METRIZABILITY OF MULTISET TOPOLOGICAL SPACES

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Abstract

In this paper, we have investigated one of the basic topological properties, called Metrizability in multiset topological space. Metrizable spaces are those topological spaces which are homeomorphic to a metric space. So, we first give the notion of metric between two multi-points in a finite multiset and studied some significant properties of a multiset metric space. The notion of metrizability is then studied by using this metric. Besides, the Urysohn's lemma which is considered to be one of the important tools in studying some metrization theorems in topology is also discussed in context with multisets.

1 Introduction

Multisets(or msets, in short) have arisen by violating one of the basic principles of the classical set theory that every mathematical object occurs without repetition. In the real world, we have to confront with many problems where there is enormous repetition. For instances, multiple roots of a polynomial, repeated statistical data, many strands of DNA etc. So to deal with problems involving multiple occurrences, multisets are more useful structures than sets. The term Multiset was first coined by N. G. de Bruijn stating it as a generalisation of a set. Besides its applications in the field computer science, physics, logic, many fields of mathematics have been explored in context with multisets. It led to the formulation of comprehensive theory of multisets Bilzard [1, 2, 12]. Girish and John [4] introduced a topology on multisets in 2011. They also studied in detail the notion of basis, closure, interior, limit points, continuity in an M-topological

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space in Girish and John [5]. Since then many developments have been made in studying many topological properties of multisets. Sobhy El-Sheikh et.al.[3]] studied different separation axioms in a multiset topological space. Shravan and Tripathy have studied the concept of generalized closed sets [7] and generalised open sets [10] in Multiset Topological spaces. The notion of quasi-coincidence in multiset [9] has also been introduced by Shravan and Tripathy which play a important role in the study of some new notions in multiset topology. The notion of multiset ideals and multiset local function was introduced in [8] whereas the study of Kuratowski closure operator on multiset topological space is done in [11]. The notion of paramaterized multiset metric space has been studied by Ibrahim et.al. [6], where a metric between two multisets in cardinality bounded multiset universe was introduced.

In this paper, we deal with metrizability of a Multiset Topological space. In section 3 metric between points in a multiset has been introduced followed by many properties of a multiset metric space have been investigated. Section 4 consist of the notion of metrizability in a Multiset Topological space and its important properties. The Urysohn's Lemma is also investigated in details in this section.

2   Definitions and Preliminaries

In this section, we procure some basic definitions, results and notations those will be used in this article.

Definition 1. *A domain X, is defined as the set of elements from which msets are constructed. The mset space $[X]^w$ is the set of all msets those are from X such that no element occurs more than w times.*

Throughout this paper, we denote a multiset drawn from the multiset space $[X]^w$ by M.

Definition 2. *A mset M drawn from the set X is represented by a count function M or $C_M : X \to N$, where N represents the set of non-negative integers.*

Here $C_M(x)$ is the number of occurrences of the element x in the mset M drawn from the set $X = \{x_1, x_2, ..., x_n\}$ as $M = \{m_1/x_1, m_2/x_2, ...m_n/x_n\}$ where m_i is the number of occurences of the element $x_i, i = 1, 2, ...n$ in the mset M. The elements which are not included in the mset M have zero count.

Consider two msets M and N drawn from a set X. The following are the operations defined on the msets, those will be used in the article.

1. $M = N$ if $C_M(x) = C_N(x)$ for all $x \in X$.

2. $M \subseteq N$ if $C_M(x) \leq C_N(x)$ for all $x \in X$.

3. $P = M \cup N$ if $C_P(x) = max\{C_M(x), C_N(x)\}$ for all $x \in X$.

4. $P = M \cap N$ if $C_P(x) = min\{C_M(x), C_N(x)\}$.

5. $P = M \oplus N$ if $C_P(x) = C_M(x) + C_N(x)$ for all $x \in X$.

6. $M^c = Z - M = \{C_{M^c}(x)/x : C_{M^c}(x) = C_Z(x) - C_M(x); x \in X\}$, Z is an mset with maximum multiplicity in the multiset space.

7. $P = M \ominus N$ if $C_P(x) = max\{C_M(x) - C_N(x), 0\}$ for all $x \in X$, where $\oplus$ and $\ominus$ represent mset addition and mset subtraction respectively.

Operations under collection of msets: Let $[X]^w$ be an mset space with $C_Z(x)$ as the multiplicities of $x \in X$ and $M_1, M_2, \ldots$ be a collection of msets drawn from $[X]^w$. Then the following operations are possible under an arbitrary collections of msets,

1. The Union
   $\cup_{i \in I} M_i = \{C_{M_i}(x)/x : C_{M_i}(x) = max\{C_{M_i}(x) : x \in X\}\}$.

2. The intersection
   $\cap_{i \in I} M_i = \{C_{\cap M_i}(x)/x : C_{\cap M_i}(x) = min\{C_{M_i}(x) : x \in X\}$.

3. The mset complement
   $M^c = Z \ominus M = \{C_{M^c}(x)/x : C_{M^c}(x) = C_Z(x) - C_M(x), x \in X\}$.

Definition 3. *Let M be an mset drawn from a set X. The support set of M denoted by M^* is a subset of X and $M^* = \{x \in X : C_M(x) > 0\}$.*

Definition 4. *An mset M is said to be an empty set if $C_M(x) = 0$, for all $x \in X$.*

Definition 5. *Let X be a support set and $[X]^w$ be the mset space defined over X. Then for any mset $M \in [X]^w$, the complement M^c of M in $[X]^w$ is an element of $[X]^w$ such that $C_{M^c} = w - C_M(x)$ for all $x \in X$.*

Since Cantor's power set theorem fails for msets, it is possible to formulate the following reasonable definition of a power mset of M for finite mset M that preserves Cantor's power set theorem.

Definition 6. *Let $M \in [X]^w$ be an mset. The power mset $P(M)$ of M is the set of all submsets of M. We have $N \in P(M)$ if and only if $N \subseteq M$. If $N = \emptyset$, then $N \in^1 P(M)$ and if $N \neq \emptyset$, then $N \in^k P(M)$, where $k = \prod_z \binom{[M]_z}{[N]_z}$, the product*

$\prod_z$ *is taken over by distinct elements z of the mset N and $|[M]_z| = m$ if and only if $z \in^m M$ and $|[N]_z| = n$ if and only if $z \in^n N$ then,*

$$\binom{[M]_z}{[N]_z} = \binom{m}{n} = \frac{m!}{n!(m-n)!}.$$

The power set of an mset is the support set of the power mset and is denoted by $P^*(M)$.

Remark 1. *Power mset is an mset but its support set is an ordinary set whose elements are msets.*

Multiset Topology

Girish and John [4] introduced the concept of Multiset topology and defined it as follows:

Let $M \in [X]^w$ and $\tau \subseteq P^*(M)$. Then τ is called a multiset topology of M if τ satisfies the following properties,

1. The mset M and the empty mset $\emptyset$ are in τ.

2. The mset union of elements of any subcollection of τ is in τ.

3. The mset intersection of the elements of any finite subcollection of τ is in τ.

Mathematically a multiset topological space is an ordered pair (M, τ) consisting of an mset $M \in [X]^w$ and a multiset topology $\tau \subseteq P^*(M)$ on M. Multiset topology is abbreviated as $M - Topology$. The elements of τ are called open mset.The complement of an open mset in a $M - Topological$ space is said to be closed mset.

The following definitions of some separation axioms studied by Sobhy El-Sheikh et.al.[6] have been stated which will be studied in the present article.

Definition 7. *A mset M is called a whole M-singleton and denoted by $\{k/x\}$ if $C_M : X \to N$ such that $C_M(x) = k$ and $C_M(x') = 0$, for all $x' \in X - \{x\}$.*

Definition 8. *($M - T_2$- space or M-Hausdorff spacse) Let (M, τ) be a M-topological space. If for every two M-singletons $\{k_1/x_1\}$, $\{k_2/x_2\} \in M$ s.t. $x_1 \neq x_2$, then there exist G and $H \in \tau$ s.t. $\{k_1/x_1\} \subseteq G$,$\{k_2/x_2\} \subseteq H$ and $G \cap H = \emptyset$. Then (M, τ) is $M - T_2$-space or M-Hausdorff space.*

Definition 9. *(M-Normal space) Let (M, τ) be a M-topological space. If for all F_1, $F_2 \in \tau^c$ s.t. $F_1 \cap F_2 = \phi$, then there exist $G, H \in \tau$ s.t. $F_1 \subseteq G$, $F_2 \subset H$ and $G \cap H = \emptyset$. Then (M, τ) is called M-normal space.*

3 Main results

Multiset Metric Space

We introduce here a metric between multipoints in a finite multiset and its different properties.

Definition 10. *A multiset M is called multipoint (m-point) on $[X]^w$, if*

$$C_M(x) = m, \ x \in M, \tag{1}$$

and

$$C_M(x) = 0, \ x \in X \setminus M. \tag{2}$$

Let M be the set of all multipoints in $[X]^w$. Then a multiset metric space can be defined as follows:

Definition 11. *Let $M = \{m_1/x_1, m_2/x_2, ...m_k/x_k\}$ be the collection of multi points in $[X]^w$. Define a function $d : M \times M \to [0, \infty)$ such that,*

1. *$d(m_1/x_1, m_2/x_2) \geq 0$, for all $m_1/x_1, m_2/x_2 \in M$.*

2. *$d(m_1/x_1, m_2/x_2) = 0$ if $x_1 = x_2$ and $m_1 = m_2$.*

3. *$d(m_1/x_1, m_2/x_2) = d(m_2/x_2, m_1/x_1)$.*

4. *$d(m_1/x_1, m_3/x_3) \leq d(m_1/x_1, m_2/x_2) + d(m_2/x_2, m_3/x_3)$.*

Then d is called a metric on M and (M, d) is multiset metric space.

Example 3.1. Let $d : M \times M \to [0, \infty)$ be defined by,

$d(m_1/x_1, m_2/x_2) = |m_1 - m_2| + |x_1 - x_2|, \ x_1, \ x_2 \in R.$

Clearly, $d(m_i/x_i, m_j/x_j) \geq 0$, for all $m_i/x_i, m_j/x_j \in M$.

$d(m_i/x_i, m_j/x_j) = 0.$
$\Leftrightarrow |m_i - m_j| + |x_i - x_j| = 0.$
$\Leftrightarrow |m_i - m_j| = 0,$ and $|x_i - x_j| = 0.$
$\Leftrightarrow m_i = m_j$ and $x_i = x_j.$

It is obvious that $d(m_i/x_i, m_j/x_j) = d(m_j/x_j, m_i/x_i)$
$d(m_i/x_i, m_k/x_k) = |m_i - m_k| + |x_i - x_k|.$
$\qquad = |(m_i - m_j) + (m_j - m_k)| + |(x_i - x_j) + (x_j - x_k)|.$
$\qquad \leq |m_i - m_j| + |m_j - m_k| + |x_i - x_j| + |x_j - x_k|.$
$\qquad = [|m_i - m_j| + |x_i - x_j|] + [|m_j - m_k| + |x_j - x_k|].$
$\qquad = d(m_i/x_i, m_j/x_j) + d(m_j/x_j, m_k/x_k).$

Thus d defined above is a metric on M and (M, d) is called a multiset metric space.

Example 3.2. Let M be a multiset in $[X]^w$. Define $d : M \times M \to [0, \infty)$ by,

$d(m_1/x_1, m_2/x_2) = 0,$ for $x_1 = x_2$ and $m_1 = m_2,$
$\qquad\qquad = 1, x_1 \neq x_2.$

It can be easily verified that, d satisfies all the axioms of a multiset metric and is the trivial metric. The space is called the *discrete multiset metric space.*

Definition 12. *Let (M, d) be a multiset metric space and $N \subseteq M$. Then a metric $d_N : N \times N \to [0, \infty)$ can be defined on N and (N, d) is called the subspace of the multiset metric space.*

Definition 13. *Let (M, d) be a multiset metric space. Then the radius of an mset $N \subseteq M$ is defined by,*

$Rad(N) = sup\{d(m_i/x_i, m_j/x_j); m_i/x_i, m_j/x_j \in N\}.$

Definition 14. *Let (M, d) be a multiset metric space and $m_i/x_i \in M$ be any m-point. Then for radius $r > 0$ the open M-sphere with centre m_i/x_i is defined as,*

$S_r(m_i/x_i) = \{m_j/x_j \in M; d(m_i/x_i, m_j/x_j) < r\}.$

Remark 2. *The closed M-sphere with radius r and centre m_i/x_i is defined by,*
$S_r(m_i/x_i) = \{m_j/x_j \in M; d(m_i/x_i, m_j/x_j) \leq r\}.$

Definition 15. *Let (M, d) be a multiset metric space. Then $N \subseteq M$ is called an open mset if there exists a real number $r > 0$ such that $S_r(m/x) \subseteq N$ for every $m/x \in N$.*

Remark 3. *Every open M-sphere in a multiset metric space is an open mset.*

Neighbourhood(nbhd) of a multipoint

Definition 16. *Let (M, d) be a multiset metric space and $N \subseteq M$. Then N is called a neighbourhood (nbhd) of a point $m/x \in M$ if there exists an open mset O such that, $m/x \in O \subseteq N$.*

Remark 4. *The empty mset and the universal mset in a metric space is always open.*

We state the following Lemma from point-set topology which is also true for multiset Topological spaces.

Lemma 1. *Let (M, d) be a multiset metric space. If $m/x \in M$ and $n/y \in S_r(m/x)$ for $r > 0$ then there exist $r_1 > 0$ such that,*

$$S_{r_1}(n/y) \subset S_r(m/x).$$

Theorem 1. *Let (M, d) be a multiset metric space. Then,*

(i) arbitrary union of open msets is open mset.
(ii) finite intersection of open mset is open mset.

Proof. (i) Let N_i be an open mset in a metric space (M, d). Then there exists a real number $r > 0$ such that $S_r(m_i/x_i) \subseteq N_i \subseteq \cup_i N_i$. Hence arbitrary union of open msets is open in a metric space.

(ii) Let us consider here two open msets N_1 and N_2 in a metric space (M, d). We show that $N_1 \cap N_2$ is also open. Let $m/x \in N_1 \cap N_2$ Since N_1 and N_2 are open,

$S_{r_1}(m/x) \subseteq N_1$ and $S_{r_2}(m/x) \subseteq N_2$

Consider r= $min\{r_1, r_2\}$. Then there exist an open M-sphere with radius r such that,

$S_r(m/x) \subseteq S_{r_i}(m/x) \subseteq N_i, i = 1, 2$.
So, $S_r(m/x) \subseteq N_1 \cap N_2$. Hence finite intersection of open mset is open. $\square$

Remark 5. *The above 1 establishes that every metric space has a topology associated with it.*

Theorem 2. *Every multiset metric space is M-Hausdorff.*

Proof. Let (M, d) be a multiset metric space. Consider two distinct m-points m/x, n/y. Then, clearly

$$d(m/x, n/y) \neq 0.$$

Let $r = \frac{d(m/x, n/y)}{2}$. Then one can find open M-spheres $S_r(m/x)$ and $S_r(n/y)$ such that $m/x \in S_r(m/x)$ and $n/y \in S_r(n/y)$.

Now we show that

$$S_r(m/x) \cap S_r(n/y) \neq \emptyset.$$

Suppose that $p/z \in S_r(m/x) \cap S_r(n/y)$.

Then,
$d(m/x, p/z) < r$ and $d(p/z, n/y) < r$.

Thus we have $2r = d(m/x, p/z) + d(p/z, n/y) > d(m/x, n/y) = 2r$, a contradiction.

Hence every multiset metric space is M-Hausdorff.

$\square$

Theorem 3. *Let (M, d) be a multiset metric space. Then the collection of open M-spheres $\mathcal{B} = \{S_r(m/x) | m/x \in M, r > 0\}$ is an M-basis.*

Proof. Since $m/x \in S_r(m/x)$ for any $m/x \in M$. Hence $\mathcal{B}$ satisfies the first condition for a basis.
For the second condition for M-basis let us assume that $m/x \in M_1 \cap M_2$. Then $m/x \in M_1$ and $m/x \in M_2$. By Lemma 1 there exist $r_1, r_2 > 0$ such that $S_{r_1}(m/x) \subset M_1$ and $S_{r_2}(m/x) \subset M_2$. Taking $r' = min\{r_1, r_2\}$, we have $m/x \in S_{r'}(m/x) \subset M_1 \cap M_2$. Hence the collection of all open M-spheres is an M-basis for a topology on X.

$\square$

Metrizability of Multiset Topology

In Theorem 3 we have observed that the collection of open M-spheres forms an M-basis from which we can generate an M-topology. We define such an topology as follows.

Definition 17. *Let (M, d) be a multiset metric space. The topology generated by the M-sphere $S_r(m/x)$ of the metric space is called the d-metric M-topology.*

Definition 18. *An M-topological space (M, d) is metrizable if there exists a metric d on M such that τ is equal to the metric topology of (M, d).*

Example 3.3. A multiset under the discrete M-topology is always metrizable.

Let M be an mset in $[X]^w$. The collection $P^*(M)$, the support set of the power mset of M is an M-topology which is called the discrete M-topology on M.

We can show that M is metrizable. Consider the trivial metric on M,

$$d(m_i/x_i, m_j/x_j) = 0, m_i/x_i = m_j/x_j,$$
$$= 1, m_i/x_i \neq m_j/x_j.$$

This metric induces the discrete M-topology on M i.e. every submset of M is open. For $r \leq 1$ and $m_i/x_i \in M$, $S_r(m_i/x_i) = \{m_i/x_i\}$, which is open since every open M-sphere is an open mset. Also for $N \subseteq M$, we have,

$$N = \cup_i \{m_i/x_i\}.$$

Hence N is open. Thus every submset of M is open giving rise to a discrete M-topology.

As we have already mentioned that as a topological property metrizability is very well-behaved, we can check this from following results.

Theorem 4. *Metrizability in an M-topological space is a hereditary property.*

Proof. Let (M, τ) be a metrizable M-topological space induced by a metric d. Let (N, τ) be a subspace of M. Then $d_N(m/x, n/y) = d(m/x, n/y)$ for every $m/x, n/y \in N$ is a metric on N by restriction. This subspace metric can induce the subspace M-topology on N and thereby making N metrizable. $\square$

In view of the above results, we state the following result without proof.

Proposition 1. *Metrizable spaces are M-Hausdorff.*

Proposition 2. *Metrizable spaces are M-normal.*

Thus, Metrizability is indeed a very nice structure and the topological spaces which are metrizable carries very rich and interesting properties. So the query which type of topological spaces are metrizable is very crucial. So, we investigate the following lemma which will provide us with tools to investigate this query.

Urysohn's Lemma

In point-set topological space, Urysohn's Lemma is of great importance in proving metrization theorems, which are theorems that yield conditions of whether a topological space is metrizable. So, motivated by its significance, we study Urysohn's Lemma in a multiset topological space which characterizes normality of M-topological space.

But, prior to the investigation of Urysohn's lemma, we state the following proposition which is also a characterisation of normal topological space.

Proposition 3. *A topological space (M, τ) is M-normal if and only if for every open mset O and every closed mset $C \subseteq O$ there is an open mset N such that,*

$$C \subseteq N \subseteq \bar{N} \subseteq O.$$

Proof. Let (M, τ) be a M-normal space and O be an open mset such that $C \subseteq O$ for every closed mset C.

Since O is open, O^c is closed. Thus O^c and C are closed mset such that,

$O^c \cap C = \emptyset.$

By definition of normality, there exist disjoint open mset N and N_1 so that,

$C \subseteq N$ and $O^c \subseteq N_1$,

i.e. $C \subseteq N \subseteq N_1^c$.

So, $\bar{N} \subseteq N_1^c$.

We have

$O^c \subseteq N_1$ and $\bar{N} \cap O^c = \emptyset.$

i.e $\bar{N} \subset O$.

Therefore,

$$C \subseteq N \subseteq \bar{N} \subseteq O.$$

$\square$

The following result is the M-set version of the Urysohn's Lemma.

Theorem 5. *Let (M, τ) be a M-normal space. Then for every pair of disjoint non empty whole submsets F_1, $F_2 \subseteq M$, there is a continuous multiset function $f : M \to [0, 1]$ such that $f(m/x) = 0$, for all $m/x \in F_1$ and $f(m/x) = 1$ for all $m/x \in F_2$.*

Proof. Let (M, τ) be a M-normal space and F_1, $F_2 \subseteq M$ be disjoint non-empty whole submsets.

Consider D to be the set of all rational numbers in [0,1]. Define for each $r \in D$, an open mset G_r such that,

$$C_{\bar{G}_r}(x) \leq C_{G_s}(x), \text{ whenever } r < s \text{ for all } x \in X.$$

Now, countability of D allows to use induction to define the sets G_r. Let us arrange the elements of D in some order i.e let $D = \{1, 0, 1/2, 1/3, 2/3, 1/4, 3/4, ...\}$.

Let $G_1 = M - F_2$, which is open mset. Then using last line of proposition 3, we have an open mset G_o such that,

$$F_1 \subseteq G_o \subseteq \bar{G}_o \subseteq G_1.$$

Let D_n be the set of first n natural numbers in the sequence and G_r be the open mset defined for all rational numbers $p \in D_n$ satisfying the condition,

$$C_{\bar{G}_r}(x) \leq C_{G_s}(x), \text{ whenever } r < s \text{ for all } x \in X. \tag{3}$$

Consider $D_{n+1} = D_n \cup \{q\}$ which is a finite subset of [0,1]. Also by our construction D_{n+1} contains 0 and 1. Thus D_{n+1} is a finite simply ordered set and so q has an immediate successor s and a predecessor r in D_{n+1}. But by induction, we have

$$C_{\bar{G}_r}(x) \leq C_{G_s}(x) \ \forall x \in X.$$

Again, by proposition 3. we have,

$$\bar{G}_r \subseteq G_q \subseteq \bar{G}_q \subseteq G_s.$$

We claim that (3) holds for every pair of elements of D_{n+1}. If both elements belongs to D_n then (3) holds by induction hypothesis. If one element is q and the

other is p from D_n then either $p \leq r$ in which we can have,

$$\bar{G}_p \subseteq G_r \subseteq G_q.$$

or $p \geq s$ for which we have,

$$\bar{G}_q \subseteq G_s \subseteq G_p.$$

Thus for every pair of elements in D_{n+1} (3) holds. So G_r can be defined for all $r \in D$.
Now, we find open msets G_r by extending r to all rational numbers set Q in R. We, therefore, define

$$G_r = \emptyset, \text{ if } r < 0,$$
$$\quad = M, \text{ if } r > 1.$$

Then (3) still holds for any pair of rational number r and s.

Let $m/x \in M$. Define,

$$Q(m/x) = \{r \in Q | m/x \in G_r\}.$$

Since for $r < 0$ there is no member $m/x \in G_r$, so this set has no number less than 0. Also for $r > 1$ and all $m/x \in G_r$. Hence $Q(m/x)$ is bounded below and has a glb. Define

$$f : M \to [0,1],$$

$$f(m/x) = inf Q(m/x),$$
$$\quad = inf\{r \in Q | m/x \in G_r\}.$$

We prove that f is the required function. We first show that f separates F_1 and F_2.

Let $m/x \in F_1$. Then $m/x \in G_o \subseteq G_r$ for all $r \geq 0$. So $m/x \in G_r$ for all $r \geq 0$ such that $Q(m/x)$ contains all non-negative rational numbers. Hence $f(m/x) = inf Q(m/x) = 0$. On the other hand, let $m/x \in F_2$. Then $m/x \notin G_1$ and therefore $m/x \notin G_r$ for all $r \leq 1$. So $Q(m/x)$ contains all rational numbers greater than 1. Hence $f(m/x) = inf Q(m/x) = 1$.

Finally, we now prove that f is continuous.

Before proving the continuity of f we claim the following to be true.

(*i*) If $m/x \in \bar{G}_r$ then $f(m/x) \leq r$.

(*ii*) If $m/x \notin G_r$ then $f(m/x) \geq r$.

To prove (*i*) let $m/x \in \bar{G}_r$. Then by our construction $m/x \in G_s$ for all $s > r$. Therefore $Q(m/x)$ contains all rational numbers greater than r.

Therefore $f(m/x) = inf Q(m/x) \leq r$.

Secondly for proving (*ii*), let $m/x \notin G_r$. Then $m/x \notin G_s$ for any $s \leq r$ such that, $f(m/x) = inf Q(m/x) \geq r$.

Let $I = (a, b)$ be an open interval in R. To prove $f^{-1}(I)$ is open in M. So for any arbitrary m-point $m/x \in f^{-1}(I)$, we find an open mset G of M such that,

$m/x \in G \subseteq f^{-1}(I)$,

i.e $f(m/x) \in f(G) \subseteq I$.

Let $m/x \in f^{-1}(I)$. Then $f(m/x) \in (a, b)$. So there exist rational number r and s such that

$a < r < f(m/x) < s < b$.

Since $r < f(m/x)$, it follows from (*i*) that $m/x \notin \bar{G}_r$. Again $f(m/x) < s$, it follows from (*ii*) that $m/x \in G_s$.

Hence $m/x \in G = G_s - \bar{G}_r$. This G is the required open mset. Now it remains to show that $f(G) \subseteq I$.

Let $m/x \in G$ such that $m/x \in G_s \subseteq \bar{G}_s$. Then (*i*) implies $f(m/x) \leq s$. Also $m/x \notin \bar{G}_r$, so $f(m/x) \geq r$ by (*i*) i,e $r \leq f(m/x) \leq s$. Thus $f(m/x) \in [r, s] \subseteq (a, b)$ as required.

Conversely, let for every disjoint msets F_1 and F_2 there exist a continuous function $f : M \to [0, 1]$ satisfying the condition stated in the theorem. Then it can be easily shown that $C \subseteq f^{-1}([0, a))$ and $D \subseteq f^{-1}((a, 1])$, where these inverse images are open and disjoint as f is continuous. $\qquad \square$

Declaration. The authors declare that the article is free from conflict of interest.

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