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LEGENDRE CURVES ON LORENTZIAN HEISENBERG SPACE

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Abstract

In this paper, we show that the Legendre curves on three-dimensional Lorentzian Heisenberg space (H_3, g) are locally ϕ- symmetric if and only if they are geodesic. Moreover, we prove that the Legendre curves on three-dimensional Lorentzian Heisenberg space are biharmonic if and only if they are pseudo-helix.

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1 Introduction

The Legendre curves play a fundamental role in 3-dimensional contact geometry. Let (M, ϕ, ξ, η) be an almost contact metric 3-manifold. Then an integral curve of the contact distribution $\ker\eta = \{X \in \Gamma(TM)|\eta(X) = 0\}$ is known as Legendre curve; $\Gamma(TM)$ being the section of tangent bundle TM of M. Legendre curves on contact manifolds have been studied by C. Baikoussis and D. E. Blair in paper ([4]). M. Belkhelfa, I. E. Hirică, R. Rosca and L. Verstraelen ([5]) have investigated Legendre curves in Riemannian and Lorentzian manifolds. As a generalization of Legendre curve, the notion of slant curves was introduced in ([8]) A curve in a contact 3-manifold is said to be slant if its tangent vector field has constant angle with the Reeb vector field. The Heisenberg group is the Lorentzian Sasakian forms with constant holomorphic sectional curvature $\mu = 3$. Heisenberg group is a unimodular Lie group with left invariant Sasakian structure.

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The present paper is organized as follows: In Section 2, we give some preliminaries of contact Lorentzian manifold. In Section 3, we also discuss about three-dimensional Lorentzian Heisenberg space. In this section it is shown that a Legendre curve in three-dimensional Lorentzian Heisenberg space is locally ϕ-symmetric if and only if is a geodesic. The concept of local ϕ- symmetric was introduced by T. Takahashi ([15]). According to Takahashi a differentiable manifold is called locally if it satisfies

$$\phi^2(\nabla_W R)(X,Y)Z = 0 \tag{1}$$

where the tangent vector fields X, Y, Z are orthogonal to the unit tangent vector field ξ and R is the Riemannian curvature tensor of type $(1,3)$ of the manifold. In Sasakian geometry locally ϕ-symmetric spaces are defined by the above curvature condition, which has several geometric interpretations, E. Boeckx and L. Vanhecke have extended the notion of locally ϕ-symmetric spaces to the broader class of contact metric manifolds using reflections with respect to characteristic curves ([4]). In Section 4, we consider biharmonic Legendre curves in three-dimensional Lorentzian Heisenberg space. Here we also prove that a biharmonic Legendre curve in three-dimensional Heisenberg space is biharmonic if and only if it is a pseudo-helix.

2 Contact Lorentzian manifold

Let M be a $(2n+1)$-dimensional differentiable manifold. M has an almost contact structure (ϕ, ξ, η) if it admits a $(1,1)$ tensor field ϕ, a vector field ξ and a $1-$form η satisfying

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1 \tag{2}$$

Suppose M has an almost contact structure (ϕ, ξ, η). Then $\phi\xi = 0$ and $\eta \circ \phi = 0$. Moreover, the endomorphism ϕ has rank $2n$

If a $(2n+1)$-dimensional smooth manifold M with almost contact structure (ϕ, ξ, η) admits a compatible Lorentzian metric such that

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y) \tag{3}$$

then we say M has an almost contact Lorentzian structure (η, ξ, ϕ, g). Setting $Y = \xi$ we have

$$\eta(X) = -g(X, \xi) \tag{4}$$

Next, if the compatible Lorentzian metric g satisfies

$$d\eta(X,Y) = g(X, \phi Y) \tag{5}$$

then η is a contact form on M, ξ the associated Reeb vector field, g an associated metric and (M, ϕ, ξ, η, g) is called a contact Lorentzian manifold.

An almost contact Lorentzian manifold (M, ϕ, ξ, η, g) is Sasakian if and only if

$$(\nabla_X \phi)Y = g(X,Y)\xi + \eta(Y)X \tag{6}$$

Let (M, ϕ, ξ, η, g) be a contact Lorentzian manifold. Then we have

$$\nabla_X \xi = \phi X - \phi h X, \quad h = \frac{1}{2} L_\xi \phi \tag{7}$$

If ξ is a killing vector field with respect to the Lorentzian metric, then we have

$$\nabla_X \xi = \phi X \tag{8}$$

An arbitrary curve $\gamma : I \longrightarrow M^3, \gamma = \gamma(s)$ in Lorentzian 3-manifolds is called spacelike, timelike or null (lightlike), if all of its velocity vectors $\gamma'(s)$ are respectively spacelike, timelike or null (lightlike). If γ is a spacelike or timelike curve, we can reparametrize it such that $g(\gamma'(s), \gamma'(s)) = \epsilon$, where $\epsilon = 1$ if γ is spacelike and $\epsilon = -1$ if γ is timelike, respectively. In this case $\gamma(s)$ is said to be unit speed or arclenght parametrization. Then the Frenet-Serret equations are the following

$$\begin{aligned} \nabla_T T &= \epsilon_2 \kappa N \\ \nabla_T N &= -\epsilon_1 \kappa T + \epsilon_3 \tau B \\ \nabla_T B &= -\epsilon_2 \tau N \end{aligned} \tag{9}$$

where $\kappa = |\nabla_T T|$ is the geodesic curvature of γ and τ is the geodesic torsion.
A Frenet curve is a geodesic if and only if $\kappa = 0$. A Frenet curve γ with constant geodesic curvature and zero geodesic torsion is called a pseudo-circle. A pseudo-helix is a Frenet curve γ whose geodesic curvature and torsion are constants.
The constant $\epsilon_1, \epsilon_2, \epsilon_3$ defined by $g(T,T) = \epsilon_1, g(N,N) = \epsilon_2, g(B,B) = \epsilon_3$, and called second causal character and third causal character of γ, respectively. Thus it satisfied $\epsilon_1 \epsilon_2 = -\epsilon_3$.

Proposition 1. *Let $\{T, N, B\}$ be an orthonomal Frame field in a Lorentzian 3-manifold. Then*

$$T \wedge N = \epsilon_3 B, \quad N \wedge B = \epsilon_1 T, \quad B \wedge T = \epsilon_2 N \tag{10}$$

3 Legendre curve on Lorentzian Heisenberg space

Definition 1. *A Frenet curve γ in a Riemannian manifold is said to be a Legendre curve if it is an integral curve of the contact distribution $\mathcal{D} = Ker(\eta)$, i.e., if $\eta(\dot{\gamma}) = 0$.*

Let us consider the three-dimensional Heisenberg group

$$H_3 = \begin{pmatrix} 1 & x & z + \frac{xy}{2} \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

Now, we take the contact form

$$\eta = dz + (ydx - xdy)$$

Then the characteristic vector field of η is $\xi = \frac{\partial}{\partial z}$.
Now, we equip the Lorentzian metric as following:

$$g = dx^2 + dy^2 - (dz + ydx - xdy)^2$$

We take a left-invariant Lorentzian orthonormal frame field (e_1, e_2, e_3) on (H_3, g) :

$$e_1 = \frac{\partial}{\partial x} - y\frac{\partial}{\partial z}, \quad e_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, \quad e_3 = \frac{\partial}{\partial z}$$

and the commutative relations are derived as follows:

$$[e_1, e_2] = 2e_3, \quad [e_2, e_3] = [e_3, e_1] = 0$$

Then the endomorphism field

$$\phi e_1 = e_2, \quad \phi e_2 = -e_1, \quad \phi e_3 = 0$$

The Levi-Civita connection ∇ of (H_3, g) is described as

$$\begin{array}{lll} \nabla_{e_1} e_3 = -e_2 & \nabla_{e_1} e_2 = e_3 & \nabla_{e_1} e_1 = 0 \\ \nabla_{e_2} e_3 = e_1 & \nabla_{e_2} e_2 = 0 & \nabla_{e_2} e_1 = -e_3 \\ \nabla_{e_3} e_3 = 0 & \nabla_{e_3} e_2 = e_1 & \nabla_{e_3} e_1 = -e_2 \end{array}$$

The contact form η satisfies $d\eta(X, Y) = g(X, \phi Y)$. Moreover structure (η, ξ, ϕ, g) is Sasakian. The Riemannian curvature tensor R of (H_3, g) is given

$$\begin{array}{ll} R(e_1, e_2)e_1 = 3e_2 & R(e_1, e_2)e_2 = -3e_1 \\ R(e_2, e_3)e_2 = -e_3 & R(e_2, e_3)e_3 = -e_2 \\ R(e_3, e_1)e_3 = e_1 & R(e_3, e_1)e_1 = e_3 \end{array} \tag{11}$$

the others are zero.
The sectional curvature is given by

$$K(\xi, e_i) = -1, \quad \text{for} \quad i = 1, 2,$$

and

$$K(e_1, e_2) = 3$$

Hence Lorentzian Heisenberg space (H_3, g) is the Lorentzian Sasakian space forms with constant holomorphic sectional curvature $\mu = 3$.

Definition 2. *A 1-dimensional integral submanifold of a contact manifold is called a Legendre curve.*

Theorem 1. *([5]) Let M be a 3-dimensional contact metric manifold. Then M is Sasakian if and only if the torsion of its Legendre curves is equal to 1.*

3.1 Locally ϕ-symmetric Legendre curves on Lorentzian Heisenberg space

Definition 3. *A Legendre curve γ on Lorentzian Heisenberg space will be called locally ϕ-symmetric if it satisfies*

$$\phi^2(\nabla_T R)(\nabla_T T, T)T = 0 \tag{12}$$

where $T = \dot{\gamma}$.

Theorem 2. *A Legendre curve on Lorentzian Heisenberg space is a locally ϕ-symmetric if and only if it is a geodesic.*

Proof. Let us consider a locally ϕ-symmetric Legendre curve on Lorentzian Heisenberg space. Let T, ϕT, ξ be a Frenet frame on Legendre curve. To maintain orientation let ϕT=N and ϕN=-T. Also, we take B=ξ. Now using Serret Frenet formula, we get

$$R(\nabla_T T, T)T = R(\varepsilon_2\kappa\phi T, T)T = \varepsilon_2\kappa R(N, T)T \tag{13}$$

Since T and N are orthogonal to ξ=e_3, we can take T=t_1, e_1+t_2e_2 and N=n_1e_1+n_2e_2. Here t_1, t_2, n_1, n_2 are scalars.
Using the definition of curvature tensor R the expression of T and N and (11) we get after straight forward calculation

$$R(N, T)T = 3t_1(-n_2t_1e_2 + n_1t_2e_2) + 3t_2(-n_1t_2e_1 + n_2t_1e_1) \tag{14}$$

Since T, ϕT and ξ=e_3 forms a right handed system. We have $t_1n_2 - t_2n_1 = \varepsilon_3$

$$R(N, T)T = 3\varepsilon_3t_2e_1 - 3\varepsilon_3t_1e_2 \tag{15}$$

Combining (13) and (14), we obtain

$$\begin{aligned} R(\nabla_T T, T)T =& 3\kappa\epsilon_2\epsilon_3t_2e_1 - 3\kappa\epsilon_2\epsilon_3t_1e_2 \\ =& - 3\kappa\epsilon_1t_2e_1 + 3\kappa\epsilon_1t_1e_2 \end{aligned} \tag{16}$$

Now

$$\begin{aligned} (\nabla_T R)(\nabla_T T, T)T &= \nabla_T R(\nabla_T T, T)T - R(\nabla_T^2 T, T)T - R(\nabla_T T, \nabla_T T)T \\ &\quad - R(\nabla_T T, T)\nabla_T T \\ &= \nabla_T R(\epsilon_2\kappa N, T)T - \epsilon_2\kappa' R(N, T)T - \epsilon_3\kappa^2 R(T, T)T \\ &\quad + \epsilon_1\kappa R(B, T)T - \kappa^2 R(N, T)N \end{aligned} \tag{17}$$

Now

$$\begin{aligned} R(B, T)T =& R(\xi, t_1e_1 + t_2e_2)(t_1e_1 + t_2e_2) \\ =& t_1t_1R(e_1, \xi)e_1 - t_1t_2R(e_2, \xi)e_1 - t_1t_2R(e_1, \xi)e_2 + t_2t_2R(e_2, \xi)e_2 \end{aligned} \tag{18}$$

Using (11) in (18), we get

$$R(B,T)T = (t_1^2 + t_2^2)e_3. \tag{19}$$

and

$$R(N,T)N = -3\epsilon_3 n_1 e_2 + 3\epsilon_3 n_2 e_1$$

Again

$$\nabla_T R(\epsilon_2 \kappa N, T)T = 3\epsilon_1 \kappa' t_1 e_2 - 3\epsilon_1 \kappa' t_2 e_1 - 3\kappa\epsilon_1 t_1 t_1 e_3 - 3\kappa\epsilon_1 t_2 t_2 e_3. \tag{20}$$

Using (19), (20) in (17), we have

$$\begin{aligned}(\nabla_T R)(\nabla_T T, T)T =& 3\epsilon_1 \kappa' t_1 e_2 - 3\epsilon_1 \kappa' t_2 e_1 - 3\kappa\epsilon_1 t_1 t_1 e_3 - 3\kappa\epsilon_1 t_2 t_2 e_3 \\ &- \epsilon_2 \kappa'(3\epsilon_3 t_2 e_1 - 3\epsilon_3 t_1 e_2) \\ &+ \epsilon_1 \kappa (t_1^2 + t_2^2) e_3 \\ &- \kappa^2(-3\epsilon_3 n_1 e_2 + 3\epsilon_3 n_2 e_1) \\ =& - 3\epsilon_1 \kappa t_1 t_1 e_3 - 3\epsilon_1 \kappa t_2 t_2 e_3 + \epsilon_1 \kappa (t_1^2 + t_2^2) e_3 \\ &+ 3\kappa^2 \epsilon_3 n_1 e_2 - 3\kappa^2 \epsilon_3 n_2 e_1\end{aligned}$$

By (2) and (3), the above equation yields

$$\phi^2(\nabla_T R)(\nabla_T T, T)T = - 3\kappa^2 \epsilon_3 n_1 e_2 + 3\kappa^2 \epsilon_3 n_2 e_1 \tag{21}$$

Let the Legendre curve be locallyϕ-symmetric. Then by definition

$$3\kappa^2 \epsilon_3 (n_2 e_1 - n_1 e_2) = 0 \tag{22}$$

In both sides of (22) taking inner product with e_1, we get

$$\kappa = 0 \tag{23}$$

□

4 Bi-harmonic Legendre curves on Lorentzian Heisenberg space

Definition 4. *([8]) A Legendre curve on a three-dimensional Heisenberg group will be called biharmonic if it satisfies the biharmonic equation*

$$\nabla_T^3 T + R(\nabla_T T, T)T = 0, \tag{24}$$

where $T = \dot{\gamma}$

Theorem 3. *A Legendre curves on Lorentzian Heisenberg space is biharmonic if and only if it is a pseudo-helix.*

Proof. Using Serret-Frennet formula, by direct computations, we have

$$\begin{aligned}\nabla_T^3 T =&\nabla_T(\nabla_T(\nabla_T T))\\ =&\nabla_T(\nabla_T \epsilon_2 \kappa N)\\ =&\epsilon_2(\nabla_T(\nabla_T \kappa N))\\ =&\epsilon_2(\nabla_T(\kappa' N + \kappa \nabla_T N))\\ =&\epsilon_2(\nabla_T(\kappa' N - \kappa^2 \epsilon_1 T + \epsilon_3 \kappa\tau B))\\ =&\epsilon_2(\kappa'' N - 2\kappa\kappa'\epsilon_1 T + \epsilon_3\kappa'\tau B + \epsilon_3\kappa\tau' B\\ &+ \kappa'\nabla_T N - \kappa^2\epsilon_1\nabla_T T + \epsilon_3\kappa\tau\nabla_T B)\\ =&3\epsilon_3\kappa\kappa' T + \epsilon_2(\kappa'' - \epsilon_3\kappa^3 - \epsilon_1\kappa\tau^2)N - \epsilon_1(2\tau\kappa' + \kappa\tau')B\end{aligned}$$

Using Theorem 1, we have

$$\begin{aligned}\nabla_T^3 T =&3\epsilon_3\kappa\kappa'(t_1e_1 + t_2e_2) + \epsilon_2(\kappa'' - \epsilon_3\kappa^3 - \epsilon_1\kappa)(n_1e_1 + n_2e_2)\\ &- 2\epsilon_1\kappa' e_3\end{aligned}$$

In view of (15) and (24), it follows that

$$\begin{aligned}\nabla_T^3 T + R(\nabla_T T, T)T =&3\epsilon_3\kappa\kappa'(t_1e_1 + t_2e_2) + \epsilon_2(\kappa'' - \epsilon_3\kappa^3 - \epsilon_1\kappa)(n_1e_1 + n_2e_2)\\ &- 2\epsilon_1\kappa' e_3 - 3\epsilon_1\kappa t_2 e_1 + 3\epsilon_1\kappa t_1 e_2 \qquad (25)\end{aligned}$$

Consider that the Legendre curve is biharmonic. Then by definition

$$\begin{aligned}0 =&3\epsilon_3\kappa\kappa'(t_1e_1 + t_2e_2) + \epsilon_2(\kappa'' + \epsilon_3\kappa^3 + \epsilon_1\kappa)(n_1e_1 + n_2e_2)\\ &- 2\epsilon_1\kappa' e_3 - 3\epsilon_1\kappa t_2 e_1 + 3\epsilon_1\kappa t_1 e_2 \qquad (26)\end{aligned}$$

In both sides of (26) taking inner product with e_3, we obtain

$$2\epsilon_1\kappa' = 0$$

which gives κ an arbitrary constant □

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