Bulletin of the *Transilvania* University of Braşov • Vol 13(62), No. 1 - 2020
Series III: Mathematics, Informatics, Physics, 219-236
https://doi.org/10.31926/but.mif.2020.13.62.1.17

ON A TYPE OF $N(k)$-QUASI EINSTEIN MANIFOLDS

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Abstract

The object of this paper is to study $N(k)$-quasi Einstein manifolds. W^*-Ricci pseudosymmetric, W_2-pseudosymmetric and Z-generalized pseudosymmetric $N(k)$-quasi Einstein manifolds are considered. Finally, we construct examples to prove the existence of such manifolds.

2000 *Mathematics Subject Classification:* 53C25, 53C35, 53D10.
Key words: Quasi Einstein manifold, k-nullity distribution, $N(k)$-quasi Einstein manifold, m-projective curvature tensor, W_2-curvature tensor, Z-tensor.

1 Introduction

Chaki and Maity[4] introduced the notion of a quasi Einstein manifold. A quasi Einstein manifold is a generalization of the Einstein manifold. A non flat Riemannian manifold $(M^n, g)(n > 2)$ is said to be quasi Einstein if the Ricci tensor S is not identically zero and satisfies

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{1}$$

$\forall X, Y \in TM$, where a and b are smooth functions, $b \neq 0$ and η is a non-zero 1-form defined by

$$g(X,\xi) = \eta(X), \quad g(\xi,\xi) = \eta(\xi), \quad \eta(\xi) = 1, \tag{2}$$

for the associated vector field ξ. Here, a and b are called the associated scalars and ξ is called the generator of the manifold. Clearly, if $b = 0$, the manifold reduces to an Einstein manifold.

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Quasi Einstein manifolds has been studied by several authors such as Chaki and Maity[4], De and De[6], De and Ghosh[7], Debnath and Konar[11] and others. The notion of quasi Einstein manifolds has been extended to generalized Einstein manifolds[1], generalized quasi Einstein manifolds([2], [8]), mixed generalized quasi Einstein manifolds[3] and others. *Özgür*[20] also studied super quasi Einstein manifolds.

In 1988, Tanno[25] defined the k-nullity distribution of a Riemannian manifold as

$$N(k) : p \rightarrow N_p(k) = \{Z \in T_pM : R(X,Y)Z = k[g(Y,Z)X - g(X,Z)Y]\}, \tag{3}$$

$\forall X, Y, Z \in T_pM$, and k is a smooth function. If the generator ξ of a quasi-Einstein manifold belongs to some k-nullity distribution, then it is called an $N(k)$-quasi Einstein manifold[26]. In an $N(k)$-quasi Einstein manifold, k is not arbitrary as given by[26]:

Lemma 1. *In an n-dimensional $N(k)$-quasi Einstein manifold,*

$$k = \frac{a+b}{n-1}. \tag{4}$$

$N(k)$-quasi Einstein manifolds have been studied by several authors such as Hui and Lemence[14], Yildiz et al.[27], Singh et al.[22] and others.

In an $N(k)$-quasi Einstein manifold, we have[26]

$$R(X,Y)\xi = k\big[\eta(Y)X - \eta(X)Y\big], \tag{5}$$

$$R(X,\xi)Y = k\big[\eta(Y)X - g(X,Y)\xi\big] = -R(\xi,X)Y, \tag{6}$$

$$\eta(R(X,Y)Z) = k\big[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\big]. \tag{7}$$

Pokhariyal and Mishra[21] introduced two types of tensors

$$W_2(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}\big[g(Y,Z)QX - g(X,Z)QY\big] \tag{8}$$

and

$$\begin{aligned} W^*(X,Y)Z &= R(X,Y)Z - \frac{1}{2(n-1)}\big[S(Y,Z)X - S(X,Z)Y \\ &+ g(Y,Z)QX - g(X,Z)QY\big], \end{aligned} \tag{9}$$

known as the W_2-curvature tensor and the m-projective curvature tensors respectively, where Q is the Ricci operator.

Mantica and Molinari[17] defined a generalized (0, 2) type tensor known as the Z-tensor as

$$Z(X,Y) = S(X,Y) + \phi g(X,Y), \tag{10}$$

where ϕ is a smooth function. The study of the Z-tensor was continued by Mantica and Molinari[17], Mantica and Suh([18], [19]), etc. In 2016, Mallick and De[16] studied the derivation conditions $R(\xi, X) \cdot Z = 0$ and $P(\xi, X) \cdot Z = 0$ in an $N(k)$-quasi Einstein manifold, where P is the projective curvature tensor. $N(k)$-quasi Einstein manifolds satisfying $C(\xi, X) \cdot R = 0$, $R(\xi, X) \cdot W^* = 0$ and $W^*(\xi, X) \cdot S = 0$, where C is the conformal curvature tensor have been studied by De et al.[10]. Also, in 2018, Chaubey[5] studied W^*-pseudosymmetric and Z-recurrent $N(k)$-quasi Einstein manifolds. These motivated us to study the properties of $N(k)$-quasi Einstein manifolds.

This paper is organized as follows: After the preliminaries, we study the m-projective curvature tensor in an $N(k)$-quasi Einstein manifold. In section 4, we consider W^*-Ricci pseudosymmetric $N(k)$-quasi Einstein manifolds and section 5 deals with W_2-pseudosymmetric $N(k)$-quasi Einstein manifolds. Z-generalized pseudosymmetric $N(k)$-quasi Einstein manifolds are studied in section 6. Finally, we construct examples to support the existence of these manifolds.

2 Preliminaries

Using equations (1) and (2), we obtain

$$S(X, \xi) = (a + b)\eta(X), \tag{11}$$

$$r = na + b, \tag{12}$$

where r is the scalar curvature of the manifold. In an n-dimensional $N(k)$-quasi Einstein manifold, we have

$$W_2(X, Y)\xi = \frac{b}{(n-1)}\big[\eta(Y)X - \eta(X)Y\big], \tag{13}$$

$$W_2(\xi, X)Y = \frac{1}{n-1)}\big[\eta(Y)QX - (a + b)\eta(Y)X\big], \tag{14}$$

$$\eta(W_2(X, Y)Z) = 0, \tag{15}$$

$$W^*(X, Y)\xi = \frac{b}{2(n-1)}\big[\eta(Y)X - \eta(X)Y\big], \tag{16}$$

$$W^*(\xi, X)Y = \frac{b}{2(n-1)}\big[g(X, Y)\xi - \eta(Y)X\big], \tag{17}$$

$$\eta(W^*(X, Y)Z) = \frac{b}{2(n-1)}\big[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\big]. \tag{18}$$

The generalized Z-tensor in an $N(k)$-quasi Einstein manifold takes the form,

$$Z(X,Y) = (a+\phi)g(X,Y) + b\eta(X)\eta(Y), \tag{19}$$

which by contraction, reduces to

$$Z = (a+\phi)n + b. \tag{20}$$

Also,

$$Z(X,\xi) = (a+b+\phi)\eta(X), \tag{21}$$

$$Z(\xi,\xi) = (a+b+\phi), \tag{22}$$

$\forall X, Y, Z \in M^n$.

3 m-projective curvature tensor in an $N(k)$-quasi Einstein manifold

Suppose an $N(k)$-quasi Einstein manifold which satisfies

$$W^*(\xi,X).W_2 = 0,$$

or,

$$\begin{aligned} &W^*(\xi,X)W_2(U,V)Z - W_2(W^*(\xi,X)U,V)Z \\ &-W_2(U,W^*(\xi,X)V)Z - W_2(U,V)W^*(\xi,X)Z = 0. \end{aligned} \tag{23}$$

Using (18), (23) it becomes

$$\begin{aligned} &\frac{b}{2(n-1)}\Big[g(X,W_2(U,V)Z)\xi - \eta(W_2(U,V)Z)X \\ &-g(X,U)W_2(\xi,V)Z + \eta(U)W_2(X,V)Z \\ &-g(X,V)W2(U,\xi)Z + \eta(V)W_2(U,X)Z \\ &-g(X,Z)W_2(U,V)\xi + \eta(Z)W_2(U,V)X\Big] = 0. \end{aligned} \tag{24}$$

Since $b \neq 0$ and $n > 1$, we have

$$\begin{aligned} &g(X,W_2(U,V)Z)\xi - \eta(W_2(U,V)Z)X \\ &-g(X,U)W_2(\xi,V)Z + \eta(U)W_2(X,V)Z \\ &-g(X,V)W_2(U,\xi)Z + \eta(V)W_2(U,X)Z \\ &-g(X,Z)W_2(U,V)\xi + \eta(Z)W_2(U,V)X = 0. \end{aligned} \tag{25}$$

Taking the inner product of (22) with respect to ξ, we have

$$\begin{aligned} &W_2'(U,V,Z,X) - \eta(W_2(U,V)Z)\eta(X) \\ &-g(X,U)\eta(W_2(\xi,V)Z) + \eta(U)\eta(W_2(X,V)Z) \\ &-g(X,V)\eta(W_2(U,\xi)Z) + \eta(V)\eta(W_2(U,X)Z) \\ &-g(X,Z)\eta(W_2(U,V)\xi) + \eta(Z)\eta(W_2(U,V)X) = 0. \end{aligned} \tag{26}$$

From (16) and (26), it follows that $W_2'(U,V,Z,X) = 0$. Thus, we can state the following theorem:

Theorem 1. *An n-dimensional $N(k)$-quasi Einstein manifold satisfies the condition $W^*(\xi, X) \cdot W_2 = 0$ if and only if the manifold is W_2-flat.*

Definition 1. *A Riemannian manifold is said to be semi-symmetric([23], [24]) if*

$$R \cdot R = 0, \tag{27}$$

where R is the Riemannian curvature tensor.

Consider an $N(k)$-quasi Einstein manifold which is W^*-semisymmetric. Then, we have

$$(R(X,Y) \cdot W^*)(U,V)Z = 0,$$

which implies that

$$\begin{aligned} &R(X,Y)W^*(U,V)Z - W^*(R(X,Y)U,V)Z \\ &-W^*(U,R(X,Y)V)Z - W^*(U,V)R(X,Y)Z = 0. \end{aligned} \tag{28}$$

Taking the inner product of (28) with respect to ξ, we have

$$\begin{aligned} &g(R(X,Y)W^*(U,V)Z,\xi) - g(W^*(R(X,Y)U,V)Z,\xi) \\ &-g(W^*(U,R(X,Y)V)Z,\xi) - g(W^*(U,V)R(X,Y)Z,\xi) = 0. \end{aligned} \tag{29}$$

Substituting $X = \xi$, (29) reduces to

$$\begin{aligned} &g(R(\xi,Y)W^*(U,V)Z,\xi) - g(W^*(R(\xi,Y)U,V)Z,\xi) \\ &-g(W^*(U,R(\xi,Y)V)Z,\xi) - g(W^*(U,V)R(\xi,Y)Z,\xi) = 0. \end{aligned} \tag{30}$$

Using equations (6) and (17) in (30), we get

$$W^*(U,V,Z,X) - \frac{b}{2(n-1)}\Big[g(U,Y)g(V,Z) - g(V,Y)g(U,Z)\Big] = 0. \tag{31}$$

Making use of (10) and (31), we obtain

$$\begin{aligned} &R'(U,V,Z,Y) - \frac{1}{2(n-1)}\Big[S(V,Z)g(U,Y) - S(U,Z)g(V,Y) \\ &+S(U,Y)g(V,Z) - S(V,Y)g(U,Z)\Big] \\ &-\frac{b}{2(n-1)}\Big[g(U,Y)g(V,Z) - g(V,Y)g(U,Z)\Big] = 0. \end{aligned} \tag{32}$$

Contracting (32) with respect to U and Y, we have

$$S(V,Z) = (a+b)g(V,Z),$$

which is a contradiction as the manifold is quasi Einstein. This leads to the theorem:

Theorem 2. *There does not exist a W^*-semisymmetric $N(k)$-quasi Einstein manifold.*

Definition 2. *A Riemannian manifold is said to be a symmetric manifold([12], [15]) if*

$$(\nabla_X R)(Y, Z)V = 0, \tag{33}$$

where ∇ is the operator of covariant differentiation with respect to metric g.

Consider an $N(k)$-quasi Einstein manifold which is W^*-symmetric. Then, we can write

$$(\nabla_X W^*)(U, V, Z, Y) = 0.$$

Using (9), we have

$$\begin{aligned}(\nabla_X R')(U, V, Z, Y) &= \frac{1}{2(n-1)}\Big[(\nabla_X S)(V, Z)g(Y, U) - (\nabla_X S)(U, Z)g(V, Y) \\ &+ (\nabla_X S)(U, Y)g(V, Z) - (\nabla_X S)(V, Y)g(U, Z)\Big]. \end{aligned}\tag{34}$$

Setting $U = Y = e_i$ and summing over $i, 1 \leq i \leq n$, we get

$$(\nabla_X S)(V, Z) = \frac{dr(X)}{n} g(V, Z). \tag{35}$$

Using (2) in (35), we obtain

$$\begin{aligned}&da(X)g(V, Z) + db(X)\eta(V)\eta(Z) + b\big[(\nabla_X \eta)(Z)\eta(V) \\ &+(\nabla_X \eta)(Z)\eta(V)\big] = \frac{dr(X)}{n} g(V, Z).\end{aligned}\tag{36}$$

Putting $Z = V = \xi$, we get

$$dr(X) = n[da(X) + db(X)]. \tag{37}$$

Also, from (12), it follows that

$$dr(X) = nda(X) + db(X). \tag{38}$$

From (37) and (38), we get

$$db(X) = 0,$$

i.e., b is constant. Therefore, we have

Theorem 3. *There exists no W^*-symmetric $N(k)$-quasi Einstein manifold unless the associated scalar b is a non-zero constant.*

From (10), we can write

$$\begin{aligned}&(divW^*)(X,Y)Z\\ =\ &(divR)(X,Y)Z-\frac{1}{2(2n-3)}\big[dr(X)g(Y,Z)-dr(Y)g(X,Z)\big],\end{aligned}\tag{39}$$

where "*div*" denotes the divergence.

We know that in a Riemannian manifold,

$$(divR)(X,Y)Z=(\nabla_X S)(Y,Z)-(\nabla_Y S)(X,Z).\tag{40}$$

Using equation (39) in (40), we get

$$\begin{aligned}(divW^*)(X,Y)Z &= (\nabla_X S)(Y,Z)-(\nabla_Y S)(X,Z)\\ &- \frac{1}{2(2n-3)}\big[dr(X)g(Y,Z)-dr(Y)g(X,Z)\big].\end{aligned}\tag{41}$$

Suppose that an $N(k)$-quasi Einstein manifold is W^*-conservative. Then,

$$(divW^*)(X,Y)Z=0,$$

or,

$$\begin{aligned}(\nabla_X S)(Y,Z)-(\nabla_Y S)(X,Z) &= \frac{1}{2(2n-3)}\Big[dr(X)g(Y,Z)\\ &- dr(Y)g(X,Z)\Big].\end{aligned}\tag{42}$$

Making use of (2) in (42), we obtain

$$\begin{aligned}&da(X)g(Y,Z)+db(X)\eta(Y)\eta(Z)-da(Y)g(X,Z)-db(Y)\eta(X)\eta(Z)\\ &+b\big[(\nabla_X\eta)(Y)\eta(Z)+(\nabla_X\eta)(Z)\eta(Y)-(\nabla_Y\eta)(X)\eta(Z)-(\nabla_Y\eta)(Z)\eta(X)\big]\\ &=\frac{1}{2(2n-3)}\Big[dr(X)g(Y,Z)-dr(Y)g(X,Z)\Big].\end{aligned}\tag{43}$$

Assume that the associated scalar b is non-zero constant. Then $db(X)=0$, from which it follows that $dr(X)=nda(X)$, $\forall X$. Therefore (43) becomes

$$\begin{aligned}&\frac{3(n-2)}{2(2n-3)}\Big[da(X)g(Y,Z)-da(Y)g(X,Z)\Big]+b\Big[(\nabla_X\eta)(Y)\eta(Z)\\ &+(\nabla_X\eta)(Z)\eta(Y)-(\nabla_Y\eta)(X)\eta(Z)-(\nabla_Y\eta)(Z)\eta(X)\Big]=0.\end{aligned}\tag{44}$$

Substituting $Y=Z=\xi$ in (44), we obtain

$$b(\nabla_\xi\eta)(X)=\frac{3(n-2)}{2(2n-3)}\Big[da(X)-da(\xi)\eta(X)\Big].\tag{45}$$

Contracting (44) over Y and Z, we have

$$b\big[(\nabla_\xi\eta)(X)+\eta(X)\sum_{i=1}^{n}(\nabla_{e_i}\eta)(e_i)\big]-\frac{3(n-1)(n-2)}{2(2n-3)}da(X)=0.\tag{46}$$

From (45) and (46), it follows that

$$b\eta(X)\sum_{i=1}^{n}(\nabla_{e_i}\eta)(e_i) = \frac{3(n-1)(n-2)}{2(2n-3)}da(X) - \frac{3(n-2)}{2(2n-3)}\Big[da(X) - da(\xi)\eta(X)\Big]. \tag{47}$$

Taking $X = \xi$, (47) becomes

$$\sum_{i=1}^{n}(\nabla_{e_i}\eta)(e_i) = \frac{3(n-1)(n-2)}{2(2n-3)}da(\xi). \tag{48}$$

Making use of (45) and (48), (46) becomes

$$da(X) = da(\xi)\eta(X). \tag{49}$$

Substituting $X = \xi$ in (44) and using (49), we get

$$b[(\nabla_X\eta)(Y) - (\nabla_Y\eta)(X)] = 0,$$

or,

$$(\nabla_X\eta)(Y) - (\nabla_Y\eta)(X) = 0.$$

which implies that the 1-form η is closed.

Setting $X = \xi$, the above equation reduces to

$$(\nabla_\xi\eta)(Y) = 0.$$

which implies that

$$\nabla_\xi\xi = 0.$$

Therefore, we can state the theorem:

Theorem 4. *On an $(n > 3)$-dimensional $N(k)$-quasi Einstein manifold which is W^*-conservative and b is non-zero constant, the associated 1-form η is closed and the integral curves of the generator ξ are geodesics.*

4 W^*-Ricci pseudosymmetric $N(k)$-quasi Einstein manifold

Definition 3. *A Riemannian manifold is said to be Ricci pseudo-symmetric [13] if the tensors $R\cdot S$ and $Q(g,S)$ are linearly dependent at every point of M^n, i. e.,*

$$R\cdot S = L_S Q(g,S),$$

where L_S is a smooth function on $A_S = \{x \in \mathbb{R} : S \neq \frac{r}{n}g \text{ at } x\}$.

Consider an $N(k)$-quasi Einstein manifold which is W^*-Ricci pseudosymmetric. Then the vectors $W^* \cdot S$ and $Q(g,S)$ are linearly dependent, i.e.,

$$(W^*(X,Y)\cdot S)(Z,U) = L_S Q(g,S)(Z,U;X,Y), \tag{50}$$

where L_S is a function on $A_S = \{x \in \mathbb{R} : S \neq \frac{r}{n} g \ at\ x\}$. Then,

$$\begin{aligned} S(W^*(X,Y)Z,U) + S(Z,W^*(X,Y)U) &= L_S\big[S((X\wedge Y)Z,U) \\ &+ S(Z,(X\wedge Y)U)\big]. \end{aligned} \tag{51}$$

Taking $X = \xi$ in (51), we have

$$\begin{aligned} S(W^*(\xi,Y)Z,U) + S(Z,W^*(\xi,Y)U) &= L_S\big[S((\xi\wedge Y)Z,U) \\ &+ S(Z,(\xi\wedge Y)U)\big]. \end{aligned} \tag{52}$$

Using (17) and

$$(X\wedge Y)Z = g(Y,Z)X - g(X,Z)Y, \tag{53}$$

equation (52) becomes

$$\begin{aligned} &\Big[\frac{b}{2(n-1)} - L_S\Big]\big[S(U,X)g(Y,Z) - S(U,Y)g(X,Z) \\ &+g(U,Y)S(Z,X) - g(U,X)S(Z,Y)\big] = 0, \end{aligned} \tag{54}$$

which implies that either

$$L_S = \frac{b}{2(n-1)},$$

or,

$$\begin{aligned} &S(U,X)g(Y,Z) - S(U,Y)g(X,Z) \\ &+g(U,Y)S(Z,X) - g(U,X)S(Z,Y) = 0. \end{aligned} \tag{55}$$

Using equation (2), (55) can be written as

$$\begin{aligned} &a\big[g(U,X)g(Y,Z) - g(U,Y)g(X,Z) \\ &+g(U,Y)g(Z,X) - g(U,X)g(Z,Y)\big] \\ &+b\big[\eta(U)\eta(X)g(Y,Z) - \eta(U)\eta(Y)g(X,Z) \\ &+g(U,Y)\eta(Z)\eta(X) - g(U,X)\eta(Z)\eta(Y)\big] = 0. \end{aligned} \tag{56}$$

Contracting equation (56) with respect to X and U, we get

$$g(Y,Z) = n\eta(Y)\eta(Z).$$

Substituting $Y = Z = \xi$ in the above equation, we have

$$n = 1,$$

which is a contradiction. Therefore,

$$L_S = \frac{b}{2(n-1)}.$$

Thus, we can state:

Theorem 5. *An n-dimensional W^*-Ricci pseudosymmetric $N(k)$-quasi Einstein manifold satisfies the relation $L_S = \frac{b}{2(n-1)}$.*

5 W_2-pseudosymmetric $N(k)$-quasi Einstein manifolds

Definition 4. *An n-dimensional Riemannian manifold is said to be pseudo-symmetric [13] if*

$$R \cdot R = LQ(g, R),$$

i.e., $R \cdot R$ and $Q(g, R)$ are linearly dependent and L is a function on $B = \{x \in \mathbb{R} : \mathbb{R} \neq 0 \ at \ x\}$.

Suppose that an $N(k)$-quasi Einstein manifold is W_2-pseudosymmetric. Then,

$$(R(X,Y) \cdot W_2)(U,V)Z = L_{W_2} Q(g, W_2)(U,V,Z;X,Y), \tag{57}$$

where L_{W_2} is a smooth function on $B_{W_2} = \{x \in \mathbb{R} : W_2 \neq 0 \ at \ x\}$.

From (57), we have

$$\begin{aligned} &R(X,Y)W_2(U,V)Z - W_2(R(X,Y)U,V)) \\ &-W_2(U,R(X,Y)V)Z - W_2(U,V)R(X,Y)Z \\ &= L_{W_2}\big[(X \wedge_{W_2} Y)W_2(U,V)Z - W_2((X \wedge_{W_2} Y)U,V)Z \\ &-W_2(U,(X \wedge_{W_2} Y)V)Z - W_2(U,V)(X \wedge_{W_2} Y)Z\big]. \end{aligned} \tag{58}$$

Put $X = \xi$ in the above equation, we have

$$\begin{aligned} &R(\xi,Y)W_2(U,V)Z - W_2(R(\xi,Y)U,V)Z) \\ &-W_2(U,R(\xi,Y)V)Z - W_2(U,V)R(\xi,Y)Z \\ &= L_{W_2}\big[(\xi \wedge_{W_2} Y)W_2(U,V)Z - W_2((\xi \wedge_{W_2} Y)U,V)Z \\ &-W_2(U,(\xi \wedge_{W_2} Y)V)Z - W_2(U,V)(\xi \wedge_{W_2} Y)Z\big]. \end{aligned} \tag{59}$$

Using (6) and (53), we get

$$\begin{aligned} &(k - L_{W_2})\big[W_2'(U,V,Z,Y)\xi - \eta(W_2(U,V)Z)Y \\ &-g(Y,U)W_2(\xi,V)Z + \eta(U)W_2(Y,V)Z \\ &-g(Y,V)W_2(U,\xi)Z + \eta(V)W_2(U,Y)Z \\ &-g(Y,Z)W_2(U,V)\xi + \eta(Z)W_2(U,V)X\big] = 0. \end{aligned} \tag{60}$$

Taking the inner product of (60) with respect to ξ, we get

$$\begin{aligned} &(k - L_{W_2})\big[W_2'(U,V,Z,Y) - \eta(W_2(U,V)Z)\eta(Y) \\ &-g(Y,U)\eta(W_2(\xi,V)Z) + \eta(U)\eta(W_2(Y,V)Z) \\ &-g(Y,V)\eta(W_2(U,\xi)Z) + \eta(V)\eta(W_2(U,Y)Z) \\ &-g(Y,Z)\eta(W_2(U,V)\xi) + \eta(Z)\eta(W_2(U,V)X)\big] = 0. \end{aligned} \tag{61}$$

By virtue of (15), (61) reduces to

$$(k - L_{W_2})W_2'(U,V,Z,Y) = 0.$$

Since $W_2 \neq 0$, we have

$$k - L_{W_2} = 0,$$

or,

$$k = L_{W_2}.$$

This leads to the theorem:

Theorem 6. *An $N(k)$-quasi Einstein manifold is W_2-pseudosymmetric provided that $k = L_{W_2}$.*

6 Z-generalized pseudosymmetric $N(k)$-quasi Einstein manifold

Definition 5. *A Riemannian manifold is said to be Ricci-generalized pseudosymmetric [13] if at every point of M^n, the tensors $R \cdot R$ and $Q(S,R)$ are linearly dependent, i. e.,*

$$R \cdot R = LQ(S,R),$$

where L is a function on $A = \{x \in \mathbb{R} : Q(S,R) \neq 0 \ at \ x\}$.

Consider an $N(k)$-quasi Einstein manifold which is Z-generalized pseudosymmetric. Then,

$$R \cdot R = L_Z Q(Z,R),$$

where L_Z is a function on $A_Z = \{x \in \mathbb{R} : Q(Z,R) \neq 0 \ at \ x\}$. Then,

$$\begin{aligned} &R(X,Y)R(U,V)W - R(R(X,Y)U,V)W - R(U,R(X,Y)V)W \\ &-R(U,V)R(X,Y)W = L_Z\big[(X \wedge_Z Y)R(U,V)W \\ &-R((X \wedge_Z Y)U,V)W - R(U,(X \wedge_Z Y)V)W - R(U,V)(X \wedge_Z Y)W\big]. \end{aligned} \tag{62}$$

Taking $X = \xi$ in (62), we have

$$\begin{aligned} &R(\xi,Y)R(U,V)W - R(R(\xi,Y)U,V)W - R(U,R(\xi,Y)V)W \\ &-R(U,V)R(\xi,Y)W = L_Z\big[(\xi \wedge_Z Y)R(U,V)W \\ &-R((\xi \wedge_Z Y)U,V)W - R(U,(\xi \wedge_Z Y)V)W - R(U,V)(\xi \wedge_Z Y)W\big]. \end{aligned} \tag{63}$$

Using (6) and

$$(X \wedge_Z Y)U = Z(Y,U)X - Z(X,U)Y,$$

in (63), we have

$$\begin{aligned}
&[k - L_Z(a+\phi)]\big[R'(U,V,W,Y)\xi - \eta(R(U,V)W)Y \\
&-g(Y,U)R(\xi,V)W + \eta(U)R(Y,V)W \\
&-g(Y,V)R(U,\xi)W + \eta(V)R(U,Y)W \\
&-g(Y,W)R(U,V)\xi + \eta(W)R(U,V)Y\big] \\
&= L_Z b\big[\eta(Y)\eta(R(U,V)W)\xi - \eta(R(U,V)W)Y \\
&-g(Y,U)R(\xi,V)W + \eta(U)R(Y,V)W \\
&-g(Y,V)R(U,\xi)W + \eta(V)R(U,Y)W \\
&-g(Y,W)R(U,V)\xi + \eta(W)R(U,V)Y\big]. \qquad (64)
\end{aligned}$$

Taking the inner product of (64) with respect to ξ, we have

$$\begin{aligned}
&[k - L_Z(a+\phi)]\big[R'(U,V,W,Y) - \eta(R(U,V)W)\eta(Y) \\
&-g(Y,U)\eta(R(\xi,V)W) + \eta(U)\eta(R(Y,V)W) \\
&-g(Y,V)\eta(R(U,\xi)W) + \eta(V)\eta(R(U,Y)W) \\
&-g(Y,W)\eta(R(U,V)\xi) + \eta(W)\eta(R(U,V)Y)\big] \\
&= L_Z b\big[\eta(Y)\eta(R(U,V)W) - \eta(R(U,V)W)\eta(Y) \\
&-g(Y,U)\eta(R(\xi,V)W) + \eta(U)\eta(R(Y,V)W) \\
&-g(Y,V)\eta(R(U,\xi)W) + \eta(V)\eta(R(U,Y)W) \\
&-g(Y,W)\eta(R(U,V)\xi) + \eta(W)\eta(R(U,V)Y)\big]. \qquad (65)
\end{aligned}$$

Using (3) and (8), (65) reduces to

$$L_Z bk[\eta(W)\eta(U)g(V,Y) - \eta(W)\eta(V)g(U,Y)] = 0,$$

which implies(since $b \neq 0$),

$$L_Z k = 0,$$

i.e., $L_Z = 0$ or $k = 0$.

This leads to the theorem:

Theorem 7. *A Z-generalized pseudosymmetric $N(k)$-quasi Einstein manifold is either semisymmetric or $k = 0$.*

7 Examples of $N(k)$-quasi Einstein manifolds

Example 1: Consider a Riemannian metric g on $\mathbb{R}^3$ by

$$ds^2 = g_{ij}dx^i dx^j = e^{x^3}cos(x^3)[(dx^1)^2 + (dx^2)^2] - (dx^3)2.$$

Then, we have

$$g_{11} = g_{22} = e^{x^3}cos(x^3), \quad g_{33} = -1,$$
$$g^{11} = g^{22} = e^{-x^3}sec(x^3), \quad g^{33} = -1.$$

Then, the non-vanishing components of the Christoffels symbols and the curvature tensors are

$$\Gamma^3_{11} = \Gamma^3_{22} = e^{x^3}\frac{(cos(x^3) - sin(x^3))}{2},$$

$$\Gamma^1_{13} = \Gamma^2_{23} = \frac{cos(x^3) - sin(x^3)}{2cos(x^3)},$$

$$R_{1221} = -e^{2x^3}\frac{(1 - sin(2x^3))}{4}, \quad R_{1331} = R_{2332} = -e^{x^3}\frac{(1 + sin(2x^3))}{4cos(x^3)}.$$

Also, the non-vanishing components of the Ricci tensors are

$$S_{11} = S_{22} = -e^{x^3} sin(x^3), \quad S_{33} = \frac{(1 + sin(2x^3))}{2cos^2(x^3)}.$$

Using these results in

$$r = g^{ij} S_{ij}, \tag{66}$$

we get

$$r = -\frac{(sec^2(x^3) - 6tan^2(x^3))}{2},$$

which is non-zero.

To show that the manifold is $N(k)$-quasi Einstein, we choose the scalar functions a and b and the 1-form η as

$$a = -tan(x^3), \quad b = \frac{1}{2} sec^2(x^3),$$

$$\eta_i(x) = \begin{cases} 1, & i = 3, \\ 0, & otherwise, \end{cases}$$

at any point $x \in \mathbb{R}^3$.

From (1), we have

$$S_{11} = ag_{11} + b\eta_1\eta_1, \tag{67}$$

$$S_{22} = ag_{22} + b\eta_2\eta_2, \tag{68}$$

$$S_{33} = ag_{33} + b\eta_3\eta_3 \tag{69}$$

and all others hold trivially.

$$\begin{aligned} \text{R. H. S of (69)} &= ag_{33} + b\eta_3\eta_3 \\ &= -tan(x^3)(-1) + \frac{1}{2}sec^2(x^3)(1) \\ &= \frac{(1+sin(2x^3))}{2cos^2(x^3)} = S_{33} \\ &= \text{L.H.S of (69).} \end{aligned}$$

Similarly, it can be shown that equations (67) and (68) hold. Using (66), we get

$$k = \frac{a+b}{n-1} = \frac{sin(2x^3)-1}{4}.$$

So, $(\mathbb{R}^3, g)$ is an $N\big(\frac{sin(2x^3)-1}{4}\big)$–quasi Einstein manifold.

Example 2: Consider $\mathbb{R}^4$ with the Riemannian metric g defined by

$$ds^2 = g_{ij}dx^idx^j = (x^3)^2[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] + (dx^4)^2.$$

Then, we have

$$g_{11} = g_{22} = g_{33} = (x^3)^2, \quad g_{44} = 1,$$
$$g^{11} = g^{22} = g^{33} = \frac{1}{(x^3)^2}, \quad g^{44} = 1.$$

The non-vanishing components of the Christoffels symbols, the curvature tensors and the Ricci tensors are

$$\Gamma^3_{11} = \Gamma^3_{22} = -\frac{1}{x^3}, \quad \Gamma^3_{33} = \Gamma^1_{13} = \Gamma^2_{23} = \frac{1}{x^3},$$

$$R_{1331} = R_{2332} = -1, \quad R_{2332} = 1, \quad S_{11} = S_{22} = S_{44} = 0, \quad S_{33} = \frac{2}{(x^3)^2}.$$

Using (66) and the above results, we get

$$r = 2,$$

which is non-vanishing. To show that the manifold under consideration is an $N(k)$-quasi Einstein manifold, we choose the scalar functions a, b and the 1-form η as

$$a = 0, \quad b = 2,$$

$$\eta_i(x) = \begin{cases} \frac{1}{x^3}, & i = 3, \\ 0, & otherwise, \end{cases}$$

at any point $x \in \mathbb{R}^4$. From (2), we have

$$S_{11} = ag_{11} + b\eta_1\eta_1, \tag{70}$$

$$S_{22} = ag_{22} + b\eta_2\eta_2, \tag{71}$$

$$S_{33} = ag_{33} + b\eta_3\eta_3, \tag{72}$$

$$S_{44} = ag_{44} + b\eta_4\eta_4 \tag{73}$$

and all others hold trivially.

$$\begin{aligned} \text{R. H. S of (72)} &= ag_{33} + b\eta_3\eta_3 \\ &= -0 + 2\big(\frac{1}{x^3}\big)\big(\frac{1}{x^3}\big) \\ &= \frac{2}{(x^3)^2} = S_{33} \\ &= \text{L.H.S of (72)}, \end{aligned}$$

Similarly, it can be shown that equations (70), (71) and (72) hold. Using (4), we get

$$k = \frac{a+b}{n-1} = \frac{2}{3}.$$

So, $(\mathbb{R}^4, g)$ is an $N\big(\frac{2}{3}\big)$-quasi Einstein manifold.

Example 3: A perfect fluid pseudo Ricci-symmetric spacetime is an $N\big(\frac{2r}{9}\big)$-quasi Einstein manifold[9].

Example 4: A four dimensional conformally flat perfect fluid (M^4, g) is an $N\Bigg(\frac{1}{3}\big(\frac{1}{2}r + f(T) + e\pi\rho^2(p+\rho)f'(T)\big)\Bigg)$-quasi Einstein manifold[5].

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