Bulletin of the *Transilvania* University of Braşov • Vol 13(62), No. 1 - 2020
Series III: Mathematics, Informatics, Physics, 101-114
https://doi.org/10.31926/but.mif.2020.13.62.1.9

GEOMETRY OF GENERALIZED F-HARMONIC MAPS

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Abstract

In this paper, we extend the definition of F-harmonic maps [1] and, we give the notion of F-biharmonic maps, which is a generalization of biharmonic maps between Riemannian manifolds [3] and f-biharmonic maps [7] and we discuss some conformal properties and the stability of F-harmonic maps. Also, we give a formula to construct some examples of proper F-biharmonic maps. Our results are extensions of [1] and [7].

2000 *Mathematics Subject Classification:* 53A45, 53C20, 58E20.
Key words: F-harmonic maps, F-biharmonic maps, Stable F-harmonic maps.

1 Introduction

Consider a smooth map $\varphi : (M, g) \to (N, h)$ between Riemannian manifolds. Let

$$F : M \times \mathbb{R} \to (0, \infty), \quad (x, r) \mapsto F(x, r), \tag{1}$$

be smooth positive function, for any compact domain D of M the L-energy functional of φ is defined by

$$E_F(\varphi; D) = \int_D F\big(x, e(\varphi)(x)\big)\, v_g, \tag{2}$$

where $e(\varphi)$ is the energy density of φ defined by

$$e(\varphi) = \frac{1}{2}\, h\big(d\varphi(e_i), d\varphi(e_i)\big), \tag{3}$$

v_g is the volume element, here $\{e_i\}$ is an orthonormal frame on (M, g).

Definition 1. *A map is called F-harmonic if it is a critical point of the F-energy functional over any compact subset D of M.*

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2 First variation formula

Let $F : M \times \mathbb{R} \to (0, \infty)$, $(x, r) \mapsto F(x, r)$, we denote by

$$\partial_r = \partial/\partial r, \quad F' = \partial_r(F), \quad F'' = \partial_r\big(\partial_r(F)\big)$$

and let $F_r,\ F'_r,\ F''_r \in C^\infty(M)$ defined by

$$F_r(x) = F\big(x, e(\varphi)(x)\big), \quad F'_r(x) = F'\big(x, e(\varphi)(x)\big), \quad F''_r(x) = F''\big(x, e(\varphi)(x)\big). \tag{4}$$

Theorem 1. *Let $\varphi : (M, g) \to (N, h)$ be a smooth map and let $\{\varphi_t\}_{t \in (-\epsilon, \epsilon)}$ be a smooth variation of φ supported in D. Then*

$$\frac{d}{dt} E_F(\varphi_t; D)\Big|_{t=0} = -\int_D h(\tau_F(\varphi), v)\, v_g, \tag{5}$$

where $v = \dfrac{\partial \varphi_t}{\partial t}\Big|_{t=0}$ denotes the variation vector field of φ,

$$\tau_F(\varphi) = F'_r\, \tau(\varphi) + d\varphi\big(\operatorname{grad}^M F'_r\big), \tag{6}$$

and $\tau(\varphi)$ is the tension field of φ given by

$$\tau(\varphi) = \operatorname{trace} \nabla d\varphi. \tag{7}$$

$\tau_F(\varphi)$ is called F-tension field of φ.

Proof. Define $\phi : M \times (-\epsilon, \epsilon) \to N$ by

$$\phi(x, t) = \varphi_t(x), \quad (x, t) \in M \times (-\epsilon, \epsilon), \tag{8}$$

let ∇^ϕ denote the pull-back connection on $\phi^{-1}TN$. Note that, for any vector field X on M considered as a vector field on $M \times (-\epsilon, \epsilon)$, we have

$$[\partial_t, X] = 0. \tag{9}$$

Using (2) we obtain

$$\frac{d}{dt} E_F(\varphi_t; D)\Big|_{t=0} = \int_D \partial_t\Big(F\big(x, e(\varphi_t)(x)\big)\Big)\Big|_{t=0} v_g, \tag{10}$$

first, note that

$$\partial_t\Big(F\big(x, e(\varphi_t)(x)\big)\Big)\Big|_{t=0} = dF\big(\partial_t(e(\varphi_t))\big)\Big|_{t=0}, \tag{11}$$

Calculating in a normal frame at $x \in M$, we have

$$\begin{aligned} \partial_t(e(\varphi_t)) &= h\big(\nabla^{\phi}_{\partial_t} d\varphi_t(e_i), d\varphi_t(e_i)\big) \\ &= h\big(\nabla^{\phi}_{e_i} d\phi(\partial_t), d\varphi_t(e_i)\big), \end{aligned} \tag{12}$$

then

$$\begin{aligned} dF\big(\partial_t(e(\varphi_t))\big)\Big|_{t=0} &= F'_r\, h\big(\nabla^{\varphi}_{e_i} v, d\varphi(e_i)\big) \\ &= e_i\big(h(v, F'_r\, d\varphi(e_i))\big) - h\big(v, \nabla^{\varphi}_{e_i} F'_r\, d\varphi(e_i)\big), \end{aligned} \tag{13}$$

where the last equality holds since $d\phi(\partial_t)\Big|_{t=0} = v$, define a 1-form on M by

$$\omega(X) = h(v, F'_r\, d\varphi(X)), \quad X \in \Gamma(TM), \tag{14}$$

by (13) and (14) we get

$$\begin{aligned} dF\big(\partial_t(e(\varphi_t))\big)\Big|_{t=0} &= \operatorname{div}\omega - h\big(v, d\varphi(\operatorname{grad}^M F'_r)\big) \\ & \quad - h\big(v, F'_r\, \tau(\varphi)\big). \end{aligned} \tag{15}$$

By substituting (11),and (15) in (10), and considering the divergence theorem, the Theorem 3.1 follows. □

Corollary 1. *A smooth map $\varphi : (M, g) \to (N, h)$ between Riemannian manifolds, is F-harmonic if and only if*

$$\tau_F(\varphi) = F'_r\, \tau(\varphi) + d\varphi\big(\operatorname{grad}^M F'_r\big) = 0. \tag{16}$$

In the case where $F(x, r) = F(r)$ we obtain the results of Ara [1]

A mapping $\varphi : (M^m, g) \longrightarrow (N^n, h)$ is called conformal if there exists a $\lambda \in C^\infty(M, \mathbb{R}^*_+)$ such that for any $X, Y \in \Gamma(TM)$ we have $h(d\varphi(X), d\varphi(Y)) = \lambda^2 g(X, Y)$. The function λ is called the dilation for the map φ. The tension field for a conformal map φ is given by (see [2]):

$$\tau(\varphi) = (2 - n) d\varphi(grad\, ln\, \lambda) \tag{17}$$

By Corollary 1 and formula (17), we obtain

Corollary 2. *Let $\varphi : (M, g) \to (N, h)$ be a smooth conformal map with dilation λ, then*

$$\tau_F(\varphi) = d\varphi\Big((2 - n)F'_r\, grad^M\, (ln\, \lambda) + \operatorname{grad}^M F'_r\Big). \tag{18}$$

From Corollary 2 we obtain

Theorem 2. *Let $\varphi : (M^m, g) \to (N^n, h)$ $(n \geq 3)$ be a conformal immersion with dilation λ, then φ is F-harmonic if and only*

$$F(x, r) = C(\lambda(x))^{(n-2)}.r. \tag{19}$$

Examples 2.1. *:*

1) If F is constant then any harmonic map is an F-harmonic map.

2) In partical, in the case where $F(x,r) = F(r)$ and φ is an isometric immersion, the following properties are equivalent:

$$\begin{cases} i) & \varphi \text{ is minimal;} \\ ii) & \varphi \text{ is harmonic;} \\ iii) & \varphi \text{ is } F\text{-harmonic.} \end{cases}$$

3) In the case where φ is an isometric harmonic immersion, the following properties are equivalent:

$$\begin{cases} i) & \varphi \text{ is } F\text{-harmonic.} \\ ii) & F = F(r) \end{cases}$$

4) In the case where φ is a harmonic map, the following properties are equivalent:

$$\begin{cases} i) & \operatorname{grad}^M F'_r \in \ker d\varphi \\ ii) & \varphi \text{ is } F\text{-harmonic.} \end{cases}$$

5) In the case where φ is a harmonic Riemannian submersion, the following properties are equivalent:

$$\begin{cases} i) & \operatorname{grad}^M F'_r \text{ is tangent to the fibers of } \varphi; \\ ii) & \varphi \text{ is } F\text{-harmonic.} \end{cases}$$

Theorem 3. *Let $\varphi : M \to N$ be a smooth map of two Riemannian manifolds and let $i : N \hookrightarrow P$ be the inclusion map of a submanifold, then φ is F-harmonic if and only if $\tau_F(i \circ \varphi)$ is normal to N, where $F \in C^\infty(M \times \mathbb{R})$ is a smooth positive function.*

Proof. The F-tension field of the composition $i \circ \varphi : M \to P$ is given by

$$\tau_F(i \circ \varphi) \quad = \quad F'_r \tau(i \circ \varphi) + di(d\varphi(\operatorname{grad}^M F_r))$$

since the tension field of the composition $i \circ \varphi$ is given by

$$\tau(i \circ \varphi) = di(\tau(\varphi)) + \operatorname{trace} \nabla di(d\varphi, d\varphi),$$

we obtain

$$\begin{aligned} \tau_F(i \circ \varphi) &= F'_r di(\tau(\varphi)) + F'_r \operatorname{trace} \nabla di(d\varphi, d\varphi) \\ &\quad + di(d\varphi(\operatorname{grad}^M F'_r)) \\ &= di(\tau_F(\varphi)) + F'_r \operatorname{trace} \nabla di(d\varphi, d\varphi). \end{aligned}$$

So $\tau_F(i \circ \varphi) - di(\tau_F(\varphi))$ is normal to N, then

$$\tau_F(\varphi) = 0 \iff \tau_F(i \circ \varphi) \perp N.$$

□

Theorem 4. *Let $\varphi : (M^m, g) \longrightarrow (N^n, h) \quad (m \geq 3)$ be a smooth map between Riemannian manifolds. we assume that $F'_r \neq 0$. Then φ is F-harmonic if and only if φ is harmonic with respect to the conformally related metric $\widetilde{g}$ given by*

$$\widetilde{g} = \left(F'_r\right)^{2/(m-2)}.g$$

Proof. Putting $\lambda(x) = F'_r(x, e(\varphi)(x)$, then the tension fields $\widetilde{\tau}(\varphi)$ with regard to the conformally related metric $\widetilde{g} = \lambda^2 g$ are given by

$$\begin{aligned} \widetilde{\tau}(\varphi) &= \frac{1}{\lambda^m}\left\{\lambda^{(m-2)}\tau(\varphi) + d\varphi(grad(\lambda^{(m-2)}))\right\} \\ &= \left(F'_r\right)^{(m-2)/m}\left\{F'_r\tau(\varphi) + d\varphi(grad(F'_r)\right\} \\ &= \left(F'_r\right)^{(m-2)/m}\tau_F(\varphi). \end{aligned}$$

□

3 Second variation formula

Theorem 5. *Let $\varphi : (M, g) \to (N, h)$ be an f-harmonic map between Riemannian manifolds and $\{\varphi_{t,s}\}_{t,s\in(-\epsilon,\epsilon)}$ be a two-parameter variation with compact support in D. Set*

$$v = \frac{\partial \varphi_{t,s}}{\partial t}\Big|_{t=s=0}, \quad w = \frac{\partial \varphi_{t,s}}{\partial s}\Big|_{t=s=0}. \tag{20}$$

Under the notation above we have the following

$$\frac{\partial^2}{\partial t \partial s} E(\varphi_{t,s}; D)\Big|_{t=s=0} = -\int_D h(J_F(v), w)\, v_g, \tag{21}$$

where $J_F(v) \in \Gamma(\varphi^{-1}TN)$ given by

$$\begin{aligned} J_F(v) &= -F'_r \text{ trace } R^N(v, d\varphi)d\varphi - \text{trace } \nabla^\varphi\, F'_r\, \nabla^\varphi\, v \\ &\quad - \text{trace } \nabla^\varphi < \nabla^\varphi\, v, d\varphi > F''_r\, d\varphi. \end{aligned} \tag{22}$$

Here $<,>$ denote the inner product on $T^*M \otimes \varphi^{-1}TN$ and R^N is the curvature tensor on (N, h).

Proof. Define $\phi : M \times (-\epsilon, \epsilon) \times (-\epsilon, \epsilon) \to N$ by

$$\phi(x, t, s) = \varphi_{t,s}(x), \quad (x, t, s) \in M \times (-\epsilon, \epsilon) \times (-\epsilon, \epsilon), \tag{23}$$

let ∇^{ϕ} denote the pull-back connection on $\phi^{-1}TN$. Note that, for any vector field X on M considered as a vector field on $M\times(-\epsilon,\epsilon)\times(-\epsilon,\epsilon)$, we have

$$[\partial_t, X] = 0, \quad [\partial_s, X] = 0, \quad [\partial_t, \partial_s] = 0, \tag{24}$$

Then, by (2) we obtain

$$\frac{\partial^2}{\partial t\partial s}E_F(\varphi_{t,s};D)\Big|_{t=s=0} = \int_D \frac{\partial^2}{\partial t\partial s}F\big(x, e(\varphi_{t,s})(x)\big)\Big|_{t=s=0} v_g, \tag{25}$$

first, note that

$$\frac{\partial}{\partial t}F\big(x, e(\varphi_{t,s})(x)\big) = dF(\partial_t(e(\varphi_{t,s}))), \tag{26}$$

$$dF(\partial_t(e(\varphi_{t,s}))) = h(\nabla^{\phi}_{\partial_t} d\phi(e_i), d\phi(e_i))\, F'_r, \tag{27}$$

when we pass to the second derivative, we get

$$\begin{aligned}\frac{\partial^2}{\partial t\partial s}F\big(x,\varphi_{t,s}(x), e(\varphi_{t,s})(x)\big) \qquad & +h(\nabla^{\phi}_{\partial_s}\nabla^{\phi}_{\partial_t} d\phi(e_i), d\phi(e_i))F'_r \\ & +h(\nabla^{\phi}_{\partial_t} d\phi(e_i), \nabla^{\phi}_{\partial_s} d\phi(e_i))F'_r \\ & +h(\nabla^{\phi}_{\partial_t} d\phi(e_i), d\phi(e_i))\, \partial_s(F'_r).\end{aligned} \tag{28}$$

by (25) and the definition of the curvature tensor of (N,h) we have

$$\begin{aligned} h(\nabla^{\phi}_{\partial_s}\nabla^{\phi}_{\partial_t} d\phi(e_i), d\phi(e_i))F'_r\Big|_{t=s=0} &= F'_r\, h(R^N(w, d\varphi(e_i))v, d\varphi(e_i)) \\ &\quad + F'_r\, h(\nabla^{\phi}_{e_i}\nabla^{\phi}_{\partial_s} d\phi(\partial_t), d\varphi(e_i))\Big|_{t=s=0}, \end{aligned} \tag{29}$$

by (29), the property of the curvature tensor of (N,h) and the compatibility of ∇^{ϕ} with the metric h we have

$$\begin{aligned} h(\nabla^{\phi}_{\partial_s}\nabla^{\phi}_{\partial_t} d\phi(e_i), d\phi(e_i))F'_r\Big|_{t=s=0} &= -F'_r\, h(R^N(v, d\varphi(e_i))d\varphi(e_i), w) \\ &\quad + e_i(h(\nabla^{\phi}_{\partial_s} d\phi(\partial_t), F'_r\, d\varphi(e_i)))\Big|_{t=s=0}, \\ &\quad - h(\nabla^{\phi}_{\partial_s} d\phi(\partial_t), \nabla^{\varphi}_{e_i} F'_r\, d\varphi(e_i))\Big|_{t=s=0}, \end{aligned} \tag{30}$$

$$h(\nabla^{\phi}_{\partial_t} d\phi(e_i), \nabla^{\phi}_{\partial_s} d\phi(e_i))F'_r\Big|_{t=s=0} = e_i(h(F'_r\nabla^{\varphi}_{e_i}v, w)) - h(\nabla^{\varphi}_{e_i}F'_r\nabla^{\varphi}_{e_i}v, w). \tag{31}$$

Note that

$$\begin{aligned} \partial_s(F'_r) &= \partial_s\big(F'_r(x, e(\varphi_{t,s})(x))\big) \\ &= +dF'_r\big(\partial_s(e(\varphi_{t,s}))\big), \end{aligned} \tag{32}$$

by a simple calculation we have

$$dF_r'\big(\partial_s(e(\varphi_{t,s}))\big)\Big|_{t=s=0} = F_r''\, h(\nabla^{\varphi}_{e_i} w, d\varphi(e_i)), \tag{33}$$

then we get

$$\begin{aligned} h(\nabla^{\phi}_{\partial_t} d\phi(e_i), d\phi(e_i))\partial_s(F_r')\Big|_{t=s=0} &= + < \nabla^{\varphi} v, d\varphi > \; F_r''\, h(\nabla^{\varphi}_{e_i} w, d\varphi(e_i)) \\ &= +e_i(h(w, < \nabla^{\varphi} v, d\varphi > \; F_r''\, d\varphi(e_i))) \\ &\quad -h(w, \nabla^{\varphi}_{e_i} < \nabla^{\varphi} v, d\varphi > \; F_r''\, d\varphi(e_i)). \end{aligned} \tag{34}$$

From formulas (25), (28), (30), (31), (34), the divergence theorem and the F-harmonicity of φ, Theorem 5 follows. □

Lemma 1.

$$-\int h(\text{trace } \nabla^{\varphi} F_r' \nabla^{\varphi} v, w) v_g = \int F_r'' < \nabla^{\varphi} v, d\varphi >< \nabla^{\varphi} w, d\varphi > .v_g. \tag{35}$$

Proof. we have:

$$\begin{aligned} -h(\text{trace } \nabla^{\varphi} F_r' \nabla^{\varphi} v, w) &= -h(\nabla^{\varphi}_{e_i} F_r' \nabla^{\varphi}_{e_i} v, w) && (36) \\ &= -e_i\big(h(F_r' \nabla^{\varphi}_{e_i} v, w)\big) + h(F_r' \nabla^{\varphi}_{e_i} v, \nabla^{\varphi}_{e_i} w) \\ &= -\operatorname{div} \omega + F_r' < \nabla^{\varphi} v, \nabla^{\varphi} w > && (37) \end{aligned}$$

where: $\omega(X) = F_r'\, h(\nabla^{\varphi}_X v, w)$.

$$\begin{aligned} - \quad h(\text{trace } \nabla^{\varphi} & < \nabla^{\varphi} v, d\varphi > F_r''\, d\varphi, w) = \\ = \quad & -h(\nabla^{\varphi}_{e_i} < \nabla^{\varphi} v, d\varphi > \; F_r''\, d\varphi(e_i), w) \\ = \quad & -e_i\big(h(< \nabla^{\varphi} v, d\varphi > \; F_r''\, d\varphi(e_i), w)\big) \\ & +h(< \nabla^{\varphi} v, d\varphi > \; F_r''\, d\varphi(e_i), \nabla^{\varphi}_{e_i} w) \\ = \quad & -\operatorname{div} \eta + F_r'' < \nabla^{\varphi} v, d\varphi >< \nabla^{\varphi} w, d\varphi > \end{aligned} \tag{38}$$

where: $\eta(X) = F_r'' < \nabla^{\varphi} v, d\varphi > \; h(d\varphi(X), w)$.
By the integration and divergence theorem we obtain (35). □

From Theorem 5 and Lemma 1 we deduce

Corollary 3.

$$\begin{aligned} \frac{\partial^2}{\partial t \partial s} E(\varphi_{t,s}; D)\Big|_{t=s=0} &= \int_D F_r''\Big(\frac{|d\varphi|^2}{2}\Big) < \nabla^{\varphi} v, d\varphi >< \nabla^{\varphi} w, d\varphi > \; v_g \\ &\quad - \int_D F_r'\Big(\frac{|d\varphi|^2}{2}\Big) h(\text{trace } R^N(v, d\varphi) d\varphi, w)\, v_g \\ &\quad + \int_D F_r'\Big(\frac{|d\varphi|^2}{2}\Big) < \nabla^{\varphi} v, \nabla^{\varphi} w > \; v_g \end{aligned} \tag{39}$$

(In the case where $F = F(r)$ we recover the result obtained by M. Ara in [1].)

Definition 2. *An F-harmonic map is called stable if $I(V,V) \geq 0$ for any compactly supported field V along φ where*

$$I(V,W) \;=\; \frac{\partial^2}{\partial t \partial s} E(\varphi_{t,s}; D)\Big|_{t=s=0}$$

From Definition 2 and Corollary 3 we obtain

Theorem 6. *Let $\varphi : M^m \longrightarrow N^n$ be an F-harmonic between Riemannian manifolds. If $F_r'' \geq 0$ and N has nonpositive curvature, then φ is stable.*

Let ${}^M\nabla$, $\widetilde{\nabla}$, ${}^R\nabla$ and ${}^S\nabla$ denote the Levi-Civita connections on M, $\varphi^{-1}TS^n$, $\mathbb{R}^{n+1}$ and S^n respectively. Let ${}^SR$, B and A denote the curvature tensor, the second fundamental form and the shape operator on S^n. If $X, Y \in \Gamma(TS^n$ and $W \in (TS^n)^\perp$, then at $x \in S^n$ we have

$$B(X,Y) = - < X, Y > .x, \quad \textit{and} \quad < A^W(X), Y >= - < X, Y >< x, W > .$$

Lemma 2. *If V is a parallel field in $\mathbb{R}^n$, then at $x \in S^n$ we have*

$$\widetilde{\nabla}_X V^\top = A^{V^\perp}(d\varphi(X)), \quad \textit{and} \quad < \widetilde{\nabla}_X V^\top, d\varphi(X) >= -|d\varphi(X)|^2 < x, V > .$$

for all $X \in \Gamma(TM)$.

Proof. We have

$$\begin{aligned}
\widetilde{\nabla}_X V^\top &= {}^S\nabla_{d\varphi(X)} V^\top \\
&= \left({}^R\nabla_{d\varphi(X)} V^\top\right)^\top \\
&= \left({}^R\nabla_{d\varphi(X)} (V - V^\perp)\right)^\top \\
&= -\left({}^R\nabla_{d\varphi(X)} V^\perp\right)^\top \\
&= A^{V^\perp}(d\varphi(X)). \\
< \widetilde{\nabla}_X V^\top, d\varphi(X) > &= < A^{V^\perp}(d\varphi(X)), d\varphi(X) > \\
&= -|d\varphi(X)|^2 < x, V^\perp > \\
&= -|d\varphi(X)|^2 < x, V >
\end{aligned}$$

□

From Lemma 2 we obtain

Lemma 3. *If V is a parallel field in $\mathbb{R}^n$, then at $x \in S^n$ we have*

$$|\widetilde{\nabla}_X V^\top|^2 = |d\varphi(X)|^2 < x, V >^2$$

for all $X \in \Gamma(TM)$.

From the sectional curvature of S^n, we obtain

Lemma 4. *If $V \in \Gamma(\mathbb{R}^n)$ then*

$$<^S R\big(V^\top, d\varphi(X)\big)d\varphi(X), V^\top >= |d\varphi(X)|^2|V^\top|^2 - < d\varphi(X), V >^2$$

for all $X \in \Gamma(TM)$.

Proposition 1. *Let $\{E_k\}_{k=1}^{n+1}$ be the canonical orthonormal frame field in $\mathbb{R}^{n+1}$, then*

$$\sum_{k=1}^{n+1} I(E_k^\top, E_k^\top) = \int_M |d\varphi|^2 \Big\{ |d\varphi|^2 F_r''(x, e(\varphi(x)) + (2-n)F_r'(x, e(\varphi(x)) \Big\} v_g.$$

Proof. Let $\{e_i\}_{i=1}^m$ be a local orthonormal frame field on M and $x \in S^n$, from lemmas 2, 3 and 4 we obtain

$$\begin{aligned} \sum_{k=1}^{n+1}\Big(\sum_{i=1}^{m} < \widetilde{\nabla}_{e_i}E_k^\top, d\varphi(e_i >\Big)^2 &= \Big(\sum_{i=1}^{m}|d\varphi(e_i)|^2\Big)^2 \sum_{k=1}^{n+1} < x, E_k >^2 \\ &= |d\varphi|^4|x|^2 \\ &= |d\varphi|^4. \end{aligned} \tag{40}$$

$$\begin{aligned} \sum_{k=1}^{n+1}\sum_{i=1}^{m} |\widetilde{\nabla}_{e_i}E_k^\top|^2 &= \sum_{k=1}^{n+1}\sum_{i=1}^{m} |d\varphi(e_i)|^2 < x, E_k >^2 \\ &= |d\varphi|^2|x|^2 \\ &= |d\varphi|^2. \end{aligned} \tag{41}$$

$$\begin{aligned} \sum_{k=1}^{n+1}\sum_{i=1}^{m} <^S R\big(E_k^\top, d\varphi(e_i)\big)d\varphi(e_i), E_k^\top > &= \sum_{k=1}^{n+1}\sum_{i=1}^{m}\Big(|d\varphi(e_i)|^2|E_k^\top|^2 - < d\varphi(e_i), E_k >^2\Big) \\ &= -|d\varphi|^2 + |d\varphi|^2 \sum_{k=1}^{n+1} |E_k^\top|^2 \\ &= -|d\varphi|^2 + |d\varphi|^2 \sum_{k=1}^{n+1} |E_k - E_k^\perp|^2 \\ &= -|d\varphi|^2 + |d\varphi|^2 \sum_{k=1}^{n+1} |E_k - < E_k, x > x|^2 \\ &= -|d\varphi|^2 + |d\varphi|^2 \sum_{k=1}^{n+1} \big(|E_k|^2 - < E_k, x > |^2\big) \\ &= -|d\varphi|^2 + |d\varphi|^2 \sum_{k=1}^{n+1} \big(1 - < E_k, x > |^2\big) \\ &= -|d\varphi|^2 + |d\varphi|^2 \big(n + 1 - \sum_{k=1}^{n+1} < E_k, x > |^2\big) \\ &= -|d\varphi|^2 + |d\varphi|^2 \big(n + 1 - |x|^2\big) \\ &= (n-1)|d\varphi|^2 \end{aligned} \tag{42}$$

By Corollary 3 and formulae (40), (41) and (42) the Proposition 1 follows. □

From Proposition 1 we obtain

Theorem 7. *Let $\varphi : M^m \longrightarrow S^n$ be an F-harmonic maps from a compact manifold M. If*

$$\int_M |d\varphi|^2 \Big\{ |d\varphi|^2 F_r''(x, e(\varphi(x)) + (2-n) F_r'(x, e(\varphi(x)) \Big\} v_g \;<\; 0$$

then φ is unstable.

Theorem 8. *Let $\varphi : M^m \longrightarrow S^n$ $(n \geq 3)$ be an F-harmonic maps from a compact manifold M. If*

$$F_r'' \leq 0, \quad \text{and} \quad F_r' > 0$$

then φ is unstable.

From Theorem 8 follows

Theorem 9. *If $F_r'' \leq 0$, $F_r' > 0$, and $n \geq 3$ or $F_r'' < 0$, and $n = 2$. Then any stable F-harmonic map from a compact manifold to S^n is constant.*

4 F-biharmonic maps.

Definition 3. *A natural generalization of F-harmonic maps is given by integrating the square of the norm of the F-tension field. More precisely, the F-bienergy functional of a smooth map $\varphi : (M, g) \to (N, h)$ is defined by*

$$E_{2,F}(\varphi; D) = \frac{1}{2} \int_D |\tau_F(\varphi)|^2 \, v_g. \tag{43}$$

A map is called F-biharmonic if it is a critical point of the F-energy functional over any compact subset D of M.

Theorem 10. *[First variation of the F-bienergy functional].*
Let $\varphi : (M, g) \to (N, h)$ be a smooth map between Riemannian manifolds, D a compact subset of M and let $\{\varphi_t\}_{t \in (-\epsilon, \epsilon)}$ be a smooth variation with compact support in D. Then

$$\frac{d}{dt} E_2(\varphi_t; D)\Big|_{t=0} = \int_D h(\tau_{2,F}(\varphi), v) \, v_g, \tag{44}$$

where

$$\begin{aligned} \tau_{2,F}(\varphi) &= -F_r' \text{ trace } R^N(\tau_F(\varphi), d\varphi)d\varphi - \text{trace } \nabla^\varphi F_r' \nabla^\varphi \tau_F(\varphi) \\ &\quad - \text{trace } \nabla^\varphi < \nabla^\varphi \tau_F(\varphi), d\varphi > F_r'' \, d\varphi. \end{aligned} \tag{45}$$

(46)

$\tau_{2,F}(\varphi)$ is called F-bitension of φ.

Proof. Define $\phi : M \times (-\epsilon, \epsilon) \to N$ by $\phi(x,t) = \varphi_t(x)$. First note that

$$\frac{d}{dt} E_{2,F}(\varphi_t; D) = \int_D h(\nabla^\phi_{\partial_t} \tau_F(\varphi_t), \tau_F(\varphi_t))\, v_g. \tag{47}$$

Calculating in a normal frame at $x \in M$ we have

$$\nabla^\phi_{\partial_t} \tau_F(\varphi_t) = \nabla^\phi_{\partial_t} \nabla^\phi_{e_i} F'_r\, d\varphi_t(e_i) \tag{48}$$

by the definition of the curvature tensor of (N,h) we have

$$\nabla^\phi_{\partial_t} \nabla^\phi_{e_i} F'_r\, d\varphi_t(e_i) = F'_r\, R^N(d\phi(\partial_t), d\varphi_t(e_i)) d\varphi_t(e_i) + \nabla^\phi_{e_i} \nabla^\phi_{\partial_t} F'_r\, d\varphi_t(e_i), \tag{49}$$

by the compatibility of ∇^ϕ with h we have

$$\begin{aligned} h(\nabla^\phi_{e_i} \nabla^\phi_{\partial_t} F'_r\, d\varphi_t(e_i), \tau_F(\varphi_t)) &= e_i\big(h(\nabla^\phi_{\partial_t} F'_r\, d\varphi_t(e_i), \tau_F(\varphi_t))\big) \\ &\quad -h(\nabla^\phi_{\partial_t} F'_r\, d\varphi_t(e_i), \nabla^\phi_{e_i} \tau_F(\varphi_t)), \end{aligned} \tag{50}$$

the second term on the left-hand side of (50) is

$$\begin{aligned} -h(\nabla^\phi_{\partial_t} F'_r\, d\varphi_t(e_i), \nabla^\phi_{e_i} \tau_F(\varphi_t)) &= -\partial_t(F'_r)\, h(d\varphi_t(e_i), \nabla^\phi_{e_i} \tau_F(\varphi_t)) \\ &\quad -F'_r\, h(\nabla^\phi_{\partial_t} d\varphi_t(e_i), \nabla^\phi_{e_i} \tau_F(\varphi_t)), \end{aligned} \tag{51}$$

be a simple calculation we have

$$\partial_t(F'_r) = d\phi(\partial_t)(F'_r) + F''_r\, h(\nabla^\phi_{e_j} d\phi(\partial_t), d\varphi_t(e_j)), \tag{52}$$

then the first term on the left-hand side of (51) is

$$\begin{aligned} -\partial_t(F'_r)\, h(d\varphi_t(e_i), \nabla^\phi_{e_i} \tau_F(\varphi_t)) &= -e_j\big(h(d\phi(\partial_t), F''_r\, h(d\varphi_t(e_i), \nabla^\phi_{e_i} \tau_F(\varphi_t))\, d\varphi_t(e_j))\big) \\ &\quad +h\big(d\phi(\partial_t), \nabla^\phi_{e_j} F''_r h(d\varphi_t(e_i), \nabla^\phi_{e_i} \tau_F(\varphi_t))\, d\varphi_t(e_j)\big), \end{aligned} \tag{53}$$

the second term on the left-hand side of (51) is

$$\begin{aligned} -F'_r\, h(\nabla^\phi_{\partial_t} d\varphi_t(e_i), \nabla^\phi_{e_i} \tau_F(\varphi_t)) &= -e_i\big(h(d\phi(\partial_t), F'_r \nabla^\phi_{e_i} \tau_F(\varphi_t))\big), \\ &\quad +h(d\phi(\partial_t), \nabla^\phi_{e_i} F'_r \nabla^\phi_{e_i} \tau_F(\varphi_t))\big), \end{aligned} \tag{54}$$

and notice that from (47), (48), (49), (50), (51), (53), (54), $v = d\phi(\partial_t)$ when $t = 0$ and the divergence theorem, we deduce Theorem 4.1. □

Corollary 4. *Let $\varphi : (M,g) \to (N,h)$ be a smooth map between Riemannian manifolds. Then φ is F-biharmonic if it satisfies the associated Euler-Lagrange equations*

$$\begin{aligned} \tau_{2,F}(\varphi) &= -F'_r \operatorname{trace}_g R^N(\tau_F(\varphi), d\varphi) d\varphi - \operatorname{trace}_g \nabla^\varphi F'_r \nabla^\varphi \tau_F(\varphi) \\ &\quad - \operatorname{trace}_g \nabla^\varphi < \nabla^\varphi \tau_F(\varphi), d\varphi > F''_r\, d\varphi \\ &= 0. \end{aligned} \tag{55}$$

From Corollary 4 and Corollary 2 we have

Theorem 11. *Let $\phi : (M^n, g) \to (N^n, h)$ be a conformal map with dilation λ. The F-bitension fields of ϕ is given by*

$$\begin{aligned}\tau_{2,F_r'}(\phi) &= (n-2)F_r' \operatorname{trace}_g \nabla^2 F_r' d\phi\,(grad \ln\lambda) - F_r' \operatorname{trace}_g \nabla^2 d\phi\left(gradF_r'\right)\\ &+ (n-2)F_r' \operatorname{trace}_g R^N\left(F_r' d\phi\,(grad\ln\lambda), d\phi\right) d\phi - F_r' \operatorname{trace}_g R^N\left(d\phi\,(gradf), d\phi\right) d\phi\\ &+ (n-2)\nabla_{gradf} F_r' d\phi\,(grad\ln\lambda) - \nabla_{gradF_r'} d\phi\left(gradF_r'\right)\\ &+ (n-2)\,F_r'' \operatorname{trace}_g \nabla^{\phi} < \nabla^{\phi} d\phi\Big(F_r'\,grad^M(ln\,\lambda) + \operatorname{grad}^M F_r'\Big), d\phi > d\phi \end{aligned} \tag{56}$$

Theorem 12. *Let $\phi : (M^n, g) \to (N^n, h)$ be a conformal map with dilation λ. Then, the F-bitension field of φ is defined by*

$$\begin{aligned}\tau_{2,F}(\phi) &= (n-2)(F_r')^2 d\phi\,(grad\,(\Delta\ln\lambda)) - (n-2)^2(F_r')^2\nabla_{grad\ln\lambda} d\phi\,(grad\ln\lambda)\\ &+ 4(n-2)F_r'\nabla_{gradf} d\phi\,(grad\ln\lambda) + (n-2)F_r'\left(\Delta F_r'\right) d\phi\,(grad\ln\lambda)\\ &- F_r' d\phi\,(grad\,(\Delta f)) + 2(n-2)(F_r')^2\left\langle\nabla d\phi, \nabla d\ln\lambda\right\rangle - 2F_r'\left\langle\nabla d\phi, \nabla dF_r'\right\rangle\\ &+ (n-2)\left|gradF_r'\right|^2 d\phi\,(grad\ln\lambda) - \nabla_{gradF_r'} d\phi\left(gradF_r'\right)\\ &+ 2(n-2)(F_r')^2 d\phi\left(Ricci^M(grad\ln\lambda)\right) - 2F_r' d\phi\left(Ricci^M\left(gradF_r'\right)\right).\\ &+ (n-2)\,F_r'' \operatorname{trace}_g \nabla^{\phi} < \nabla^{\varphi} d\phi\Big(F_r'\,grad^M(ln\,\lambda) + \operatorname{grad}^M F_r'\Big), d\phi > d\phi\end{aligned}$$

Proof. Fix a point $x_0 \in M$ and let $\{e_i\}_{1\le i\le m}$ be an orthonormal frame, such that $\nabla_{e_i} e_j = 0$, at x_0 for all i, j. Then calculating at x_0, we have

$$\begin{aligned} Tr_g\nabla^2 F_r' d\phi\,(grad\ln\lambda) &= \nabla_{e_i}\nabla_{e_i} F_r' d\phi\,(grad\ln\lambda)\\ &+ e_i\left(F_r'\right)\nabla_{e_i} d\phi\,(grad\ln\lambda) + e_i\left(e_i\left(F_r'\right)\right) d\phi\,(grad\ln\lambda)\\ &= F_r' Tr_g\nabla^2 d\phi\,(grad\ln\lambda) + 2\nabla_{gradF_r'} d\phi\,(grad\ln\lambda)\\ &+ \left(\Delta F_r'\right) d\phi\,(grad\ln\lambda). \end{aligned} \tag{57}$$

Note that (see [3])

$$\begin{aligned} Tr_g\nabla^2 d\phi\,(grad\ln\lambda) &= d\phi\,(grad\,(\Delta\ln\lambda)) + 2d\phi\left(Ricci^M(grad\ln\lambda)\right)\\ &+ (2-n)\nabla_{grad\ln\lambda} d\phi\,(grad\ln\lambda) + 2\left\langle\nabla d\phi, \nabla d\ln\lambda\right\rangle\\ &- Tr_g R^N\left(d\phi\,(grad\ln\lambda), d\phi\right) d\phi, \end{aligned} \tag{58}$$

where

$$\left\langle\nabla d\phi, \nabla d\ln\lambda\right\rangle = \nabla d\phi\,(e_i, e_j)\,\nabla d\ln\lambda\,(e_i, e_j).$$

On substituting (58) in (57), we conclude that

$$\begin{aligned} Tr_g\nabla^2 F_r' d\phi\,(grad\ln\lambda) &= F_r' d\phi\,(grad\,(\Delta\ln\lambda)) + 2F_r' d\phi\left(Ricci^M(grad\ln\lambda)\right)\\ &+ (2-n)F_r'\nabla_{grad\ln\lambda} d\phi\,(grad\ln\lambda) + 2F_r'\left\langle\nabla d\phi, \nabla d\ln\lambda\right\rangle\\ &+ F_r' Tr_g R^N\left(d\phi\,(grad\ln\lambda), d\phi\right) d\phi + 2\nabla_{gradf} d\phi\,(grad\ln\lambda)\\ &+ \left(\Delta F_r'\right) d\phi\,(grad\ln\lambda). \end{aligned} \tag{59}$$

$$\begin{aligned} Tr_g\nabla^2 d\phi\,(gradf) = {} & d\phi\left(grad\left(\Delta F'_r\right)\right) + 2d\phi\left(Ricci^M\,(gradf)\right) \\ & + (2-n)\nabla_{gradF'_r} d\phi\,(grad\ln\lambda) + 2\left\langle \nabla d\phi, \nabla dF'_r\right\rangle \\ & + Tr_g R^N\left(d\phi\left(gradF'_r\right), d\phi\right) d\phi. \end{aligned} \tag{60}$$

Finally, we have

$$\begin{aligned} \nabla_{gradF'_r} F'_r d\phi\,(grad\ln\lambda) = {} & F'_r \nabla_{gradF'_r} d\phi\,(grad\ln\lambda) \\ & + \left|gradF'_r\right|^2 d\phi\,(grad\ln\lambda)\,. \end{aligned} \tag{61}$$

On substituting (59), (60) and (61) in (56), Theorem 12 follows.

□

In particular, we obtain

Corollary 5. *Let (M^m, g) be a flat Riemannian manifold. Then $Id_M : M \to M$ is proper F-biharmonic if and only if function F satisfied the equation*

$$\begin{cases} F'_r grad(\Delta F'_r) + \frac{1}{2} grad\left(|gradF'_r|^2\right) + F''_r grad(\Delta F'_r) = 0. \\ grad\ F'_r \neq 0. \end{cases}$$

Remark: From Corollary 5, we obtain many examples of proper F-biharmonic maps.

For example if $F(x, r) = h(x_i)f(r)$, then Id_{R^m} is proper F-biharmonic if and only $h(x_i) = \frac{C}{K}e^{Kx_i}$ where $x = (x_1, ..., x_m)$, $C = const$ *and* $K = -\frac{2f'(m/2)+f''(m/2)}{f'(m/2)}$.

Acknowledgements

The authors would like to thank Professor M. Djaa for his guidelines, his helpful comments and suggestions.

The authors were supported by the Laboratory of Geometry Analysis Control and Applications of Saida University and PRFU National Agency Scientific Research of Algeria.

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