{F}$ is a Riemannian metric and F is Randers metric. □

Remark 3. *In the Riemannian case, Theorem 5 becomes the Szabò's result in Riemannian geometry ([16]).*

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